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THE PERIODS OF ABELIAN VARIETIES WITH COMPLEX
MULTIPLICATION AND THE SPECIAL VALUES
OF CERTAIN ZETA FUNCTIONS

by

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Let $K$ be a CM-field of degree $2n$ and $I_K$ the free $\mathbb{Z}$-module generated by all embeddings of $K$ into $\mathbb{C}$. Given a CM-type $\varphi = \sum_{i=1}^{n} \tau_i$ of $K$, take a $\mathcal{O}$-rational abelian variety of type $(K, \varphi)$ and a $\mathcal{O}$-rational holomorphic 1-form $\omega_i$ on $A$ such that $\omega_i \cdot a = a^i \omega_i$ for all $a \in K$. As shown in [2, p. 383], there is a non-zero complex number $p_K(\tau_i, \varphi)$ depending only on $K, \varphi$, and $\tau_i$ such that

$$[\pi.p_K(\tau_i, \varphi)]^{-1} \int_{c} \omega_i \in \mathcal{O}$$

for every $c \in H_1(A, \mathbb{Z})$. The quantity $p_K(\tau_i, \varphi)$ can actually be chosen to be a positive real number; it is also given as the value of a certain $\mathcal{O}$-rational (meromorphic) Hilbert modular form at a CM-point (see [2]). Now denote by $\rho$ the complex conjugation, and put $\Gamma_K(\tau_i, \varphi) = p_K(\tau_i, \varphi)^{-1}$. Then we have
Theorem 1: If $\phi_1, \ldots, \phi_m$ are CM-types of $K$ and $\tau$ is an embedding of $K$ into $\mathbb{C}$, the product $\prod_{i=1}^m p_K(\tau, \phi_i)^{s_i}$ with $s_i \in \mathbb{Z}$, up to algebraic factors, depends only on $\tau$ and $\sum_{i=1}^m s_i \phi_i$. Moreover, if $L$ is a CM-field containing $K$ and $\psi$ is a CM-type of $L$ whose restriction to $K$ is $\sum_{i=1}^m s_i \phi_i$, then the above product equals, up to algebraic factors, to $\prod_{\sigma} p_L(\sigma, \psi)$, where $\sigma$ runs over all embeddings of $L$ into $\mathbb{C}$, which coincide with $\tau$ on $K$.

The proof is given in [3]. To express this theorem in a different way, we consider two linear maps

$$\text{Res}_{L/K} : I_L \to I_K, \quad \text{Inf}_{L/K} : I_K \to I_L.$$

Here $\text{Res}_{L/K}(\sigma)$ is the sum of all restrictions of $\sigma$ to $K$; $\text{Inf}_{L/K}(\tau)$ is the sum of all extensions of $\tau$ to $L$.

Theorem 2: The above $p_K$ can be extended to a bilinear map of $I_K \times I_K$ into $\mathbb{C}^\times / \mathbb{Q}^\times$ with the following properties:

1) $p_K(\alpha \beta) = p_K(\alpha, \beta) = p_K(\alpha, \beta)^{-1}$ for $\alpha, \beta \in I_K$;

2) $p_K(\alpha, \text{Res}_{L/K}(\beta)) = p_L(\text{Inf}_{L/K} \alpha, \beta)$, $p_K(\text{Res}_{L/K}(\beta), \alpha) = p_L(\beta, \text{Inf}_{L/K} \alpha)$ for $\alpha \in I_K$, $\beta \in I_L$, and $K \subset L$;

3) $p_M(\gamma \alpha, \gamma \beta) = p_K(\alpha, \beta)$ if $\gamma$ is an isomorphism of $M$ onto $K$.

Theorem 3: If $(L, \psi)$ is the reflex of $(K, \phi)$, we have $p_K(\sigma, \phi) = p_L(\psi \sigma, \text{id}_L)$ for every embedding $\sigma$ of $K$ into $\mathbb{C}$.

These theorems imply various algebraic relations among the periods. For example, we have:
Theorem 4: For $\alpha \in \mathcal{I}_K$, let $t(\alpha)$ denote the rank of the module $\sum_{\gamma \in G} \mathbb{Z} \alpha \gamma$, where $G$ is the Galois group over $\mathbb{Q}$ of the Galois closure of $K$. If $\sum_{i=1}^{n} \tau_i$ is a CM-type of $K$, then for every $\beta \in \mathcal{I}_K$, the module

$$\{(e_1, \ldots, e_n) \in \mathbb{Z}^n \mid \prod_{i=1}^{n} p_K(\tau_i, \beta) e_i = 1\}$$

has rank at least $n - t(\beta - \beta_0)$.

If $\beta$ is a CM-type, we have $t(\beta - \beta_0) = t(\beta) - 1$. Theorems 2, 3 and 4 will be proved in [4].

The quantities $p_K$ occur as the values of an $L$-function of a CM-field with an algebraic valued Hecke character of infinite order (see [1, Theorem 2]).

As a new example of a zeta function whose values are given by $p_K$, we consider

$$D(s) = \sum_{\mathfrak{a} \in \mathcal{A}(\lambda)} \mu(\mathcal{K}_\mathfrak{a}(\mathbb{Q})) x^{\phi(x \tau)} (x \tau)^{-k} x^2 s^{-2 s} (s \in \mathbb{C}).$$

Here $\mathcal{A}$ is a lattice in $K$ and $\alpha \in \mathcal{K} = \mathbb{Z}$; $\tau$ is an embedding of $K$ into $\mathbb{C}$; $\mu$ denotes the Fourier coefficients of an elliptic modular form $g(z) = \sum_{b} \mu(b) e^{2\pi i b z}$; $\gamma$ is a real element of $K$ such that $\gamma^\tau$ is its only positive conjugate; $\phi$ is an element of $\mathcal{I}_K$ with non-negative coefficients.

Theorem 5: The series $D$ is convergent for sufficiently large $\text{Re}(s)$ and can be continued to a meromorphic function on the whole plane.

Theorem 6: Suppose that $g$ is a cusp form of weight $l$, $\mu(b)$ are all algebraic, and $\tau$ and $\tau_0$ occur in $\phi$ with the same multiplicity, say $q$. Let $m$ be an integer such that

$$(2n - 1 - k + l + \text{deg}(\phi))/2 < m \leq q.$$ 

Then $D(m)$ is $\pi^r p_K(k \tau - \phi, 2 \tau)$ times an algebraic number.
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A more general result holds for a series of a similar type with a Hilbert modular form (which is not necessarily a cusp form) in place of $g$. The details will be given in [4].

References


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