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SOME FUNCTIONS RELATED TO THE DERIVATIVES OF THE L-SERIES OF AN ELLIPTIC CURVE AT $s = 1$.

by M. J. Razar

If E is an elliptic curve defined over the rationals and having complex multiplication then it is well known that its L-series may be identified with a Hecke L-series with Grössencharakter. In their paper [1], Birch and Swinnerton-Dyer state the beautiful conjecture which now bears their names concerning the relationship between the arithmetic of the curve and the initial term in the Taylor expansion of $L_E(s)$ about $s = 1$. Their paper gives extensive evidence for the correct value of $L_E(1)$ in the above named case.

Further evidence (e.g. [4], [5], [7]) has accumulated over the years but certainly the most striking work is due to Coates and Wiles [2] who prove that if the group of rational points $E(\mathbb{Q})$ is infinite then $L_E(1) = 0$. Moreover, they give considerable insight into the (p-adic) correctness of the Birch-Swinnerton-Dyer predictions when $L_E(1) \neq 0$.

Considerably less is known about the first term in the expansion of $L_E(s)$ about $s = 1$ when $L_E(1) = 0$. Stephens ([7]) provides some numerical evidence. Birch (unpublished, to the best of my knowledge) has given a rather precise prediction in the rank 1 case ($L_E'(1) \neq 0$), by expressing the canonical height in terms of the Weierstrass sigma function.

One of the major obstacles in the study of $L_E(s)$ when $L_E(1) = 0$ has been the lack of suitable formulas for proving algebraicity and p-adic results about, say, $L_E'(1)$. In my talk I propose to discuss a close relation existing between the 'higher terms" of Kronecker's limit formulas (e.g. the Laurent expansion of $\Sigma\Sigma\,|mz + n|^{-2s}$ about $s = 0$ or $s = 1$) and the expansion of $L_E(s)$. Not surprisingly some of the functions arising in this context are explicable in terms of Bessel functions -but in a case where they simplify substantially. In addition, we are led to introduce some non meromorphic elliptic functions.

There are some analogies and connection between the above mentioned functions and the functions associated to the full Taylor expansions of Hurwitz-type zeta functions at non positive integers. In the latter situation some rather precise rationality statements can be proved. Koblitz and Ogus and, independently, Kubert have proved one such result in a slightly different formulation and I believe Gross will be discussing the p-adic significance of some of these ideas. For a fairly general "rationality" statement concerning the first non zero coefficient in the expansion of certain linear combinations of Dirichlet L-functions at non positive integers, see [6].

REFERENCES

[1] B. J. Birch and H.P.F. Swinnerton-Dyer : Notes on elliptic curves II, J. Reine u.angew. Math. vol. 218 (1965) pp.79-108.

[2] J. Coates and A. Wiles : On the conjecture of Birch and Swinnerton-Dyer, Inventiones Math. 39 (1977), pp.223-253.

[3] D. Goldfeld : The class number of quadratic fields and the conjectures of Birch and Swinnerton-Dyer, Annali Scuola Normale Superiore.

[4] M. J. Razar : The non vanishing of L(1) for certain elliptic curves, American Journal of Math., 96 (1974), pp.104-126.

[5] M. J. Razar : A relation between the two component of the Tate-Safarevic group and L(1) for certain elliptic curves, American Journal of Math. 96 (1974), pp.127-144.

[6] M. J. Razar : The value of the first non vanishing derivative at non positive integers of linear combinations of Dirichlet L-series (preliminary draft).

[7] N. Stephens : The conjectures of Birch and Swinnerton-Dyer about $X^3 + Y^3 = DZ^3$.

[8] A. Weil : Elliptic functions according to Eisenstein and Kronecker Springer Verlag, New York, 1976.

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