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THE PERIODS OF ABELIAN VARIETIES WITH COMPLEX MULTIPLICATION AND THE SPECIAL VALUES OF CERTAIN ZETA FUNCTIONS

by

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Let K be a CM-field of degree 2n and I_K the free $\mathbb{Z}$-module generated by all embeddings of K into $\mathbb{C}$. Given a CM-type $\varphi = \sum_{i=1}^{n} \tau_i$ of K, take a $\overline{\mathbb{Q}}$-rational abelian variety of type (K,φ) and a $\overline{\mathbb{Q}}$-rational holomorphic 1-form ω_i on A such that $\omega_i \circ a = a^{\tau_i} \omega_i$ for all $a \in K$. As shown in [2, p.383], there is a non-zero complex number $p_K(\tau_i, \varphi)$ depending only on K, φ, and τ_i such that

$$[\pi . p_K(\tau_i, \varphi)]^{-1} \int_c \omega_i \in \overline{\mathbb{Q}}$$

for every $c \in H_1(A, \mathbb{Z})$. The quantity $p_K(\tau_i, \varphi)$ can actually be chosen to be a positive real number; it is also given as the value of a certain $\overline{\mathbb{Q}}$-rational (meromorphic) Hilbert modular form at a CM-point (see [2]). Now denote by ρ the complex conjugation, and put $p_K(\tau_i \rho, \varphi) = p_K(\tau_i, \varphi)^{-1}$. Then we have

Theorem 1 : If $\varphi_1, \dots, \varphi_m$ are CM-types of K and τ is an embedding of K into $\mathbb{C}$, the product $\prod_{i=1}^{m} p_K(\tau, \varphi_i)^{s_i}$ with $s_i \in \mathbb{Z}$, up to algebraic factors, depends only on τ and $\sum_{i=1}^{m} s_i \varphi_i$. Moreover, if L is a CM-field containing K and ψ is a CM-type of L whose restriction to K is $\sum_{i=1}^{m} s_i \varphi_i$, then the above product equals, up to algebraic factors, to $\prod_{\sigma} p_L(\sigma, \psi)$, where σ runs over all embeddings of L into $\mathbb{C}$, which coincide with τ on K.

The proof is given in [3] . To express this theorem in a different way, we consider two linear maps

$$\mathrm{Res}_{L/K} : I_L \longrightarrow I_K \quad , \quad \mathrm{Inf}_{L/K} : I_K \longrightarrow I_L .$$

Here $\mathrm{Res}_{L/K}(\sigma)$ is the sum of all restrictions of σ to K; $\mathrm{Inf}_{L/K}(\tau)$ is the sum of all extensions of τ to L.

Theorem 2 : The above p_K can be extended to a bilinear map of $I_K \times I_K$ into $\mathbb{C}^\times/\overline{\mathbb{Q}}^\times$ with the following properties :

1) $p_K(\alpha\rho, \beta) = p_K(\alpha, \beta\rho) = p_K(\alpha, \beta)^{-1}$ for $\alpha, \beta \in I_K$;

2) $p_K(\alpha, \mathrm{Res}_{L/K}\beta) = p_L(\mathrm{Inf}_{L/K}\alpha, \beta)$, $p_K(\mathrm{Res}_{L/K}\beta, \alpha) = p_L(\beta, \mathrm{Inf}_{L/K}\alpha)$ for $\alpha \in I_K$, $\beta \in I_L$, and $K \subset L$;

3) $p_M(\gamma\alpha, \gamma\beta) = p_K(\alpha, \beta)$ if γ is an isomorphism of M onto K.

Theorem 3 : If (L, ψ) is the reflex of (K, φ), we have $p_K(\sigma, \varphi) = p_L(\psi\sigma, \mathrm{id}_L)$ for every embedding σ of K into $\mathbb{C}$.

These theorems imply various algebraic relations among the periods. For example, we have :

<u>Theorem 4</u> : <u>For</u> $\alpha \in I_K$, <u>let</u> $t(\alpha)$ <u>denote the rank of the module</u> $\sum_{\gamma\in G} \mathbb{Z}\alpha\gamma$, <u>where</u> G <u>is the Galois group over</u> $\mathbb{Q}$ <u>of the Galois closure of</u> K. <u>If</u> $\sum_{i=1}^{n} \tau_i$ <u>is a CM-type of</u> K, <u>then for every</u> $\beta \in I_K$, <u>the module</u>

$$\{(e_1,\ldots,e_n) \in \mathbb{Z}^n \mid \prod_{i=1}^{n} p_K(\tau_i,\beta)^{e_i} = 1\}$$

<u>has rank at least</u> $n - t(\beta - \beta\rho)$.

If β is a CM-type, we have $t(\beta-\beta\rho) = t(\beta) - 1$. Theorems 2, 3 and 4 will be proved in [4].

The quantities p_K occur as the values of an L-function of a CM-field with an algebraic valued Hecke character of infinite order (see [1, Theorem 2]). As a new example of a zeta function whose values are given by p_K, we consider

$$D(s) = \sum_{0\neq x \equiv a(\Lambda)} \mu\left(\mathrm{Tr}_{K/\mathbb{Q}}(yxx^{\rho})\right) x^{\Phi}(x^{\tau})^{-k} |x^{\tau}|^{-2s} \qquad (s \in \mathbb{C}).$$

Here Λ is a lattice in K and $a \in K$; $0 < k \in \mathbb{Z}$; τ is an embedding of K into $\mathbb{C}$; μ denotes the Fourier coefficients of an elliptic modular form $g(z) = \sum \mu(b) e^{2\pi i b z}$; y is a real element of K such that y^{τ} is its only positive conjugate; Φ is an element of I_K with non-negative coefficients.

<u>Theorem 5</u> : <u>The series</u> D <u>is convergent for sufficiently large</u> $\mathrm{Re}(s)$ <u>and can be continued to a meromorphic function on the whole plane</u>.

<u>Theorem 6</u> : <u>Suppose that</u> g <u>is a cusp form of weight</u> ℓ, $\mu(b)$ <u>are all algebraic, and</u> τ <u>and</u> $\tau\rho$ <u>occur in</u> Φ <u>with the same multiplicity, say</u> q. <u>Let</u> m <u>be an integer such that</u>

$$(2n - 1 - k + \ell + \deg(\Phi))/2 < m \leq q .$$

<u>Then</u> $D(m)$ <u>is</u> $\pi^k p_K(k\tau - \Phi, 2\tau)$ <u>times an algebraic number</u>.

A more general result holds for a series of a similar type with a Hilbert modular form (which is not necessarily a cusp form) in place of g. The details will be given in [4].

References

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