G. V. CHUDNOVSKY

Constructive and non-constructive methods of proofs of irrationality and transcendency and algebraic independence of periods of abelian varieties


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CONSTRUCTIVE AND NON-CONSTRUCTIVE METHODS OF
PROOFS OF IRRATIONALITY AND TRANSCENDENCY
AND ALGEBRAIC INDEPENDENCE OF
PERIODS OF ABELIAN VARIETIES.

by G. V. Chudnovsky

The methods announced in the title are based on the deformation
twoy theory for Fuchsian linear differential equations. They include effective
constructions of the Padé approximations to generalized hypergeometric
functions giving the periods of certain algebraic varieties, of the
Padé approximations to systems of Abelian integrals of the first and
second kind, and of the Padé approximations to systems of Abelian functions
on certain Abelian varieties. We shall describe the arithmetic structure
of the coefficients of the polynomials in the Padé approximations, their
sizes and the remainder term.

Example 1: Let $k$ be an algebraic number such that $0 < |k| < 1$, and
consider the equation: $y^2 = (1 - x^2)(1 - k^2x^2)$. If $|k| < H(k^2)^{-\delta}$ for some
positive $\delta$, then

$$|x_1 + y_1 + z_1 x_\eta| > H^{-C}$$

where $C = C(\delta)$.

If now $k^2 = 1/q$ and $q \geq q_0(\epsilon)$, then

$$|x_1 + y_1 + z_1 x_\eta| > H^{-3-\epsilon}$$

In these formulae, $x$, $y$, $z$ are integers, $0 < \max(|x|,|y|,|z|) = H$ and $\eta$ and $\eta_1$ are respectively the period and quasi period of the corresponding
elliptic curve.

The same type of result can be proved for $n$ elliptic curves.
Example 2: If \( E: y^2 = 4x^3 - g_2x - g_3 \) is defined over \( \mathbb{Q} \) and \( P(u) = 1/q \) for \( q \geq q_0(\varepsilon) \), then

\[
|uP'(u) - r/s| > |s|^{-2-\varepsilon}.
\]

We obtain similar results, under much milder restrictions, for \( \zeta(u) + \alpha u, \alpha \in \mathbb{Q} \), as well.

These are the most elementary examples. It should be noted that we can compute all the constants explicitly.

We shall also present results on algebraic independence for sets of numbers connected with the exponential function and Abelian functions. We propose new bounds for the types of transcendence of two numbers, as \( \eta/\omega \) and \( \zeta(u) - (\eta/\omega)u \), the algebraic independence of which was earlier proved by the author. Finally, we consider an elliptic function \( \rho(z) \) with complex multiplication over \( \mathbb{K} \), and obtain the bound

\[
|\rho(\rho(z),\rho(\rho(z)))| > H(P)^{-C_2d(P)^2}
\]

where \( \log \log H(P) \geq d(P)^4 \) and \( C_2 = C_2([\mathbb{K}(\alpha, \beta): \mathbb{Q}]) > 0 \).