C. JOURON

On some structural design problems

*Mémoires de la S. M. F.*, tome 60 (1979), p. 87-93

<http://www.numdam.org/item?id=MSMF_1979__60__87_0>
ON SOME STRUCTURAL DESIGN PROBLEMS

C. JOURON

Analyse Numérique et Fonctionnelle
C.N.R.S. et Université de Paris-Sud Département de Mathématiques
Bâtiment 425, 91405 - Orsay

Conférence au Colloque d’Analyse non convexe - Pau, Mai 1977 -

1. INTRODUCTION

Some structural design problems may be stated as

\[(\mathcal{P}) \quad \inf_{u \in C} J(u) \quad \lambda(u) = \lambda_1\]

where \(u\) is the thickness of the structure, \(J(u)\) its weight, \(\lambda(u)\) its fundamental frequency of vibration, \(\lambda_1\) the fundamental frequency of the structure with uniform thickness and \(C\) is the convex of constraints on \(u\).

Here we consider the problems \((\mathcal{P})\) when \(J\) and \(\lambda\) have the following properties

\(J\) is convex on \(C\)
\(\lambda\) is pseudoconcave on \(C\).

We shall state necessary and sufficient conditions of optimality for an abstract problem \((\mathcal{P})\). And we shall apply these results in structural design. We refer to \([4]\) \([5]\) for the proof of these results and for more details.

2. PROBLEM \((\mathcal{P})\) : ABSTRACT CASE.

Let \(E\) a locally convex Hausdorff space, \(C\) a convex of \(E\), \(J\) and \(\lambda\)
two real valued functions defined on $C$. We consider the two following problems

\begin{align*}
(\mathcal{S}) & \quad \inf_{u \in C} J(u) \\
(\mathcal{Q}) & \quad \inf_{u \in C} J(u) \\
\end{align*}

\begin{align*}
\lambda(u) = \lambda_1 \\
\lambda(u) \geq \lambda_1 \\
\text{where } \lambda_1 = \lambda(u_1) \quad u_1 \in C.
\end{align*}

**Proposition 2.1.** Assume

\begin{enumerate}
\item \( J \) pseudoconvex on $C$
\item \( \lambda \) continuous on $C$
\item \( \exists u_\alpha \in C \) such that \( \lambda(u_\alpha) < \lambda_1 \) and
\end{enumerate}

\[ J(u_\alpha) < J(u) \quad \forall u \in C, \; u \neq u_\alpha \]

then the problems \((\mathcal{S})\) and \((\mathcal{Q})\) are equivalent (i.e. if \( u \) is a solution of \((\mathcal{S})\) \( u \) is a solution of \((\mathcal{Q})\) and reciprocally).

Let's recall the definition of a pseudoconvex function. \( J \) is a pseudoconvex on $C$ if \( J \) is Gateaux-differentiable on $C$ and if

\[ \forall (u,v) \in C \times C, \quad J'(u)(v-u) \geq 0 \implies J(v) \geq J(u) .\]

The reader can find the properties of pseudoconvex and quasiconvex functions in MANGASARIAN [6].

We make use of these properties for the proof of Proposition 2.1.

**Remark.** \((\mathcal{Q})\) is a convex problem.

**Proposition 2.2.** Assume \((2.1)\),

\begin{enumerate}
\item \( \lambda \) pseudoconcave on $C$
\end{enumerate}

and
\( \exists \ u_\gamma \in C \) \text{ such that } \lambda(u_\gamma) > \lambda_1.

Then if \( \bar{u} \in C \) and if \( \lambda(\bar{u}) \geq \lambda_1 \), \( \bar{u} \) is a solution of \((Q)\) if and only if

\[ \exists \ \eta \leq 0 \ \text{ such that } \]

\( J'(\bar{u}).(v-\bar{u}) + \eta \lambda'(\bar{u}).(v-\bar{u}) \geq 0 \) \( \forall v \in C \)

\( \lambda(\bar{u}) = \lambda_1 \).

The proof of the necessary condition is based on the results of HALKIN [2] and (2.4)-(2.5) assumptions.

The proof of the sufficiency is easy.

**Corollary 2.1.** Assume (2.1)-(2.3)-(2.4)-(2.5). Let \( \bar{u} \in C \), \( \lambda(\bar{u}) = \lambda_1 \) then \( \bar{u} \) is a solution of \((Q)\) if and only if

\[ \exists \ \eta < 0 \ \text{ such that } \]

\[ J'(\bar{u}).(v-\bar{u}) + \eta \lambda'(\bar{u}).(v-\bar{u}) \geq 0 \ \forall v \in C. \]

**3. THE PROBLEM \((\mathcal{G})\) IN STRUCTURAL DESIGN.**

Let \( \Omega \) be an open bounded connected set in \( \mathbb{R}^n \) let \( a \) and \( b \) two functions

\[ a : [0, +\infty[ \rightarrow [0, +\infty[ \quad \mathcal{C}^2, \text{ concave} \]

\[ b : [0, +\infty[ \rightarrow [0, +\infty[ \quad \mathcal{C}^2, \text{ convex}. \]

Denote by \( \langle ., . \rangle \) the scalar product of \( L^2(\Omega) \)

\[ \langle ., . \rangle \]

the bilinear canonical pairing over \( H^1_0(\Omega) \times H^{-1}(\Omega) \).

\[ U = \{ u \in L^\infty(\Omega) ; \int_\Omega (u) > 0 \ \text{ such that } \ u(x) > a(u) \ a.e. \ in \ \Omega \} \]

\[ C = \{ u \in L^\infty(\Omega) ; \ 0 < a \leq u(x) \leq b \ a.e. \ in \ \Omega \} \alpha, \beta \ \text{ given with} \]

\[ 0 < \alpha < 1 < \beta < +\infty \]

\( a \) and \( b \) the maps from \( U \) to \( U \) defined by

\[ a(u)(x) = a(u(x)), \ b(u)(x) = b(u(x)). \]

\( A_u \) the isomorphism from \( H^1_0(\Omega) \) to \( H^{-1}(\Omega) \) defined by
The operator of \( L^2(\Omega) \) defined by

\[
B_u w = b(u) w.
\]

### 3.1. The spectral problem.

It is known that the first eigenvalue \( \lambda(u) \) of

\[
A_u w = \lambda B_u w
\]

is simple and that the associated eigenvector \( w(u) \) satisfies

\[
w(u;x) \geq 0 \quad \text{a.e. in } \Omega
\]

\[
\lambda(u) = \inf_{w \in H_0^1(\Omega), \|w\|_0^2 = 1} \frac{\langle A_u w, w \rangle - \langle A_u w(u), w(u) \rangle}{\langle B_u w, w \rangle - \langle B_u w(u), w(u) \rangle}
\]

We suppose that \( w(u) \) is normalized in \( L^2(\Omega) \).

Using the Implicit Function Theorem we can prove that the function

\[
u \in U \mapsto \lambda(u)
\]

is differentiable on \( U \) and that

\[
\lambda'(u) \cdot v = \frac{\int_{\Omega} (a'(u) \cdot v) w_0 x |w(u;x)|^2 - \lambda(u) (b'(u) \cdot v) x w^2(u;x)) \, dx}{\int_{\Omega} b(u) w^2(u;x) \, dx}
\]

From the concavity of \( a \) and \( -b \), it follows that:

\[
\lambda(v) \leq \lambda(u) + \lambda'(u) \cdot (v-u) \frac{\int_{\Omega} b(u)(x) w^2(u;x) \, dx}{\int_{\Omega} b(v)(x) w^2(u;x) \, dx}
\]

Using this inequality we get

**Theorem 3.1.** \( \lambda \) is pseudo concave on \( U \). \( \lambda \) is upper semicontinuous on \( C \) for the weak\( ^* \) topology of \( L^\infty(\Omega) \).
3.2. Statement of the problem and results.

\( \Omega \) is the form of the structure, \( u \) is the thickness and therefore the weight is
\[
J(u) = \int_{\Omega} u(x) \, dx.
\]
Let \( u_1 \) be the function in \( C \) defined by
\[
u_1(x) = 1 \quad \forall \, x \in \Omega,
\]
and
\[
\lambda_1 = \lambda(u_1).
\]
We shall consider the following optimisation problem:

minimize \( J(u) \) subject to \( \lambda(u) = \lambda_1 \) and \( u \in C \) i.e.
\[
\text{(Q)} \quad \inf_{\lambda(u) = \lambda_1, \, u \in C} J(u).
\]
Let \( u_\alpha \) be the function defined on \( C \) by
\[
u_\alpha(x) = \alpha \quad \forall \, x \in \Omega.
\]
This function satisfies
\[
J(u_\alpha) < J(u) \quad \forall \, u \in C, \, u \neq u_\alpha.
\]
Introduce the problem
\[
\text{(Q)} \quad \inf_{\lambda(u) \leq \lambda_1, \, u \in C} J(u).
\]
Then using the previous results it is easy to prove

Theorem 3.2. Assume
\[
(3.5) \quad \lambda(u_\alpha) < \lambda_1
\]
\[
(3.6) \quad \exists \, u_\gamma \in C \text{ such that } \lambda(u_\gamma) > \lambda_1
\]
then
i) \( (f) \) and \( (Q) \) are equivalent
ii) \((\mathcal{G})\) has at least one solution;

iii) \(\bar{u} \in C\) is a solution of \((\mathcal{G})\) if \(\lambda(\bar{u}) = \lambda_1\) and

\[
(3.7) \quad \exists \, \eta < 0 \text{ such that } \quad J(v) - J(\bar{u}) + \eta \lambda'(u). (v - \bar{u}) > 0 \quad \forall \, v \in C; \]

iv) the set \(S(\mathcal{G})\) of solutions of \((\mathcal{G})\) is convex and \(\{w(u); u \in S(\mathcal{G})\}\) is reduced to one function.

Remark. The condition (3.7) is equivalent to (3.8) \(\exists \, e > 0\) such that

\[g(\bar{u};x) \leq e \text{ a.e. in } \Omega_\beta = \{x \in \Omega; \bar{u}(x) < \beta\}\]

and

\[g(\bar{u},x) \geq e \text{ a.e. in } \Omega_\alpha = \{x \in \Omega; \bar{u}(x) > \alpha\}\]

where

\[g(\bar{u};x) = A'(u(x)) |w(u;x)|^2 - \lambda_1 B'(u(x)) w^2(u;x).\]

3.3. Examples.

The previous results can be applied to numerous examples and in particular to

Example 1 \(a(u) = u\) \(b(u) = 1\)

Example 2 \(a(u) = u\) \(b(u) = u + \delta\) where \(\delta\) is a constant positive

Example 3 \(a(u) = 1\) \(b(u) = \frac{1}{u^2}\).

We refer to [1] and [3] for the motivation of these problems.

In these examples the function \(\bar{u}_\alpha\) defined by (3.4) satisfies (3.5) and the function \(u_\gamma\) defined by

\[u_\gamma(x) = \gamma \text{ where } 1 < \gamma < \delta\]

satisfies (3.6) and we can apply Theorem 3.2. In particular in Example 3, (3.8) gives \(\exists \, e > 0\) such that
and this implies that, in this example (33) has only one solution \( \bar{u} \). Moreover if \( \bar{\Omega} = ]0,1[ \) we can construct this solution.

Remark. We have used the same method to study problems of the following form

\[
\text{Sup } \lambda(u).
\]

\[
\text{J(u)=J}_1
\]

\[
\text{with } u \in C
\]

REFERENCES.


