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ON SOME STRUCTURAL DESIGN PROBLEMS

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1. INTRODUCTION

Some structural design problems may be stated as

\[(\mathcal{P}) \quad \inf_{u \in C} J(u) \quad \lambda(u) = \lambda_1\]

where \(u\) is the thickness of the structure, \(J(u)\) its weight, \(\lambda(u)\) its fundamental frequency of vibration, \(\lambda_1\) the fundamental frequency of the structure with uniform thickness and \(C\) is the convex of constraints on \(u\).

Here we consider the problems \((\mathcal{P})\) when \(J\) and \(\lambda\) have the following properties

- \(J\) is convex on \(C\)
- \(\lambda\) is pseudoconcave on \(C\).

We shall state necessary and sufficient conditions of optimality for an abstract problem \((\mathcal{P})\). And we shall apply these results in structural design. We refer to [4] [5] for the proof of these results and for more details.

2. PROBLEM \((\mathcal{P})\) : ABSTRACT CASE.

Let \(E\) a locally convex Hausdorff space, \(C\) a convex of \(E\), \(J\) and \(\lambda\)
two real valued functions defined on $C$. We consider the two following problems

\[(\mathcal{O}) \quad \inf_{u \in C} J(u) \quad \lambda(u) = \lambda_1 \]

\[(Q) \quad \inf_{u \in C} J(u) \quad \lambda(u) \geq \lambda_1 \]

where $\lambda_1 = \lambda(u_1)$, $u_1 \in C$.

**Proposition 2.1.** Assume

(2.1) $J$ pseudoconvex on $C$

(2.2) $\lambda$ continuous on $C$

(2.3) $\exists \ u_\alpha \in C$ such that $\lambda(u_\alpha) < \lambda_1$ and $J(u_\alpha) < J(u)$ $\forall \ u \in C$, $u \neq u_\alpha$

then the problems $(\mathcal{O})$ and $(Q)$ are equivalent (i.e. if $u$ is a solution of $(\mathcal{O})$ $u$ is a solution of $(Q)$ and reciprocally).

Let's recall the definition of a pseudoconvex function. $J$ is a pseudoconvex on $C$ if $J$ is Gateaux-differentiable on $C$ and if

\[\forall \ (u,v) \in C \times C, \ J'(u).(v-u) \geq 0 \implies J(v) \geq J(u).\]

The reader can find the properties of pseudoconvex and quasiconvex functions in MANGASARIAN [6].

We make use of these properties for the proof of Proposition 2.1.

**Remark.** $(Q)$ is a convex problem.

**Proposition 2.2.** Assume (2.1),

(2.4) $\lambda$ pseudoconcave on $C$ and
(2.5) \[ \exists \ u \in C \text{ such that } \lambda(u_1) > \lambda_1. \]

Then if \( u \in C \) and if \( \lambda(u) \geq \lambda_1 \), \( u \) is a solution of (Q) if and only if

\[ \exists \ n < 0 \text{ such that} \]

(2.6) \[ J'(u_1)(v-u) + n \lambda'(u_1)(v-u) > 0 \quad \forall \ v \in C \]

(2.7) \[ n(\lambda(u_1) - \lambda_1) = 0. \]

The proof of the necessary condition is based on the results of HALKIN [2] and (2.4)-(2.5) assumptions.

The proof of the sufficiency is easy.

Corollary 2.1. Assume (2.1)-(2.3)-(2.4)-(2.5). Let \( u \in C \), \( \lambda(u) = \lambda_1 \) then \( u \) is a solution of (Q) if and only if

\[ \exists \ n < 0 \text{ such that } J'(u_1)(v-u) + n \lambda'(u_1)(v-u) > 0 \quad \forall \ v \in C. \]

3. THE PROBLEM (Q) IN STRUCTURAL DESIGN.

Let \( \Omega \) be an open bounded connected set in \( \mathbb{R}^n \) let \( a \) and \( b \) two functions

\[ a : [0, +\infty[ \rightarrow ]0, +\infty[ \quad C^2, \text{ concave} \]

\[ b : [0, +\infty[ \rightarrow ]0, +\infty[ \quad C^2, \text{ convex}. \]

Denote by \( (.,.) \) the scalar product of \( L^2(\Omega) \)

\[ <.,.> \]

the bilinear canonical pairing over \( H^1_0(\Omega) \times H^{-1}(\Omega) \).

\( U = \{ u \in L^\infty(\Omega) ; \Phi(u) > 0 \text{ such that } u(x) > \alpha(u) \text{ a.e. in } \Omega \} \)

(3.1) \[ C = \{ u \in L^\infty(\Omega) ; \ 0 < \alpha \leq u(x) \leq \beta \text{ a.e. in } \Omega \} \quad \alpha, \beta \text{ given with} \]

0 < \alpha < 1 < \beta < +\infty

\( a \) and \( b \) the maps from \( U \) to \( U \) defined by

\[ a(u)(x) = a(u(x)) \quad b(u)(x) = b(u(x)). \]

\( A_u \) the isomorphism from \( H^1_0(\Omega) \) to \( H^{-1}(\Omega) \) defined by
\[ \langle A_u, w, \phi \rangle = \int_\Omega a(u)(x) \nabla w(x) \cdot \nabla \phi(x) \, dx, \quad (w, \phi) \in H^1_0(\Omega)^2 \]

\( B_u \) the operator of \( L^2(\Omega) \) defined by
\[ B_u w = b(u) w. \]

3.1. The spectral problem.

It is known that the first eigenvalue \( \lambda(u) \) of
\[ A_u w = \lambda B_u w \]
is simple and that the associated eigenvector \( w(u) \) satisfies
\[ w(u;x) > 0 \quad \text{a.e. in } \Omega \]

\[ \lambda(u) = \inf_{w \in H^1_0(\Omega) \setminus \{0\}} \frac{\langle A_u w, w \rangle}{\langle B_u w, w \rangle} \]

We suppose that \( w(u) \) is normalized in \( L^2(\Omega) \).

Using the Implicit Function Theorem we can prove that the function
\[ u \in U \mapsto \lambda(u) \]
is differentiable on \( U \) and that
\[ \lambda'(u).v = \int_\Omega (a'(u)(x)v(x)|w(u;x)|^2 - \lambda(u)(b'(u)(x)v)w^2(u;x)) \, dx \]
\[ \int_\Omega b(u)w^2(u;x) \, dx \]

From the concavity of \( a \) and \(-b\), it follows that:
\[ \lambda(v) \leq \lambda(u) + \lambda'(u)(v-u) \int_\Omega \frac{b(u)(x)w^2(u;x) \, dx}{\int_\Omega b(v)(x)w^2(u;x) \, dx} \quad \forall (u,v) \in U \times U. \]

Using this inequality we get

Theorem 3.1. \( \lambda \) is pseudo concave on \( U \). \( \lambda \) is upper semicontinuous on \( C \) for the weak* topology of \( L^\infty(\Omega) \).
3.2. Statement of the problem and results.

Ω is the form of the structure, \( u \) is the thickness and therefore the weight is

\[
J(u) = \int_\Omega u(x) \, dx.
\]

Let \( u_1 \) be the function in \( C \) defined by

\[
u_1(x) = 1 \quad \forall x \in \Omega,
\]

and

\[
\lambda_1 = \lambda(u_1).
\]

We shall consider the following optimisation problem:

minimize \( J(u) \) subject to \( \lambda(u) = \lambda_1 \) and \( u \in C \) i.e.

\[
(\mathcal{P}) \quad \inf_{\lambda(u_1) = \lambda_1} J(u).
\]

Let \( u_\alpha \) be the function defined on \( C \) by

\[
u_\alpha(x) = \alpha \quad \forall x \in \Omega.
\]

This function satisfies

\[
J(u_\alpha) < J(u) \quad \forall u \in C, u \neq u_\alpha.
\]

Introduce the problem

\[
(\mathcal{Q}) \quad \inf_{\lambda(u) \leq \lambda_1} J(u).
\]

Then using the previous results it is easy to prove

**Theorem 3.2.** Assume

\[
(3.5) \quad \lambda(u_\alpha) < \lambda_1
\]

\[
(3.6) \quad \exists u_\gamma \in C \text{ such that } \lambda(u_\gamma) > \lambda_1
\]

then i) \((\mathcal{P})\) and \((\mathcal{Q})\) are equivalent ;
ii) \((\mathcal{G})\) has at least one solution;

iii) \(u \in C\) is a solution of \((\mathcal{G})\) if \(\lambda(\bar{u}) = \lambda_1\) and

\[
\exists \bar{\eta} < 0 \text{ such that } \\
J(v) - J(\bar{u}) + \bar{\eta} \lambda'(u)(v - \bar{u}) > 0 \quad \forall v \in C;
\]

iv) the set \(S(\mathcal{G})\) of solutions of \((\mathcal{G})\) is convex and \(\{w(u); u \in S(\mathcal{G})\}\) is reduced to one function.

Remark. The condition (3.7) is equivalent to (3.8) \(\exists \varepsilon > 0\) such that

\[
g(\bar{u};x) \leq \varepsilon \text{ a.e. in } \Omega_\beta = \{x \in \Omega; \bar{u}(x) < \beta\}
\]

and

\[
g(\bar{u};x) > \varepsilon \text{ a.e. in } \Omega_\alpha = \{x \in \Omega; \bar{u}(x) > \alpha\}
\]

where

\[
g(\bar{u};x) = \notw(u(x)) |\notw(u; x)|^2 - \lambda_1 \notb'(u(x)) \notw(u; x).
\]

3.3. Examples.

The previous results can be applied to numerous examples and in particular to

Example 1 \(a(u) = u, b(u) = 1\)

Example 2 \(a(u) = u, b(u) = u + \delta\) where \(\delta\) is a constant positive

Example 3 \(a(u) = 1, b(u) = \frac{1}{u^2}\).

We refer to [1] and [3] for the motivation of these problems.

In these examples the function \(\bar{u}_\alpha\) defined by (3.4) satisfies (3.5) and the function \(u_\gamma\) defined by

\[
u_\gamma(x) = \gamma \quad \text{where} \quad 1 < \gamma \leq \delta
\]

satisfies (3.6) and we can apply Theorem 3.2. In particular in Example 3, (3.8) gives \(\exists \varepsilon > 0\) such that
\[
\frac{w^2(\bar{u};x)}{\bar{u}^3(x)} \leq e \text{ a.e. in } \Omega_\beta
\]
\[
\frac{w^2(\bar{u};x)}{\bar{u}^3(x)} > e \text{ a.e. in } \Omega_\alpha
\]
and this implies that, in this example (33) has only one solution \( \bar{u} \). Moreover if \( \Omega = [0,1[ \) we can construct this solution.

Remark. We have used the same method to study problems of the following form

\[
\sup_{\Omega} \lambda(u), \quad J(u) = J_1
\]

REFERENCES.


