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PERTURBED OPTIMIZATION PROBLEMS

IN BANACH SPACES

Gérard LEBOURG

ABSTRACT -
Families of perturbed (non-convex) optimization problems in Banach spaces have been shown to be densely solvable ; this paper tries to present a unified approach of such density results and highlights their intimate connection with the geometry of Banach spaces.

INTRODUCTION -
The paper is concerned with families of optimization problems regularly depending on a parameter, the domain of which is an open subset of a Banach space. Such families often arise as "natural" perturbations of a given optimization problem ; beside trivial situations, mostly encountered in finite dimensional spaces, the so-defined perturbed problems may have no solution.

However, typical examples of those natural perturbations have been shown to be densely solvable. Roughly speaking this means that the original problem can be approximated by perturbed problems, all of which do have solutions. This paper tries to provide a starting point for the understanding of such density results.

Section 1 is introductory and deals with our starting example of a density result : the well-known theorem of Bishop-Phelps, considered here through the frame of optimization theory ; section 2 sketches out a proof of a non-convex variant of this theorem and suggests a possible approach to density results, which is developped in section 3 and leads to the fundamental result of section 4;section 5 emphazises the role played by the space of parameters and states an abstract density result in uniformly convexifiable Banach spaces.

Examples are taken from former works ([1], [2] and [5]), and must be thought of just as guide marks.

Of course, the paper does not supply a survey of what should be termed a perturbation theory of optimization problems. As a matter of facts, it brings out more questions than answers.

I. PERTURBATIONS OF OPTIMIZATION PROBLEMS

Throughout the following, E is a Banach space, E^* its dual space ; $\| \ \|$ represents indifferently the norm of E or E^*. Unless further specifications are given, the topological notions used always refer to the norm topology on E or E^*.

1.1. Let X be a subset of a Banach space, f a lower semi-continuous mapping from X into $\mathbb{R} \cup \{+\infty\}$, bounded from below, and consider the minimization problem : minimize f on X , which we write :

$$\mathcal{P} : \inf_{x \in X} f(x)$$

A perturbation of the problem $\mathcal{P}$, with domain of parameters an open subset Ω of a Banach space, will be the datum of a family of minimization problems :

$$\mathcal{P}_v : \inf_{x \in X} F(x,v)$$

where F is a mapping from $X \times \Omega$ into $\mathbb{R} \cup \{+\infty\}$ such that :

i) for every $v \in \Omega$, F_v ; $x \to F(x,v)$ is lower semi-continuous on X and bounded from below.

ii) for some $u \in \Omega$, $F_u = f$.

Such a perturbation will be said continuous if the value $V(v) = \inf_{x \in X} F(x,v)$ of the problem $\mathcal{P}_v$ continuously depends on the parameter v on Ω.

1.2. Central to the duality theory of Banach spaces, the minimization problem of a continuous linear functional on a bounded subset provides an elementary example of a continuous perturbation : define the initial problem $\mathcal{P}$ by : minimize a continuous linear functional u on a closed bounded subset X of a Banach space E :

$$\mathcal{P} : \inf_{x \in X} u(x)$$

This problem is naturally immersed in the family of optimization problems :

$$(1.1) \qquad \mathcal{P}_v : \inf_{x \in X} v(x)$$

with domain of parameters an open neighbourhood of u in Ω .

But for weakly compact subsets X , $\mathcal{P}_v$ may have no solution. Nethertheless, in the case of a convex subset X , the Bishop-Phelps theorem ([7] asserts that $\mathcal{P}_v$ is solvable for each v in a dense subset of Ω .

THEOREM 1 (Bishop-Phelps)

Let X be a closed bounded convex subset of a Banach space E ; the set of continuous linear functionals which achieve their minimum on X is dense in E^* .

Thus the initial problem $\mathcal{P}$ can be approximated by perturbed problems which have solutions : we shall say that the perturbation defined in (1.1) is densely solvable.

1.3. A little more work allows us to derive from the Bishop-Phelps theorem that, for every closed bounded convex subset X of any Banach space E , and every lower semi-continuous convex function $f : E \to \mathbb{R} \cup \{+\infty\}$ bounded from below on X , the "convex" minimization problem " $\mathcal{P}$: minimize f on X" can be slightly perturbed in a way that it becomes solvable. As a matter of fact, consider the perturbation :

$\mathcal{P}(\alpha,v)$: minimize the continuous linear functional $(\lambda,s) \to \alpha\lambda + v(x)$ on the epigraph of f above X with domain of parameters $]0,+\infty[\times E^*$.

By the Bishop-Phelps theorem, this perturbation is densely solvable, and, for every $(\alpha,v) \in]0,+\infty[\times E^*$ $\mathcal{P}(\alpha,v)$ is equivalent to the minimization problem :

$$\mathcal{P}_v : \inf_{x \in X} f(x) + \alpha^{-1}v(x)$$

Thus the perturbation $\mathcal{P}_v$ is again densely solvable since $(\alpha,v) \to \alpha^{-1}v$ is a continuous mapping from $]0,+\infty[\times E^*$ onto E^* .

The next section sketches out the proof of a non-convex variant of the Bishop-Phelps theorem which provides a similar result for some non-convex optimization problems.

2. A NON-CONVEX BISHOP-PHELPS THEOREM IN ASPLUND SPACES

2.1. To release the convexity assumption in theorem 1 , we need a geometric assumption on the space of parameters ; recall that an Asplund space ([6]) is :

(2.1) A Banach space E such that every continuous convex function defined on an open (convex) subset $\Omega \in E$ is Fréchet-derivable at a dense G_δ subset in Ω .

THEOREM 2

Let X be a closed bounded subset of a Banach space E , the dual or which is an Asplund space E^* , and f a lower semi-continuous mapping from E into $\mathbb{R} \cup \{+\infty\}$, bounded from below on X .

The perturbation :

$$\mathcal{P}_v : \inf_{x \in X} f(x) + v(x)$$

with domain of parameters E^* , is densely solvable ; moreover, there exists a dense G_δ subset in E^* , at every point v of which :

i) $\mathcal{P}_v$ admets a unique solution x(v)

ii) the minimizing sequences of $\mathcal{P}_v$ converge to x(v) .

2.2. The proof follows geometrical intuition : the involved perturbation is generated by the deformation :

$$F : X \times \Omega \to \mathbb{R} \cup \{+\infty\} : (x,v) \to f(x) + v(x)$$

of the criterion f . The value function now appears as the pointwise infimum of the family of functions $F_x : v \to F(x,v)$. Let j be the canonical injection from X into E^{**} ; a <u>contact unit</u> associated with the family of functions F_x will be an element $(v,p,F_x(v))$ in $E^* \times E^{**} \times \mathbb{R}$ where $p = j(x)$ stands for the "gradient" of F_x at v . The proof relies on the fact that, subject to Fréchet-differentiability of the value function V at $u \in E^*$, there exists, for every minimizing sequence x_n of the problem $\mathcal{P}_u$, a sequence

$$(u_n,p_n,F_{x_n}(u_n)) = (u_n,j(x_n),f(x_n) + u_n(x_n))$$

of contact units which converges to $(u,v'(u),v(u))$ in $E^* \times E^{**} \times \mathbb{R}$ (the claim will be proved in section 4).

Since j is an isomorphism from X onto $j(x)$, the minimizing sequence x_n converges to some $x \in X$, which satisfies :

$$j(x) = v'(u) \quad \text{and} \quad f(x) + u(x) = F_x(u) = v(u)$$

Thus $\mathcal{P}_u$ admits a unique solution $x(u)$ characterized by :

$$j(x(u)) = v'(u)$$

Now, we only need the geometrical assumption on E^* , since v is a continuous concave function, hence Frěchet-derivable on a dense G_δ subset in E^* .

2.3. Therefore, every minimization problem on a closed bounded subset of a predual of an Asplund space can be slightly perturbed in a way that it becomes solvable. We point out that many classical spaces of functional analysis, among which reflexive Banach spaces are preduals of Asplond spaces.

The approach used in theorem 2 can be carried on to other types of perturbations ; however, it first requires to single out a class of suitable perturbations.

3. <u>A SELECTION OF SUITABLE PERTURBATIONS</u>

3.1. Let us consider a perturbation of some initial minimization problem $\mathcal{P}$, defined as in 1.1. :

$$\mathcal{P}_v : \inf_{x \in X} F(x,v)$$

with domain of parameters an open subset Ω of a Banach space E .

From now on, we shall assume that :

(3.1) for every $x \in X$, $F_x : v \to F(x,v)$ is locally Lipschitzian on its effective domain $\{v \in \Omega \ / \ F(x,v) < +\infty\}$.

Then we can still define a <u>contact unit</u> associated with the family of functions F_x as an element $(v,p,F_x(v))$ in $\Omega \times E^* \times \mathbb{R}$, with p a continuous linear functional on the space E of parameters, which belongs to the generalized gradient $\partial F_x(v)$ of F_x at $v \in \Omega$ ([3]) .

p can still be thought of as the "gradient" of a tangent hyperplane in $E \times \mathbb{R}$ to the epigraph of F_x above Ω , at the point $(v,F_x(v))$.

The following criterion provides a convenient substitute to the situation encountered in section 2.

The perturbation :

$$\mathcal{P}_v : \inf_{x \in X} F(x,v)$$

with domain of parameters an open subset Ω of a Banach space E , is said to be <u>proper</u> if :

(3.2) for every sequence $(u_n,p_n,F_{x_n}(u_n))$ of contact units converging in $\Omega \times E^* \times \mathbb{R}$, the sequence x_n has at least one cluster point $x \in X$.

The linear perturbation introduced in theorem 2 is clearly a proper one. Other easy examples suit this "rough" criterion ; for instance, the elementary perturbation derived from the nearest points problem :

3.2. The initial problem is to minimize the distance of a given point u in a Banach space E to the elements of a closed subset $X \subset E$:

$$\mathcal{P} : \inf_{x \in X} \|x - u\|$$

The family of associated perturbed problems is :

$$\mathcal{P}_v : \inf_{x \in X} \|x - v\|$$

with domain of parameters a neighbourhood Ω of u in E . This define a natural perturbation of the problem $\mathcal{P}$ which is clearly continuous (cf. 1.1). We claim that, for a weakly relatively compact closed subset X , it is also a proper one, as soon as the norm of E satisfies the following geometric condition :

(3.3) every weakly convergent sequence in the unit sphere of E

converges in fact in the norm topology.

We shall prove in fact a little more sophisticated result :

PROPOSITION 1

Let X be a weakly relatively compact closed subset of a Banach space E , f a lower semi-continuous mapping from E into $\mathbb{R}\cup\{+\infty\}$, bounded from below on X , and φ a lower semi-continuous mapping from $\mathbb{R}$ into $\mathbb{R}\cup\{+\infty\}$, locally Lipschitzian on its effective domain $\{t\in\mathbb{R} \ / \ \varphi(t) < +\infty\}$.

Consider the minimization problem :

$$\mathcal{P} : \inf_{x\in X} f(x) + \varphi(\|x-u\|)$$

and assume that the generalized gradient of φ at $t \neq 0$ never contains 0 , and the norm of E satisfies (3.3) , then the family of minimization problems :

$$\mathcal{P}_v : \inf_{x\in X} f(x) + \varphi(\|x-v\|)$$

with domain of parameters an open neighbourhood Ω of u in E defines a proper perturbation of $\mathcal{P}$.

3.3. Proof : Denote by F the mapping $(x,u) \to f(x) + \varphi(\|x-v\|)$ from $X\times\Omega$ into $\mathbb{R}\cup\{+\infty\}$. Then, for every $v\in\Omega$, $F_v : X \to F(x,v)$ is lower semi-continuous on X and bounded from below ; thus, according to 1.1. the family of minimization problems $\mathcal{P}_v$ defines a perturbation of the problem $\mathcal{P}$. Moreover, it satisfies (3.1). Therefore, we only have to check condition (3.2). Let $(u_n, p_n, F_{x_n}(u_n))$ be a sequence of contact units which converges to (u,p,α) in $\Omega\times E^*\times\mathbb{R}$. By assumption, there always exists a subsequence $(u_m, p_m, F_{x_m}(u_m))$ such that x_m weakly converges to some $X\in E$, and $\|x_m-u_m\|$ converges to β in $\mathbb{R}$.

Now, for every $m\in\mathbb{N}$, $p_m = r_m\cdot q_m$ where $r_m\in\partial\varphi(\|x_m-u_m\|)$ and q_m belongs to the generalized gradient of the norm at $x_m - u_m$, that is :

$$q_m(x_m-u_m) = \|x_m-u_m\| \ , \ \|q_m\| \leqq 1 \ .$$

If $\beta = 0$, x_m converges to $x = u$ in X ; otherwise , r_m converges to some $r\in\partial\varphi(\beta)$ and $q_m = r_m^{-1}.p_m$ converges to $r^{-1}.p$. It follows :

$$q(x-u) = \lim_n \|x_m-u_m\| = \beta \geqslant \|x-u\| \ , \ \|q\| \leqq 1$$

whence : $\lim_n \|x_m-u_m\| = \|x-u\| > 0$ and $\dfrac{x_m-u_m}{\|x_m-u_m\|}$ is a sequence in the unit sphere of E weakly converging to $\frac{x-u}{\|x-u\|}$. Thus, we derive from (3.3) that x_m converges to X in the norm topology, and $x \in X$. Q.E.D.

3.4. This proposition arouses some comments :

i) the geometric assumption on the norm is satisfied in L.U.C. Banach spaces, among which Hilbert spaces and L^p spaces $(1 < p < +\infty)$, although $\mathbb{R}^2$ has an equivalent norm which satisfies this assumption and is not even strictly convex ([2] 2.1).

ii) the proposition applies, in particular, with φ any function $t \to t^\alpha$ $(\alpha \neq 0)$ (with the convention $0^\alpha = +\infty$, for negative α) or any strictly increasing locally Lipschitzian function on $\mathbb{R}$.

iii) If E is a reflexive Banach space and $\liminf_{t\to+\infty} \varphi(t) = +\infty$ the assumption X is weakly relatively compact can be dropped ; as a matter of fact, if $f(x_n) + \varphi(\|x_n - u_n\|)$ converges in $\mathbb{R}$, then $\|x_n - u_n\|$ is necessarily bounded hence $x_n - u_n$ remains in a weakly compact subset of E .

3.5. Finally, adding these remarks, we point out that in reflexive L.U.C. Banach spaces (in particular in Hilbert spaces and L^p spaces $(1 < p < +\infty)$) the proposition gives, setting $f = 0$ and $\varphi = Id_{\mathbb{R}}$, a proper continuous natural perturbation of the nearest points problem :

$$\mathcal{P} : \inf_{x \in X} \|x-u\|$$

for every closed subset $X \subset E$. It is a well-known result of convex optimization that, for a convex subset X , $\mathcal{P}$ is always solvable. This is no longer true if the convexity assumption is dropped ; however, we shall give in section 4 necessary conditions under which this problem can be approximated by perturbed problems which have solutions ; in fact, we shall give necessary conditions under which every proper continuous perturbation is densely solvable.

4. THE FUNDAMENTAL RESULT

4.1. Let us say that a function f which maps an open subset Ω of a Banach space E into $\mathbb{R}$ is strictly derivable at $u \in \Omega$ if it is Lipschitzian on some neighbourhood of u in Ω and its generalized gradient ∂f is reduced to a singleton and continuous at u ([3]) ; then f must be Frechet-derivable at u and $\partial f(u) = \{f'(u)\}$; the converse holds for a continuous convex function although it fails in general .

However, a locally Lipschitzian function is strictly derivable on an open subset Ω if it is continuously Fréchet-derivable on Ω (Then, it will be termed a C^1 function on Ω).

If f maps an open subset Ω of a Banach space into $\mathbb{R} \cup \{+\infty\}$ we shall say that a mapping h from Ω into $\mathbb{R}$ is an exact minorant of f at $u \in \Omega$ if :

i) $h(v) \leq f(v)$ for every $v \in \Omega$

ii) $h(u) = f(u)$

We can now state our fundamental result :

THEOREM 3. Let us consider a proper perturbation :

$$\mathcal{P}_v : \inf_{x \in X} F(x,v)$$

of some minimization problem $\mathcal{P}$, as defined in 1.1., with domain of parameters an open subset Ω of a Banach space E.

Assume that the value function admits at some $u \in \Omega$, an exact minorant, strictly derivable at u, then :

i) the set $S(u)$ of solutions of the problem $\mathcal{P}_u$ is non-vacuous.

ii) Every minimizing sequence of the problem $\mathcal{P}_u$ admits at least one cluster point in $S(u)$; in particular, it converges to the unique solution of $\mathcal{P}_u$, if $S(u)$ is reduced to a singleton.

iii) $\bigcap_{x \in S(u)} \partial F_x(u) \neq \emptyset$

iv) For every sequence u_n converging to u in Ω, and every sequence $x_n \in S(u_n)$, the set of the cluster points of x_n is non-vacuous and contained in $S(u)$.

Proof : Let h be an exact minorant of the value function at $u \in \Omega$, strictly derivable at u. The proof relies on the :

Lemma : for every sequence $(x_n,u_n) \in X \times \Omega$ satisfying :

$$(4.1) \qquad \lim_n \|u_n - u\| = 0 \ , \ \lim_n [F(x_n,u_n) - v(u_n)] = 0$$

there exists a sequence of contact units $(v_n, p_n, F_{x_n}(v_n))$, $p_n \in \partial F_{x_n}(v_n)$, which converges to $(u, h'(u), h(u))$ in $\Omega \times E^* \times \mathbb{R}$.

Proof of the lemma : the proof is divided in two steps :

construction of the sequence $(v_n, p_n, F_{x_n}(v_n))$:

First notice that : $h(u) = v(u) = \inf_{x \in X} F(x,u) < +\infty$. Thus, for every $\delta > 0$, there exists $x \in X$ such that :

$$F(x,u) < v(u) + \delta \ .$$

Then, for every $n \in \mathbb{N}$, $v(u_n) \leq F(x,u_n)$ and : $\limsup_n v(u_n) \leq \limsup_n F(x,u_n)$.

Since $F(x,u) < +\infty$, we deduce from (3.1) that :

$$\limsup_n v(u_n) \leq F(x,u) < v(u) + \delta \ .$$

Therefore, v is upper semi-continuous at u and $v(u_n)-h(u_n)$ converges to 0.

Now, let $\varepsilon_n^2 = F(x_n,u_n)-h(u_n)$, then by (4.1), ε_n converges to 0 .

If $\varepsilon_n = 0$, $F(x_n,u_n) = h(u_n) \leq v(u_n)$ thus x_n is a solution of the probem $\mathcal{P}_{u_n}$ and :

$$0 = F(x_n-u_n)-h(u_n) = \inf_{v \in \Omega} F(x,v) - h(v)$$

Therefore, $0 \in \partial F_{x_n}(u_n) - \partial h(u_n)$ ([3] prop.6 and prop.8) and we can choose :

$$(4.2) \qquad v_n = u_n \ \text{ and } \ p_n \in \partial F_{x_n}(u_n) \cap \partial h(u_n) \ .$$

If $\varepsilon_n > 0$, the mapping from E into $\mathbb{R} \cup \{+\infty\}$ defined by :

$$v \to G(v) = \begin{cases} F(x_n,v_n)-h(v) & \text{if } v \in \Omega \\ 0 & \text{if } v \notin \Omega \end{cases}$$

is lower semi-continuous, bounded from below, and satisfies :

$$G(u_n) = \varepsilon_n^2 \leq \inf G + \varepsilon_n^2 \ .$$

Thus, we can apply theorem 1.1 of [4] (I.Ekeland), whence we derive :

$$(4.3) \qquad \exists\, v_n \in E \qquad \begin{aligned} & G(v_n) \leq G(u_n) \\ & \|v_n - u_n\| \leq \varepsilon_n \\ & \forall v \in E \quad G(v) \geq G(v_n) - \varepsilon_n \|v_n - v\| \end{aligned}$$

(4.1) and (4.3) show that, for n large enough, $v_n \in \Omega$ and thus the mapping $v \to G(v) + \varepsilon_n \|v_n - v\|$ from E into $\mathbb{R} \cup \{+\infty\}$ is Lipschitzian on some neighbourhood of v_n in Ω , and achieves its minimum at v_n . Hence, it follows from [3] (prop.6 and prop.8) :

$$(4.4) \qquad \exists\, q_n \in \partial G(v_n) \qquad \|q_n\| \leq \varepsilon_n$$

But $\partial G(v_n) \subset \partial F_{x_n}(v_n) - \partial h(v_n)$, therefore we can choose :

(4.5) $\quad v_n$ as in (4.3) and $p_n \in \partial F_{x_n}(v_n) \cap (q_n + \partial h(v_n))$

<u>Convergence of the sequence</u> $(v_n, p_n, F_{x_n}(v_n))$:

The sequence v_n converges to u by (4.3) or (4.2) and (4.1). The sequence p_n converges to $h'(u)$ since, by assumption, $\partial h(u) = \{h'(u)\}$ and ∂h is continuous at u ,, using (4.5) and (4.4) or (4.2). Finally :

$$h(v_n) \leq F_{x_n}(v_n) = G(v_n) + h(v_n) \leq G(u_n) + h(v_n) = F_{x_n}(u_n) + h(v_n) - h(u_n)$$

(using (4.3) for $\varepsilon_n > 0$)

and $\quad \limsup_n F_{x_n}(u_n) \leq \lim_n [F_{x_n}(u_n) - v(u_n)] + \limsup_n v(u_n) \leq v(u)$

together implie :

$$V(u) = h(u) = \liminf_n h(v_n) \leq \liminf_n F_{x_n}(v_n) \leq \limsup_n F_{x_n}(v_n)$$

$$\leq \limsup_n F_{x_n}(u_n) + \limsup_n [h(v_n) - h(u_n)] \leq v(u)$$

whence : $h(u) = \lim_n F_{x_n}(v_n)$. Q.E.D.

Now we can achieve the proof of theorem 3. :

For every minimizing sequence x_n of the problem $\mathcal{P}_n$, the sequence (x_n, u) satisfies (4.1) ; then it's a straightforward consequence of the previous lemma and the definition of a proper perturbation that the sequence x_n has at least one cluster point $x \in X$. By the semi-continuity of F_u , x satisfies : $F_x(u) \leq h(u) = v(u)$ Moreover, we derive from [3] prop. 6 that $h'(u) \in \partial F_x(u)$.

The set $S(u)$ of the solutions of the problem $\mathcal{P}_u$ is non-vacuous ; every minimizing sequence of the problem $\mathcal{P}_u$ has at least one cluster point, which belongs to $S(u)$, and :

$$h'(u) \in \bigcap_{x \in S(u)} \partial F_x(u)$$

It remains to prove that, for every sequence u_n which converges to u in Ω , and every sequence $x_n \in S(u_n)$, the sequence x_n has at least one cluster point which belongs to $S(u)$; but the sequence (x_n, u_n) again satisfies (4.1). The claim follows thus from the lemma and the definition of a proper perturbation.Q.E.D.

4.2. According to theorem 3., a necessary condition for a proper perturbation to be densely solvable is that the value function densely admits an exact strictly derivable minorant ; the condition is trivially satisfied if the value function is itself densely strictly derivable. Thus the next density result appears as a straightforward consequence of theorem 3 :

COROLLARY : Let us consider a continuous and proper perturbation :

$$\mathcal{P}_v : \inf_{x\in X} F(x,v)$$

of some minimization problem $\mathcal{P}$, with domain of parameters an open subset Ω of an Asplund space.

If for every $x \in X$, $F_x : v \to F(x,v)$ is a concave function on Ω, then the perturbation is densely solvable.

In fact, assertions i) to iv) of theorem3. hold for each v in a dense G_δ subset of Ω.

By theorem 3. we only need to prove that the value function is strictly derivable on a dense G_δ subset in Ω, but this follows directly from assumptions since it is obviously a concave continuous function on Ω, and Ω is an open subset of an Asplund space (cf.(2.1)).

Theorem 2. is a trivial consequence of this latter corollary ; another interesting application is given in the following :

PROPOSITION 2 (Baranger-Temam)
Let x be a closed bounded subset of a reflexive Banach space E, f a lower semi-continuous mapping from E into $\mathbb{R}\cup\{+\infty\}$, bounded from below on x, and φ a continuous strictly decreasing concave function on $[0,+\infty[$.

Assume that the norm of E satisfies (3.3), then the perturbation :

$$\mathcal{P}_v : \inf_{x\in X} f(x) + \varphi(\|x-v\|)$$

with domain of parameters an open subset $\Omega\subset E$, is densely solvable. Moreover, for each v in a dense G_δ subset of E, assertions i) to iv) of theorem 3 hold.

φ is uniformly continuous on each bounded subset, hence the perturbaton is continuous ; moreover, we proved in proposition 1 that perturbations of this type are proper ones. Now according to the corollary of theorem 3., assertions i) to iv) of theorem 3 hold for a dense G_δ subset in Ω.

The comments on the technical assumptions involved in proposition 1 still apply to proposition 2.

We point out that theorem 3. provides a complete description of the "typical" situation encountered in proposition 2. For instance, if, moreover, the norm of E is assumed to be strictly convex, then assertion iii) of theorem 3. implies the unicity of the solution (the technical argument is left to the "interested" reader) ; thus, the problem $\mathcal{P}_v$ densely admits a unique solution and every minimizing sequence converges to this solution.

Proposition 2. applies, in particular, with $f = 0$ and $\varphi = - Id_{\mathbb{R}}$. Then the perturbation becomes :

$$\mathcal{P}_v : \sup_{x \in X} \|x-v\|$$

which is the natural perturbation of the "farthest points problem", clearly matched with be nearest points problem introduced in section 3.

In order to apply theorem 3. at its best, one is led to investigate the class of functions mapping open subsets of Banach spaces into $\mathbb{R} \cup \{+\infty\}$ which densely admit an exact strictly derivable minorant. Clearly, such functions must be lower semi-continuous and lower semi-continucus convex functions are suitable. The natural question is : does each lower semi-continuous function defined on an open subset of a Banach space have this property ? In fact, the answer turns out to depend on the geometry of the underlying space of parameters :

5. DENSITY RESULTS IN UNIFORMLY CONVEXIFIABLE BANACH SPACES

5.1. Let us introduce the class of Banach spaces E which satisfy the following structural condition :

(5.1) Every lower semi-continuous mapping f from E into $\mathbb{R} \cup \{+\infty\}$, bounded from below, admits, at a dense set of points in its effective domain $\{U \in E / f(v) < +\infty\}$ an exact minorant of type C^1 .

Then, we derive obviously from theorem 3. :

THEOREM 4.

Every continuous proper perturbation with domain of parameters an open subset of a Banach space which satisfies (5.1) is densely solvable.

It should be observed that the converse holds, at least in reflexive spaces :

PROPOSITION 3.

Let E be a reflexive Banach space such that every continuous proper perturbation with domain of parameters an open subset of E is densely solvable, then E satisfies (5.1).

We shall deduce the proposition from two lemmas :

Lemma 1. Let E be a reflexive Banach space the norm of which satisfies (3.3) and f a lower semi-continuous mapping from E into $\mathbb{R} \cup \{+\infty\}$ bounded from below, the effective domain of which $\{v \in E / f(v) < +\infty\}$ is non-vacuous. For every positive real number C , the perturbation :

$$\mathcal{P}_v : \inf_{x \in E} f(x) + C\|x-v\|^2$$

with domain of parameters the whole space E , is continuous and proper.

The value function is always finite, since the effective domain of f is non-vacuous, and upper semi-continuous as a pointwise infimum of continuous functions. Thus, for every $u \in E$, there exists a constant $K(u)$ such that $V(v) < K(u)$ for every v in an open ball $B(u)$ of center u and radius r in E . Now let v and w belong to $B(u)$; for every $x \in E$, satisfying :

$$(5.2) \qquad f(x) + C\|x-v\|^2 \leq K(u)$$

we derive :

$$\|x-v\| < \left(\frac{K(u) - \inf f}{C}\right)^{1/2}$$

whence :

$$F_x(v) - F_w(w) = C\left(\|x-v\|^2 - \|x-w\|^2\right)$$

$$\leq C(\|x-v\| + \|x-w\|).\ \|v-w\|$$

$$\leq 2\ C\ (M + r)\ \ \|v+w\|$$

and :

$$V(w) \leq F_x(w) \leq F_x(v) + 2C(M+2).\ \|v-w\|\ .$$

Taking the infimum along the X satisfying (5.2), we deduce :

$$V(w) \leq V(v) + 2C\ (M+2).\ \|v-w\|$$

Therefore V is Lipschitzian on $B(u)$ and the perturbation is continuous ; finally the perturbation is proper by proposition 1 (and the third following remark).

Lemma 2. Let E be a Banach space, the norm of which is Frechet-derivable at each non-zero point, and f a lower semi-continuous mapping from E into $\mathbb{R} \cup \{+\infty\}$, bounded from below, the effective domain of which is non-vacuous. Assume that for every positive real number C
the perturbation :

$$\mathcal{P}_v : \inf_{x \in X} f(x) + C\|x-v\|^2$$

with domain the whole space E , is densely solvable, then f admits an exact minorant of type $\mathcal{C}^1$ at each point of a dense subset in its effective domain.

Let u be any point in the effective domain of f , and ε a positive real number ; we shall exhibit a minorant of f which is a $\mathcal{C}^1$ function on E and is exact at some point of the open ball of center u and radius ε in E . Define :

$$(5.3) \qquad C = 8.\ \varepsilon^{-2}[f(u) - \inf f] \geq 0$$

- If $C = 0$, f achieves its minimum on E at u and the claim is trivial.
- Otherwise, there exists, by assumption, $v \in E$ such that :

$$(5.4) \qquad \|v-u\| < \varepsilon\ \sqrt{2}/4$$

and the problem $\mathcal{P}_v$ admits a solution $x \in E$. Thus, it follows :

$$(5.5) \qquad \forall y \in E \qquad f(x) + C\|x-v\|^2 + C\|y-v\|^2 \leq f(y)$$

which shows that the function :

$$h : y \to f(x) + C\ \|x-v\|^2 - C\|y-v\|^2$$

is a minorant of f which is of type $\mathcal{C}^1$ and is exact at $x \in E$.

(5.5) moreover implies :

$$f(x) + C\|x-v\|^2 + C\|u-v\|^2 \leq f(u)$$

hence :

$$C\|x-v\|^2 \leq f(u) - \inf f + C\|u-v\|^2 .$$

Using now (5.3) and (5.4), we get :

$$\|x-v\|^2 \leq \varepsilon^2/8 + \varepsilon^2/8 = \varepsilon^2/4$$

whence :

$$(5.6) \qquad \|x-v\| \leq \varepsilon/2$$

Finally, combining (5.6) and (5.4) :

$$\|x-u\| \leq \|x-v\| + \|x-u\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

Therefore, X belongs to the open ball of center u and radius ε in E , which ends the proof .

Now the previous proposition is a straightforward consequence of the lemmas, since every reflexive Banach space can be renormed both in a Frechet-derivable and locally uniformly convex manner ([7] V.5 Cor.3) ; such a norm satisfies (3.3) ([7] II.2. THM.4).

5.2. To end this discussion, we shall exhibit a particular class of reflexive Banach spaces satisfying (5.1) :

THEOREM 5.
Every uniformly convexifiable Banach space satisfies (5.1).

Proof : Every uniformly convexifiable Banach space E can be renormed in a way that E and E^* are both uniformly convex ([7] III Notes and remarks) ; the norm of E is then Frechet-derivable at every non-zero point ([7] II. 4. THM 1.). According to lemma 2. we only need to prove that, for every lower semi-continuous mapping f from E into $\mathbb{R}\cup\{+\infty\}$, bounded from below, the effective domain of which is non-vacuous, and every positive real number C , the perturbation :

$$\mathcal{P}_v : \inf_{x\in E} f(x) + C\|x-v\|^2$$

with domain of parameters the whole space E , is densely solvable. Let u any point in E and K(u) a constant such that V(v) < K(u) for every v in an open ball B(u) of center u and radius r in E :

$$f(x) + C\|x-v\|^2 < K(u) \Longrightarrow f(x) \leq K(u) , \quad \|x-u\| \leq \left(\frac{K(u)-\inf f}{C}\right)^{1/2} + r$$

For every $v\in B(u)$, the problem $\mathcal{P}_v$ is thus equivalent to :

$$\mathcal{P}'_v : \inf_{x\in X(u)} f(x) + C\|x-v\|^2$$

where $X(u) = \{x\in E \ / \ f(x) \leq K(u) , \ \|x-u\| \leq \left(\frac{K(u) - \inf f}{C}\right)^{1/2} + r\}$

is a closed bounded subset in E . The claim is then a direct application of [5] proposition 3.14.

COROLLARY : Let E be a uniformly convexifiable Banach space ; then every continuous proper perturbation with domain of parameters an open subset of E is densely solvable.

In particular :

PROPOSITION 4. (Ekeland-Lebourg)
Let X be a closed subset of a uniformly convexifiable Banach Space E , f a lower semi-continuous mapping from E into $\mathbb{R}\cup\{+\infty\}$, bounded from below on X ,

and φ a lower semi-continuous mapping from $[0,+\infty[$ into $\mathbb{R}\cup\{+\infty\}$, locally Lipschitzian on its effective domain.

Assume that :

i) X is bounded or $\lim_{t\to+\infty} \varphi(t) = +\infty$

ii) the generalized gradient of φ at $t \neq 0$ never contains 0

iii) the norm of E satisfies (3.3)

Then the perturbation :

$$\mathcal{P}_v : \inf_{x\in X} f(x) + \varphi(\|x-v\|)$$

with domain of parameters the whole space E is densely solvable.

Note that E is reflexive, hence the perturbation is proper by proposition 3. (and the following remarks). Furthermore, the value function is easily checked to be locally Lipschitzian on its effective domain ; the perturbation is thus continuous on the effective domain of the value function and the proposition follows directly from the latter corollary.

Theorem 3. together with the assumption that (5.1) is satisfied gives in fact much more information on the perturbation ; for instance, if moreover the norm of E is assumed to be strictly convex, then $\mathcal{P}_v$ densely admits a unique solution and every minimizing sequence converges to this solution.

Proposition 4. applies, in particular, with $f = 0$ and $\varphi = Id_{\mathbb{R}}$. Then the perturbation becomes :

$$\mathcal{P}_v : \inf_{x\in X} \|x-v\|$$

Thus proposition 4. asserts that, in every uniformly convexifiable Banach space E, and for every closed subset $X\subset E$, the set of the points $U\in E$ which have (at least) a nearest point in X is dense in E. We point out that E can be equipped with a not even strictly convex norm, inasmuch as it satisfies (3.3).

Taking now $f = 0$ and $\varphi = -Id_{\mathbb{R}}$ in proposition 4 . we get a similar result as well for the farthest points problem.

To conclude this section, we notice that a standard argument shows that Banach spaces which satisfy (5.1) are Asplund spaces ; so it should come as no surprise that Asplund spaces appeared in the statement of density results such in theorem 2. or the corollary of theorem 3. However, it seems to be unknown if there exist Asplund spaces which do not satisfy (5.1).