REMI SENTIS

Discretized feedback for differential games

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§1. Introduction

Let us recall a result of minimization for optimal control (see SENTIS [1]). For any initial condition $(t,x)$ of $[0,T] \times \mathbb{R}^d$, we call $\mathcal{U}_{t,x}$ the set of the controls $b$ of $L^\infty(0,T;\mathbb{R}^d)$ such that (1) admits a solution (which is denoted $y_b$):

$$
\begin{align*}
&y'(s) = b(s) \quad b(s) \in B(s,y(s)) \quad a.e. s \in [t,T] \\
&y(t) = x
\end{align*}
$$

with the hypothesis:

$$
B \text{ is a Lipschitzian multivalued mapping from } [0,T] \times \mathbb{R}^d \\
\text{with convex compact values in the sphere of radius } Q_0.
$$

For fixed $(t,x)$ we will minimize on $\mathcal{U}_{t,x}$ the following cost

$$
J_{t,x}(b) + F(y_b(T))
$$

where $F$ is Lipschitzian and has no property of convexity. This problem admits an optimal open-loop control, but we look for a feedback which approaches the optimum for any initial condition. For that purpose we discretize the interval of time defining:

$$
\begin{align*}
&h_n = T/n \\
&t^k_n = kh_n \\
&\theta_n(t) \text{ the unique integer such that } t \in [t^k_n,t^{k+1}_n[
\end{align*}
$$

And there exist multivalued (m.v.) mappings $v_{n}^{0}, v_{n}^{1}, \ldots, v_{n}^{n-1}$ from $\mathbb{R}^d$ to $\mathbb{R}^d$ such that

$$
v_{n}^{k}(z) \subset B(t_{n}^{k},z) \\
\forall z \in \mathbb{R}^d
$$

and such that for any initial condition $(t,x)$, if we define a trajectory $y_{n}$ (linear on any interval $[t_{n}^{k}, t_{n}^{k+1}]$) by $y_{n}(t_{n}^{k}) = x_{n}^{k}$ with...
then any accumulation point $y$ in $C^0(0,T;\mathbb{R}^d)$ of $(y^n)$ is a solution of (1) and is optimal, that is to say:

$$F(y(T)) = J_{t,x}(y(t)) = \min_{b \in \mathcal{F}_{t,x}} J_{t,x}(b)$$

Let us now consider the following differential game. For any initial condition $(t,x)$ the admissible trajectories are the solutions of

$$\begin{cases}
y'(s) \in A(s,y(s)) + B(s,y(s)) & \text{a.e. } s \in [t,T] \\
y(t) = x
\end{cases}$$

where $A$ and $B$ satisfy (2). Let $F$ be a Lipschitzian function on $\mathbb{R}^d$.

Heuristically, if $u$ and $v$ are two sections of $A$ and $B$ such that there exists a solution (denoted $y_{u,v}$) of:

$$\begin{cases}
y'(s) = u(s,y(s)) + v(s,y(s)) \\
y(t) = x
\end{cases}$$

then we look for $u^*$ and $v^*$, sections of $A$ and $B$, such that

$$(9) \quad F(y_{u^*,v^*}(T)) \leq F(y_{u,v}(T)) \leq F(y_{u^*,v^*}(T))$$

for any $u$ and $v$ section of $A$ and $B$.

In general, there do not exist sections $u^*$ and $v^*$ verifying (9) and such that $u^*$ and $v^*$ are continuous with respect to the state variable. (Obviously there do not exist open-loop controls $u^*$ and $v^*$ verifying (9).)

The topic of this paper is to find a couple of strategies which is a saddle-point for the differential game in a certain class of strategies. For that purpose we must first define the class of admissible strategies (we use the notations (4) except $h_n = T/2^n$ and we write $\overline{n}$ for $2^n$)

Definition 1. An admissible strategy for the player $U$ [or $V$] is a sequence $(u^n)_{n \geq 1}$ [or $(v^n)_{n \geq 1}$] of elements $u^n$ [or $v^n$] (which are called discretized feedbacks) with:

\[
\begin{align*}
x^n_0 &= x \\
x^n_k &= \begin{cases} x^n_0 & \text{if } k = 0 \\
\frac{1}{h^n} \left( \sum_{i=0}^{k-1} x^n_i + v^n(x^n_k) h^n \right) & \text{if } 0 < k < h^n \\
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\{ u_n = (u_0, u_1, u_2, \ldots, u_{n-1}) \in \bigcap_{k=0}^{n-1} U_k \} \\
\text{where } U_n^k \text{ is the set of the m.v. mappings } u \text{ on } \mathbb{R}^d \text{ verifying :} \\
\bigcup (\tau^k_n, z) \\
\end{align*}
\]

\[
\begin{align*}
\{ v_n = (v_0, v_1, v_2, \ldots, v_{n-1}) \in \bigcap_{k=0}^{n-1} V_k \} \\
\text{where } V_n^k, k \text{ is the set of the m.v. mappings } v \text{ on } \mathbb{R}^d \text{ verifying :} \\
\bigcup (\tau^k_n, z) \\
\end{align*}
\]

In §2, we exhibit particular discretized feedbacks associated to each \( h_n \) and in §3, let \( n \) go to infinity, to show that the sequences of such discretized feedbacks constitutes a saddle point in the class of admissible strategies (for detailed proofs, see SENTIS [2]).

§2. Definition of the discretized feedbacks \( u_n \) and \( v_n \).

The following proposition justifies the term admissible in definition 1.

**Proposition 1.** Let us fix \((t, x)\). If \((u_n)\) and \((v_n)\) are admissible strategies and if we define a trajectory \( y_n \) linear on each interval \([t_n^k, t_n^{k+1}]\) by \( y_n(t_n^k) = x_n^k \) and \( x_n^k \) given by (11)

\[
\begin{align*}
\theta k = x \\
\{ x_n^k + h_n (u_n(x_n^k) + v_n(x_n^k)) \text{ for } k \geq \theta t_n \}
\end{align*}
\]

then any accumulation point \( y \) in \( C^0 \) verifies (7).

Now let us give two definitions for the cost of a game with initial conditions \((t, x)\).

**Definition 2.** The cost of the game for the two discretized feedbacks \( u_n \) and \( v_n \) is the subset of \( \mathbb{R} \) defined by:
Definition 3. The cost of the game for the two admissible strategies \( (u)^n \) and \( (v)^n \) is the subset of \( \mathbb{R} \) denoted by \( J((u)^n,(v)^n) \) and containing the accumulation points of all the sequences \( (a)^n \) verifying \( a^n \in J((u)^n,(v)^n) \).

Let us yet define the lower and upper optimal cost-functions \( W_n^k \) and \( \bar{W}_n^k \) as Friedman [1] by decreasing induction:

\[
\begin{aligned}
W_n^0(x) &= F(x) \\
W_n^k(x) &= \max_{u \in A(t_n^k,x)} \bar{Z}_n^k(x,u) \quad \text{and} \quad \bar{W}_n^k(x) = \min_{v \in B(t_n^k,x)} W_n^{k+1}(x+(u+v)h_n)
\end{aligned}
\]

and:

\[
\begin{aligned}
\bar{W}_n^0(x) &= F(x) \\
\bar{W}_n^k(x) &= \min_{v \in B(t_n^k,x)} \bar{Z}_n^k(x,v) \quad \text{and} \quad \bar{W}_n^k(x) = \max_{u \in A(t_n^k,x)} W_n^{k+1}(x+(u+v)h_n)
\end{aligned}
\]

Now we can exhibit the m.v. mappings \( \bar{u}_n^k \) and \( \bar{v}_n^k \), which do not depend on the initial conditions.

\[
\begin{aligned}
\bar{u}_n^k(x) &= \arg \max_{u \in A(t_n^k,x)} \bar{Z}_n^k(x,u) \\
\bar{v}_n^k(x) &= \arg \min_{u \in A(t_n^k,x)} \bar{Z}_n^k(x,u)
\end{aligned}
\]

We can prove easily by induction the following:

**Proposition 2.** All the mappings \( \bar{W}_n^k, \bar{Z}_n^k, \bar{W}_n^k, \bar{Z}_n^k \) are Lipschitzian (with respect to \( x \)) with constant \( K \) (independent of \( n \) and \( k \)).
§3. **Saddle point theorem**

**Proposition 3** We have when \( n \) goes to infinity:

\[
\begin{align*}
\theta^n_t \ & W_n^t (x) \rightarrow W^t(t,x) \\
\hat{\theta}^n_t \ & \hat{W}_n^t (x) \rightarrow \hat{W}^t(t,x)
\end{align*}
\]

moreover:

\[
W^-(t,x) \leq W^t(t,x)
\]

**principle of the proof.**

First we show by decreasing induction on \( k \) that

\[
W^n_k (x) \rightarrow W^2 (x) \leq (n-k) C_0 (h_n)^2 \quad \forall k
\]

And as \( \rho_{n+1} \) is equal to \((2\rho_n)\) or \((2\rho_{n+1})\) we have according to proposition 2:

\[
(14) \quad \theta^n_t W_n^t (x) - \theta^{n+1}_t W_{n+1}^t (x) \leq C_1 h_n \quad \text{with} \quad C_1 = C_0 T + 2KQ_0
\]

Hence if we denote:

\[
W^-(t,x) = \lim_{n} \sup \theta^n_t W_n^t (x)
\]

we can show easily according to \((14)\) that \( W_n^t (x) \rightarrow W^-(t,x) \). We show exactly the same way that \( W_n^t (x) \rightarrow W^+(t,x) \). The end of the proposition is a consequence of the following fact:

\[
W^n_k (x) \leq \hat{W}_n^k (x) \quad \forall x, n, k \quad \text{Q.E.D.}
\]

The following proposition is fundamental and is proved in FRIEDMAN [1], using the m.v. mappings:
Proposition 4

We have

\[ W^-(t,x) = W^+(t,x) \]

We write thus \( W(t,x) \) instead of \( W^-(t,x) \). This number is called the value of the game.

Proposition 5

For any \( u \in \prod_{k=0}^{n-1} \mathcal{U}^k \), we have

\[
\text{(15)} \quad J_{t,x}(u_n, v_n) < W_n(x) 
\]

(This means that any element of the left-hand side is smaller than the right-hand side.)

Proof

Using the notations (11) (changing \( v_n \) into \( \tilde{v}_n \)), we note that there exist \( q_n^k \in x_n^k \) such that

\[
x_n^{k+1} = x_n^k + h_n(u_n^k(x_n^k) + q_n^k), \quad \forall k \geq \theta_n t
\]

Thus we have:

\[
\tilde{u}_n^k(x_n^k) = \tilde{u}_n^k(x_n^k, q_n^k) > \tilde{u}_n^{k+1}(x_n^{k+1})
\]

Rewriting this inequality for \( k \) from \( \theta_n t \) to \( \bar{n} \), we obtain (15). Q.E.D.

We have evidently also:

\[
\text{(16)} \quad J_{t,x}(\tilde{u}_n^k, v_n) > W_n(x)
\]
Let $n$ go to infinity in (15) and (16), we deduce immediately from the propositions 3 and 4 the following:

**Theorem**

For any admissible strategy $(u_n)$ and $(v_n)$, we have:

$$ J_{t,x}(u_n, v_n) \leq W(t,x) \leq J_{t,x}(\tilde{u}_n, \tilde{v}_n) $$

Thus we have:

$$ W(t,x) = \min_{(u_n)} \max_{(v_n)} J_{t,x}(u_n, v_n) = \max_{(u_n)} \min_{(v_n)} J_{t,x}(u_n, v_n) $$

And if $y$ is an accumulation point in $C^0$ of trajectories $y_n$ associated to $\tilde{u}_n$ and $\tilde{v}_n$ we have

$$ W(t,x) = J_{t,x}(\tilde{u}_n, \tilde{v}_n) = F(y(T)) $$

**References:**

