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THE PERIODS OF ABELIAN VARIETIES WITH COMPLEX
MULTIPLICATION AND THE SPECIAL VALUES
OF CERTAIN ZETA FUNCTIONS

by

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Let $K$ be a CM-field of degree $2n$ and $I_K$ the free $\mathbb{Z}$-module generated by all embeddings of $K$ into $\mathbb{C}$. Given a CM-type $\varphi = \sum_{i=1}^{n} \tau_i$ of $K$, take a $\mathbb{Q}$-rational abelian variety of type $(K, \varphi)$ and a $\mathbb{Q}$-rational holomorphic 1-form $\omega_i$ on $A$ such that $\omega_i \cdot a = a_i^* \omega_i$ for all $a \in K$. As shown in [2, p.383], there is a non-zero complex number $p_K(\tau_i, \varphi)$ depending only on $K, \varphi$, and $\tau_i$ such that

$$[\pi.p_K(\tau_i, \varphi)]^{-1} \int_c \omega_i \in \overline{ \mathbb{Q} }$$

for every $c \in H_1(A, \mathbb{Z})$. The quantity $p_K(\tau_i, \varphi)$ can actually be chosen to be a positive real number; it is also given as the value of a certain $\mathbb{Q}$-rational (meromorphic) Hilbert modular form at a CM-point (see [2]). Now denote by $\rho$ the complex conjugation, and put $\Gamma_K(\tau_i, \varphi) = p_K(\tau_i, \varphi)^{-1}$. Then we have
Theorem 1: If \( \varphi_1, \ldots, \varphi_m \) are CM-types of \( K \) and \( \tau \) is an embedding of \( K \) into \( \mathbb{C} \), the product \( \prod_{i=1}^m p_K(\tau, \varphi_i) \) with \( s_i \in \mathbb{Z} \), up to algebraic factors, depends only on \( \tau \) and \( \sum s_i \varphi_i \). Moreover, if \( L \) is a CM-field containing \( K \) and \( \psi \) is a CM-type of \( L \) whose restriction to \( K \) is \( \sum_s \varphi_s \), then the above product equals, up to algebraic factors, to \( \prod_{\sigma} p_L(\sigma, \psi) \), where \( \sigma \) runs over all embeddings of \( L \) into \( \mathbb{C} \), which coincide with \( \tau \) on \( K \).

The proof is given in [3]. To express this theorem in a different way, we consider two linear maps

\[
\text{Res}_{L/K} : I_L \mapsto I_K , \quad \text{Inf}_{L/K} : I_K \rightarrow I_L.
\]

Here \( \text{Res}_{L/K}(\sigma) \) is the sum of all restrictions of \( \sigma \) to \( K \); \( \text{Inf}_{L/K}(\tau) \) is the sum of all extensions of \( \tau \) to \( L \).

Theorem 2: The above \( p_K \) can be extended to a bilinear map of \( I_K \times I_K \) into \( \mathbb{C} \) with the following properties:

1) \( p_K(\alpha \beta) = p_K(\alpha, \beta \alpha) = p_K(\alpha, \beta)^{-1} \) for \( \alpha, \beta \in I_K \);

2) \( p_K(\alpha, \text{Res}_{L/K} \beta) = p_L(\text{Inf}_{L/K} \alpha, \beta), p_K(\text{Res}_{L/K} \beta, \alpha) = p_L(\beta, \text{Inf}_{L/K} \alpha) \) for \( \alpha \in I_K, \beta \in I_L \), and \( K \subseteq L \);

3) \( p_M(\gamma \alpha, \gamma \beta) = p_K(\alpha, \beta) \) if \( \gamma \) is an isomorphism of \( M \) onto \( K \).

Theorem 3: If \( (L, \psi) \) is the reflex of \( (K, \varphi) \), we have \( p_K(\alpha, \varphi) = p_L(\psi \alpha, \text{id}_L) \) for every embedding \( \sigma \) of \( K \) into \( \mathbb{C} \).

These theorems imply various algebraic relations among the periods. For example, we have:

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Theorem 4: For $\alpha \in I_K$, let $t(\alpha)$ denote the rank of the module $\sum_{\gamma \in \mathcal{C}} \mathbb{Z}\alpha\gamma$, where $G$ is the Galois group over $\mathfrak{q}$ of the Galois closure of $K$. If $\sum_{i=1}^{n} \tau_i$ is a CM-type of $K$, then for every $\beta \in I_K$, the module

$\{(e_1, \ldots, e_n) \in \mathbb{Z}^n \mid \prod_{i=1}^{n} p_K(\tau_i, \beta) e_i = 1\}$

has rank at least $n - t(\beta - \beta_0)$.

If $\beta$ is a CM-type, we have $t(\beta - \beta_0) = t(\beta) - 1$. Theorems 2, 3 and 4 will be proved in [4].

The quantities $p_K$ occur as the values of an $L$-function of a CM-field with an algebraic valued Hecke character of infinite order (see [1, Theorem 2]). As a new example of a zeta function whose values are given by $p_K$, we consider

$$D(s) = \sum_{\mathfrak{a} \in \mathcal{A}(A)} \mu(T\mathcal{K}/\mathbb{Q}(\mathfrak{a})) x^k |x^\tau|^{-2s} \quad (s \in \mathbb{C}).$$

Here $\Lambda$ is a lattice in $K$ and $a \in K$; $0 < k \in \mathbb{Z}$; $\tau$ is an embedding of $K$ into $\mathbb{C}$; $\mu$ denotes the Fourier coefficients of an elliptic modular form $g(z) = \sum_{b} \mu(b) e^{2\pi ibz}$; $Y$ is a real element of $K$ such that $Y^\tau$ is its only positive conjugate; $\phi$ is an element of $I_K$ with non-negative coefficients.

Theorem 5: The series $D$ is convergent for sufficiently large $\text{Re}(s)$ and can be continued to a meromorphic function on the whole plane.

Theorem 6: Suppose that $g$ is a cusp form of weight $\ell$, $\mu(b)$ are all algebraic, and $\tau$ and $\tau_0$ occur in $\phi$ with the same multiplicity, say $q$. Let $m$ be an integer such that

$$(2n - 1 - k + \ell + \deg(\phi))/2 < m < q.$$ 

Then $D(m)$ is $\pi p_K(k\tau - \phi, 2\tau)$ times an algebraic number.
A more general result holds for a series of a similar type with a Hilbert modular form (which is not necessarily a cusp form) in place of $g$. The details will be given in [4].

References


