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ON SOME STRUCTURAL DESIGN PROBLEMS

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1. INTRODUCTION

Some structural design problems may be stated as

$$(\mathcal{P}) \qquad \inf_{\substack{u\in C \\ \lambda(u) = \lambda_1}} J(u)$$

where u is the thickness of the structure, $J(u)$ its weight, $\lambda(u)$ its fundamental frequency of vibration, λ_1 the fundamental frequency of the structure with uniform thickness and C is the convex of constraints on u.

Here we consider the problems $(\mathcal{P})$ when J and λ have the following properties

$$J \text{ is convex on } C$$
$$\lambda \text{ is pseudoconcave on } C .$$

We shall state necessary and sufficient conditions of optimality for an abstract problem $(\mathcal{P})$. And we shall apply these results in structural design. We refer to [4] [5] for the proof of these results and for more details.

2. PROBLEM $(\mathcal{P})$: ABSTRACT CASE.

Let E a locally convex Hausdorff space, C a convex of E, J and λ

two real valued functions defined on C . We consider the two following problems

$$(\mathcal{P}) \qquad \inf_{\substack{u \in C \\ \lambda(u) = \lambda_1}} J(u)$$

$$(\mathcal{Q}) \qquad \inf_{\substack{u \in C \\ \lambda(u) \geqslant \lambda_1}} J(u)$$

where $\lambda_1 = \lambda(u_1) \quad u_1 \in C$.

<u>Proposition 2.1.</u> <u>Assume</u>

(2.1) $\qquad J$ <u>pseudoconvex on</u> C

(2.2) $\qquad \lambda$ <u>continuous on</u> C

(2.3) $\qquad \exists\, u_\alpha \in C$ <u>such that</u> $\lambda(u_\alpha) < \lambda_1$ <u>and</u>

$$J(u_\alpha) < J(u) \qquad \forall\, u \in C \;,\; u \neq u_\alpha$$

<u>then the problems</u> $(\mathcal{P})$ <u>and</u> $(\mathcal{Q})$ <u>are equivalent</u> (i.e. if u is a solution of $(\mathcal{P})$ u is a solution of $(\mathcal{Q})$ and reciprocally).

Let's recall the definition of a pseudoconvex function. J is a pseudoconvex on C if J is Gateaux-differentiable on C and if

$$\forall\, (u,v) \in C \times C \quad , \quad J'(u).(v-u) \geqslant 0 \implies J(v) \geqslant J(u) \; .$$

The reader can find the properties of pseudoconvex and quasiconvex functions in MANGASARIAN [6].

We make use of these properties for the proof of Proposition 2.1.

<u>Remark</u>. $(\mathcal{Q})$ is a convex problem.

<u>Proposition 2.2.</u> <u>Assume</u> (2.1),

(2.4) $\qquad \lambda$ <u>pseudoconcave on</u> C $\qquad$ and

(2.5) $\exists\ u_\gamma \in C$ <u>such that</u> $\lambda(u_\gamma) > \lambda_1$.

<u>Then if</u> $\bar{u} \in C$ <u>and if</u> $\lambda(\bar{u}) \geqslant \lambda_1$, $\bar{u}$ <u>is a solution of</u> (Q) <u>if and only if</u>

$\exists\ \bar{\eta} \leqslant 0$ <u>such that</u>

(2.6) $$J'(\bar{u}).(v-\bar{u}) + \bar{\eta}\ \lambda'(\bar{u}).(v-\bar{u}) \geqslant 0 \qquad \forall\ v \in C$$

(2.7) $$\bar{\eta}(\lambda(\bar{u})-\lambda_1) = 0 \ .$$

The proof of the necessary condition is based on the results of HALKIN [2] and (2.4)-(2.5) assumptions.

The proof of the sufficiency is easy.

<u>Corollary 2.1</u>. <u>Assume</u> (2.1)-(2.3)-(2.4)-(2.5). <u>Let</u> $\bar{u} \in C$, $\lambda(\bar{u}) = \lambda_1$ <u>then</u> $\bar{u}$ <u>is a solution of</u> $(\mathcal{P})$ <u>if and only if</u>

$$\exists\ \bar{\eta} < 0 \ \text{such that}\ J'(\bar{u}).(v-\bar{u}) + \bar{\eta}\ \lambda'(\bar{u}).(v-\bar{u}) \geqslant 0 \qquad \forall\ v \in C \ .$$

3. <u>THE PROBLEM</u> $(\mathcal{P})$ <u>IN STRUCTURAL DESIGN</u>.

Let Ω be an open bounded connected set in $\mathbb{R}^n$ let $\mathbb{a}$ and $\mathbb{b}$ two functions

$$\mathbb{a} : \ ]0,+\infty[\ \longrightarrow \ ]0,+\infty[\qquad \mathcal{C}^2 \text{, concave}$$

$$\mathbb{b} : \ ]0,+\infty[\ \longrightarrow \ ]0,+\infty[\qquad \mathcal{C}^2 \text{, convex} .$$

Denote by $(.,.)$ the scalar product of $L^2(\Omega)$

$<.,.>$ the bilinear canonical pairing over $H^1_0(\Omega) \times H^{-1}(\Omega)$.

$U = \{u \in L^\infty(\Omega) \ ; \exists \alpha(u) > 0$ such that $u(x) \geq \alpha(u)$ a.e. in $\Omega\}$

(3.1) $C = \{u \in L^\infty(\Omega) \ ; \ 0 < \alpha \leq u(x) \leqslant \beta$ a.e. in $\Omega\}$ α, β given with

$$0 < \alpha < 1 < \beta < +\infty$$

a and b the maps from U to U defined by

$$a(u)(x) = \mathbb{a}(u(x)) \ , \ b(u)(x) = \mathbb{b}(u(x)).$$

A_u the isomorphism from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$ defined by

$$< A_u \, w, \phi > = \int_\Omega a(u)(x) \nabla w(x) \, . \nabla\phi\,(x) \, dx \;, \quad (w,\phi) \in H_0^1(\Omega)^2$$

B_u the operator of $L^2(\Omega)$ defined by

$$B_u \, w = b(u) \, w \, .$$

3.1. <u>The spectral problem</u>.

It is known that the first eigenvalue $\lambda(u)$ of

$$A_u \, w = \lambda \, B_u \, w$$

is simple and that the associated eigenvector $w(u)$ satisfies

$$w(u;x) \geqslant 0 \qquad \text{a.e. in } \Omega$$

$$\lambda(u) = \operatorname*{Inf}_{\substack{w \in H_0^1(\Omega) \\ w \neq o}} \frac{< A_u \, w,w >}{(B_u \, w,w)} = \frac{< A_u \, w(u),w(u) >}{(B_u \, w(u),w(u))}$$

We suppose that $w(u)$ is normalized in $L^2(\Omega)$.

Using the Implicit Function Theorem we can prove that the function

$$u \in U \longrightarrow \lambda(u)$$

is differentiable on U and that

$$(3.2) \quad \lambda'(u).v = \frac{\int_\Omega \{(a'(u).v)(x) \, |\nabla w(u;x)|^2 - \lambda(u)(b'(u).v)(x) \, w^2(u;x)\} \, dx}{\int_\Omega b(u) \, w^2(u;x) \, dx} \, .$$

From the concavity of a and $(-b)$, it follows that :

$$(3.3) \qquad \lambda(v) \leqslant \lambda(u) + \lambda'(u).(v-u) \frac{\int_\Omega b(u)(x) \, w^2(u;x) \, dx}{\int_\Omega b(v)(x) \, w^2(u;x) \, dx} \qquad \forall \, (u,v) \in U \times U \, .$$

Using this inequality we get

<u>Theorem 3.1.</u> λ <u>is pseudo concave on</u> U . λ <u>is upper semicontinuous on</u> C <u>for the weak</u>$\ast$ <u>topology of</u> $L^\infty(\Omega)$.

3.2. Statement of the problem and results.

Ω is the form of the structure, u is the thickness and therefore the weight is

$$J(u) = \int_\Omega u(x)\,dx \;.$$

Let u_1 be the function in C defined by

$$u_1(x) = 1 \quad \forall\, x \in \Omega \,,$$

and

$$\lambda_1 = \lambda(u_1) \;.$$

We shall consider the following optimisation problem :

minimize $J(u)$ subject to $\lambda(u) = \lambda_1$ and $u \in C$ i.e.

$$(\mathcal{P}) \qquad \inf_{\substack{\lambda(u) = \lambda_1 \\ u \in C}} J(u) \;.$$

Let u_α be the function defined on C by

$$(3.4) \qquad u_\alpha(x) = \alpha \quad \forall\, x \in \Omega \;.$$

This function satisfies

$$J(u_\alpha) < J(u) \qquad \forall\, u \in C \,,\; u \neq u_\alpha \;.$$

Introduce the problem

$$(\mathcal{Q}) \qquad \inf_{\substack{u \in C \\ \lambda(u) \leq \lambda_1}} J(u) \;.$$

Then using the previous results it is easy to prove

Theorem 3.2. Assume

$$(3.5) \qquad \lambda(u_\alpha) < \lambda_1$$

$$(3.6) \qquad \exists\, u_\gamma \in C \text{ such that } \lambda(u_\gamma) > \lambda_1$$

then i) $(\mathcal{P})$ and $(\mathcal{Q})$ are equivalent ;

ii) $(\mathcal{P})$ <u>has at least one solution</u> ;

iii) $\bar{u} \in C$ <u>is a solution of</u> $(\mathcal{P})$ <u>if</u> $\lambda(\bar{u}) = \lambda_1$ <u>and</u>

(3.7) $\exists\ \bar{\eta} < 0$ <u>such that</u>

$$J(v) - J(\bar{u}) + \bar{\eta}\ \lambda'(u).(v-\bar{u}) \geqslant 0 \quad \forall\ v \in C \ ;$$

iv) <u>the set</u> $S(\mathcal{P})$ <u>of solutions of</u> $(\mathcal{P})$ <u>is convex and</u> $\{w(u);u \in S(\mathcal{P})\}$ <u>is reduced to one function.</u>

<u>Remark</u>. The condition (3.7) is equivalent to (3.8) $\exists\ e > 0$ such that

$$g(\bar{u};x) \leqslant e \quad \text{a.e. in} \quad \Omega_\beta = \{x \in \Omega;\bar{u}(x) < \beta\}$$

and

$$g(\bar{u},x) \geqslant e \quad \text{a.e. in} \quad \Omega_\alpha = \{x \in \Omega;\bar{u}(x) > \alpha\}$$

where

$$g(\bar{u};x) = \mathfrak{a}'(u(x))\ |\nabla w(u;x)|^2 - \lambda_1\ \mathfrak{b}'(u(x))\ w^2(u;x)\ .$$

3.3. <u>Examples</u>.

The previous results can be applied to numerous examples and in particular to

Example 1	$a(u) = u$	$b(u) = 1$
Example 2	$a(u) = u$	$b(u) = u + \delta$ where δ is a constant positive
Example 3	$a(u) = 1$	$b(u) = \frac{1}{u^2}$.

We refer to [1] and [3] for the motivation of these problems.

In these examples the function u_α defined by (3.4) satisfies (3.5) and the function u_γ defined by

$$u_\gamma(x) \equiv \gamma \quad \text{where} \quad 1 < \gamma \leqslant \beta$$

satisfies (3.6) and we can apply Theorem3.2. In particular in Example 3, (3.8) gives $\exists\ e > 0$ such that

$$\frac{w^2(\bar{u};x)}{\bar{u}^3(x)} \leqslant e \quad \text{a.e. in } \Omega_\beta$$

$$\frac{w^2(\bar{u};x)}{\bar{u}^3(x)} \geqslant e \quad \text{a.e. in } \Omega_\alpha$$

and this implies that, in this example $(\mathcal{P})$ has only one solution $\bar{u}$. Moreover if $\Omega =]0,1[$ we can construct this solution.

Remark. We have used the same method to study problems of the following form

$$\sup_{\substack{J(u)=J_1 \\ u\in C}} \lambda(u) .$$

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