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ON SOME STRUCTURAL DESIGN PROBLEMS

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1. INTRODUCTION

Some structural design problems may be stated as

\[
(\mathcal{P}) \quad \inf_{u \in C} J(u) \quad \lambda(u) = \lambda_1
\]

where \( u \) is the thickness of the structure, \( J(u) \) its weight, \( \lambda(u) \) its fundamental frequency of vibration, \( \lambda_1 \) the fundamental frequency of the structure with uniform thickness and \( C \) is the convex of constraints on \( u \).

Here we consider the problems \((\mathcal{P})\) when \( J \) and \( \lambda \) have the following properties

\( J \) is convex on \( C \)
\( \lambda \) is pseudoconcave on \( C \).

We shall state necessary and sufficient conditions of optimality for an abstract problem \((\mathcal{P})\). And we shall apply these results in structural design. We refer to \([4]\) \([5]\) for the proof of these results and for more details.

2. PROBLEM \((\mathcal{P})\) : ABSTRACT CASE.

Let \( E \) a locally convex Hausdorff space, \( C \) a convex of \( E \), \( J \) and \( \lambda \)
two real valued functions defined on C. We consider the two following problems

\[(\mathcal{S}) \quad \inf_{u \in C} J(u) \quad \lambda(u) = \lambda_1\]

\[(Q) \quad \inf_{u \in C} J(u) \quad \lambda(u) \geq \lambda_1\]

where \(\lambda_1 = \lambda(u_1)\) \(u_1 \in C\).

Proposition 2.1. Assume

(2.1) \(J\) pseudoconvex on C

(2.2) \(\lambda\) continuous on C

(2.3) \(\exists u_\alpha \in C\) such that \(\lambda(u_\alpha) < \lambda_1\) and

\[J(u_\alpha) < J(u) \quad \forall u \in C, u \neq u_\alpha\]

then the problems \((\mathcal{S})\) and \((Q)\) are equivalent (i.e. if \(u\) is a solution of \((\mathcal{S})\) \(u\) is a solution of \((Q)\) and reciprocally).

Let's recall the definition of a pseudoconvex function. \(J\) is a pseudoconvex on C if \(J\) is Gateaux-differentiable on C and if

\[\forall (u,v) \in C \times C, J'(u). (v-u) \geq 0 \implies J(v) \geq J(u)\]

The reader can find the properties of pseudoconvex and quasiconvex functions in MANGASARIAN [6].

We make use of these properties for the proof of Proposition 2.1.

Remark. \((Q)\) is a convex problem.

Proposition 2.2. Assume (2.1),

(2.4) \(\lambda\) pseudoconcave on C and
(2.5) \[ \exists \, u \in C \text{ such that } \lambda(u) > \lambda_1. \]

Then if \( \bar{u} \in C \) and if \( \lambda(\bar{u}) \geq \lambda_1 \), \( \bar{u} \) is a solution of (Q) if and only if

\[ \exists \, \bar{n} \leq 0 \text{ such that } \]

(2.6) \[ J'(\bar{u}).(v-\bar{u}) + \bar{n} \lambda'(\bar{u})(v-\bar{u}) \geq 0 \quad \forall \, v \in C \]

(2.7) \[ \bar{n}(\lambda(\bar{u})-\lambda_1) = 0. \]

The proof of the necessary condition is based on the results of HALKIN [2] and (2.4)-(2.5) assumptions.

The proof of the sufficiency is easy.

Corollary 2.1. Assume (2.1)-(2.3)-(2.4)-(2.5). Let \( \bar{u} \in C \), \( \lambda(\bar{u}) = \lambda_1 \) then \( \bar{u} \) is a solution of (Q) if and only if

\[ \exists \, \bar{n} < 0 \text{ such that } J'(\bar{u}).(v-\bar{u}) + \bar{n} \lambda'(\bar{u})(v-\bar{u}) \geq 0 \quad \forall \, v \in C. \]

3. THE PROBLEM (Q) IN STRUCTURAL DESIGN.

Let \( \Omega \) be an open bounded connected set in \( \mathbb{R}^n \) let \( a \) and \( b \) two functions

\[ a : [0,+\infty[ \rightarrow ]0,+\infty[, \quad C^2, \text{ concave} \]

\[ b : [0,+\infty[ \rightarrow ]0,+\infty[, \quad C^2, \text{ convex}. \]

Denote by \( (\cdot,\cdot) \) the scalar product of \( L^2(\Omega) \)
\[ <\cdot,\cdot> \text{ the bilinear canonical pairing over } H^1_0(\Omega) \times H^{-1}(\Omega). \]

\( U = \{ u \in L^\infty(\Omega) \mid \int_\Omega u > 0 \text{ such that } u(x) \geq a(u) \text{ a.e. in } \Omega \} \)

(3.1) \[ C = \{ u \in L^\infty(\Omega) ; 0 < \alpha \leq u(x) \leq \beta \text{ a.e. in } \Omega \} \quad \alpha, \beta \text{ given with } 0 < \alpha < 1 < \beta < +\infty \]

\( a \) and \( b \) the maps from \( U \) to \( U \) defined by

\[ a(u)(x) = a(u(x)), b(u)(x) = b(u(x)). \]

\( A_u \) the isomorphism from \( H^1_0(\Omega) \) to \( H^{-1}(\Omega) \) defined by
\begin{align*}
\langle A_u, w, \phi \rangle &= \int_\Omega a(u)(x) \nabla w(x) \cdot \nabla \phi(x) \, dx, \quad (w, \phi) \in H_0^1(\Omega)^2 \\
B_u \text{ the operator of } L^2(\Omega) \text{ defined by} \\
B_u w &= b(u) w.
\end{align*}

3.1. The spectral problem.

It is known that the first eigenvalue \( \lambda(u) \) of

\[ A_u w = \lambda B_u w \]

is simple and that the associated eigenvector \( w(u) \) satisfies

\( w(u; x) > 0 \) \ a.e. in \( \Omega \)

\[ \lambda(u) = \inf \left\{ \frac{\langle A_u, w, w \rangle - \langle A_u, w(u), w(u) \rangle}{(B_u w, w) - (B_u w(u), w(u))} \right\} \]

We suppose that \( w(u) \) is normalized in \( L^2(\Omega) \).

Using the Implicit Function Theorem we can prove that the function

\[ u \in U \rightarrow \lambda(u) \]

is differentiable on \( U \) and that

\[ \lambda'(u).v = \frac{\int_\Omega \left( (a'(u).v)(x) \right) \langle w(u; x) \rangle^2 - \lambda(u)(b'(u).v)(x) \right) w^2(u; x) \, dx}{\int_\Omega b(u) w^2(u; x) \, dx} \]

From the concavity of \( a \) and \(-b\), it follows that:

\[ \lambda(v) \leq \lambda(u) + \lambda'(u)(v-u) \frac{\int_\Omega b(u)(x) w^2(u; x) \, dx}{\int_\Omega b(v)(x) w^2(u; x) \, dx} \quad \forall (u, v) \in U \times U. \]

Using this inequality we get

Theorem 3.1. \( \lambda \) is pseudo concave on \( U \). \( \lambda \) is upper semicontinuous on \( C \) for the weak* topology of \( L^\infty(\Omega) \).
3.2. Statement of the problem and results.

$\Omega$ is the form of the structure, $u$ is the thickness and therefore the weight is

$$J(u) = \int_{\Omega} u(x) \, dx.$$ 

Let $u_1$ be the function in $C$ defined by

$$u_1(x) = 1 \quad \forall \ x \in \Omega,
$$

and

$$\lambda_1 = \lambda(u_1).$$

We shall consider the following optimisation problem:

$$\text{minimize } J(u) \text{ subject to } \lambda(u) = \lambda_1 \text{ and } u \in C \text{ i.e.}$$

$$(\mathcal{Q}) \quad \inf_{u \in C} J(u) \quad \lambda(u) = \lambda_1.$$ 

Let $u_\alpha$ be the function defined on $C$ by

$$(3.4) \quad u_\alpha(x) = \alpha \quad \forall \ x \in \Omega.$$ 

This function satisfies

$$J(u_\alpha) < J(u) \quad \forall \ u \in C, \ u \neq u_\alpha.$$ 

Introduce the problem

$$(Q) \quad \inf_{u \in C} J(u) \quad \lambda(u) \leq \lambda_1.$$ 

Then using the previous results it is easy to prove

Theorem 3.2. Assume

$$\lambda(u_\alpha) < \lambda_1 \quad (3.5)$$

$$\exists \ u_\gamma \in C \text{ such that } \lambda(u_\gamma) > \lambda_1 \quad (3.6)$$

then i) $(\mathcal{Q})$ and $(Q)$ are equivalent ;
ii) \((\mathfrak{G})\) has at least one solution;

iii) \(\overline{u} \in C\) is a solution of \((\mathfrak{G})\) if \(\lambda(\overline{u}) = \lambda_1\) and

\[
(3.7) \quad \exists \pi < 0 \quad \text{such that} \quad J(v) - J(\overline{u}) + \pi \lambda'(u)(v-\overline{u}) > 0 \quad \forall v \in C;
\]

iv) the set \(S(\mathfrak{G})\) of solutions of \((\mathfrak{G})\) is convex and \((w(u); u \in S(\mathfrak{G}))\) is reduced to one function.

Remark. The condition (3.7) is equivalent to (3.8) \(\exists e > 0\) such that

\[g(\overline{u};x) \leq e \quad \text{a.e. in} \quad \Omega_{\beta} = \{x \in \Omega; \overline{u}(x) < \beta\}\]

and

\[g(\overline{u};x) \geq e \quad \text{a.e. in} \quad \Omega_{\alpha} = \{x \in \Omega; \overline{u}(x) > \alpha\}\]

where

\[g(\overline{u};x) = \lambda'(u(x)) |w(u;x)|^2 - \lambda_1 B'(u(x)) w^2(u;x).\]

3.3. Examples.

The previous results can be applied to numerous examples and in particular to

Example 1 \quad a(u) = u \quad b(u) = 1

Example 2 \quad a(u) = u \quad b(u) = u + \delta \quad \text{where} \ \delta \ \text{is a constant positive}

Example 3 \quad a(u) = 1 \quad b(u) = \frac{1}{u^2}.

We refer to [1] and [3] for the motivation of these problems.

In these examples the function \(u_\infty\) defined by (3.4) satisfies (3.5) and the function \(u_\gamma\) defined by

\[u_\gamma(x) = \gamma \quad \text{where} \quad 1 < \gamma < 8\]

satisfies (3.6) and we can apply Theorem 3.2. In particular in Example 3, (3.8) gives \(\exists e > 0\) such that
and this implies that, in this example (33) has only one solution \( \bar{u} \). Moreover if \( \alpha = ]0,1[ \) we can construct this solution.

**Remark.** We have used the same method to study problems of the following form

\[
\sup_{u \in C} \lambda(u).
\]

REFERENCES.


