REMI SENTIS

Discretized feedback for differential games

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§1. Introduction

Let us recall a result of minimization for optimal control (see SENTIS [1]). For any initial condition \((t, x)\) of \([0, T] \times \mathbb{R}^d\), we call \(\mathcal{U}_{t, x}\) the set of the controls \(b\) of \(L^\infty(0, T; \mathbb{R}^d)\) such that (1) admits a solution (which is denoted \(y_b\)):

\[
y'(s) = b(s) \quad b(s) \in B(s, y(s)) \quad a.e. s \in [t, T]
\]

(1) \(y(t) = x\)

with the hypothesis:

\[
B \text{ is a Lipschitzian multivalued mapping from } [0, 1] \times \mathbb{R}^d
\]

with convex compact values in the sphere of radius \(Q_0\).

For fixed \((t, x)\) we will minimize on \(\mathcal{U}_{t, x}\) the following cost

\[
J_{t, x}(b) + F(y_b(T))
\]

where \(F\) is Lipschitzian and has no propriety of convexity. This problem admits an optimal open-loop control, but we look for a feedback which approaches the optimum for any initial condition. For that purpose we discretize the interval of time defining:

\[
\begin{align*}
h_n &= T/n \\
t_k^n &= kh_n \\
\theta_t &= \text{the unique integer such that } t \in [t_k^n, t_{k+1}^n]
\end{align*}
\]

(4)

And there exist multivalued (m.v.) mappings \(v_n^0, v_n^1, ..., v_n^{n-1}\) from \(\mathbb{R}^d\) to \(\mathbb{R}^d\) such that

\[
v_n^k(z) \subseteq B(t_n^k, z) \quad \forall z \in \mathbb{R}^d
\]

and such that for any initial condition \((t, x)\), if we define a trajectory \(y_n^k\) (linear on any interval \([t_n^k, t_{n+1}^k]\)) by \(y_n^k(t_n^k) = x_n^k\) with
\[
\begin{aligned}
&\begin{cases}
  x_n^t = x \\
  n
  k+1 \leq x_n + v(x_n) h_n \\
  k \geq t
\end{cases}
\end{aligned}
\]

then any accumulation point \( y \) in \( C^0(0,T;\mathbb{R}^d) \) of \( (y_n) \) is a solution of (1) and is optimal, that is to say:

\[
F(y(T)) = J_{t,x}(y) = \min_{b \in \mathbb{R}^d_{t,x}} J_{t,x}(b)
\]

Let us now consider the following differential game. For any initial condition \((t,x)\) the admissible trajectories are the solutions of

\[
\begin{aligned}
  y'(s) &\in A(s,y(s)) + B(s,y(s)) \quad \text{a.e. } s \in [t,T] \\
  y(t) &= x
\end{aligned}
\]

where \( A \) and \( B \) satisfy (2). Let \( F \) be a Lipschitzian function on \( \mathbb{R}^d \).

Heuristically, if \( u \) and \( v \) are two sections of \( A \) and \( B \) such that there exists a solution (denoted \( y_{u,v} \)) of:

\[
\begin{aligned}
  y'(s) &= u(s,y(s)) + v(s,y(s)) \\
  y(t) &= x
\end{aligned}
\]

then we look for \( u^* \) and \( v^* \), sections of \( A \) and \( B \), such that

\[
(9) \quad F(y_{u,v}(T)) \leq F(y_{u^*,v^*}(T)) \leq F(y_{u^*,v}(T))
\]

for any \( u \) and \( v \) section of \( A \) and \( B \).

In general, there do not exist sections \( u^* \) and \( v^* \) verifying (9) and such that \( u^* \) and \( v^* \) are continuous with respect to the state variable. (Obviously there do not exist open-loop controls \( u^* \) and \( v^* \) verifying (9).)

The topic of this paper is to find a couple of strategies which is a saddle-point for the differential game in a certain class of strategies. For that purpose we must first define the class of admissible strategies (we use the notations (4) except \( h_n = T/2^n \) and we write \( n \) for \( 2^n \)).

**Definition 1.** An admissible strategy for the player \( U \) (or \( V \)) is a sequence \((u_n) \) (or \((v_n)\)) of elements \( u_n \) (or \( v_n \)) (which are called discretized feedbacks) with:
\[
\begin{cases}
    u_n = (u_n^0, u_n^1, u_n^2, \ldots, u_n^{n-1}) \in \bigcap_{k=0}^{n-1} U^n_k \\
    u(z) \subseteq A(t_k^n, z)
\end{cases}
\]

(10)

where \( U^n_k \) is the set of the m.v. mappings \( u \) on \( \mathbb{R}^d \) verifying:

\[
\begin{cases}
    v_n = (v_n^0, v_n^1, v_n^2, \ldots, v_n^{n-1}) \in \bigcap_{k=0}^{n-1} V^n_k \\
    v(z) \subseteq B(t_k^n, z)
\end{cases}
\]

(10')

where \( V^n_k \) is the set of the m.v. mappings \( v \) on \( \mathbb{R}^d \) verifying:

In §2, we exhibit particular discretized feedbacks associated to each \( h_n \) and in §3, let \( n \) go to infinity, to show that the sequences of such discretized feedbacks constitutes a saddle point in the class of admissible strategies (for detailed proofs, see SENTIS [2]).

§2. Definition of the discretized feedbacks \( \bar{u}_n \) and \( \bar{v}_n \).

The following proposition justifies the term admissible in definition 1.

Proposition 1. Let us fix \((t, x)\). If \((u_n)\) and \((v_n)\) are admissible strategies and if we define a trajectory \( y_n \) linear on each interval \([t^k_n, t^{k+1}_n]\) by \( y_n(t^k_n) = x_n^k \) and \( x_n^k \) given by (11)

\[
\begin{cases}
    x_n^0 = x \\
    x_n^k = x_n^{k-1} + h_n(u_n^k(x_n^{k-1}) + v_n^k(x_n^{k-1})) \quad k \geq 0 \quad t \\
\end{cases}
\]

(11)

then any accumulation point \( y \) in \( C^0 \) verifies (7).

Now let us give two definitions for the cost of a game with initial conditions \((t, x)\).

Definition 2. The cost of the game for the two discretized feedbacks \( u_n \) and \( v_n \) is the subset of \( \mathbb{R} \) defined by:
Let us yet define the lower and upper optimal cost-functions $\widehat{w}_n$ and $\underline{w}_n$ as FRIEDMAN [1] by decreasing induction:

\[
\begin{align*}
\widehat{w}_n(x) &= F(x) \\
\underline{w}_n(x) &= \max_{u \in A(t_k_n, x)} \min_{v \in B(t_k_n, x)} \widehat{Z}_n(x, u + v) \\
\widehat{Z}_n(x, u) &= \min_{v \in B(t_k_n, x)} \max_{u \in A(t_k_n, x)} \widehat{Z}_n(x, u + v) \\
\underline{Z}_n(x, v) &= \max_{u \in A(t_k_n, x)} \min_{v \in B(t_k_n, x)} \underline{Z}_n(x, u + v)
\end{align*}
\]

Now we can exhibit the m.v. mappings $\hat{u}_n$ and $\hat{v}_n$, which do not depend on the initial conditions.

\[
\begin{align*}
\hat{u}_n(x) &= \arg \max_{u \in A(t_k_n, x)} \widehat{Z}_n(x, u) \\
\hat{v}_n(x) &= \arg \min_{v \in B(t_k_n, x)} \underline{Z}_n(x, v)
\end{align*}
\]

We can prove easily by induction the following:

**Proposition 2.** All the mappings $\hat{w}_n, \hat{z}_n, \hat{w}_n, \hat{z}_n$ are Lipschitzian (with respect to $x$) with constant $K$ (independent of $n$ and $k$).
§3. **Saddle point theorem**

**Proposition 3** We have when $n$ goes to infinity:

\[ \hat{W}_n^{\theta_n t}(x) \to W^{-}(t,x) \quad \hat{W}_n^{\theta_n t}(x) \to W^{+}(t,x) \]

moreover:

\[ W^{-}(t,x) \leq W^{+}(t,x) \]

**principle of the proof.**

First we show by decreasing induction on $k$ that

\[ W_{n}^{k}(x) - W_{n+1}^{2k}(x) \leq (n-k) C_0 (h_n)^2 \quad \forall k \]

And as $\theta_{n+1} t$ is equal to $(2\theta_n t)$ or $(2\theta_n t+1)$ we have according to proposition 2:

\[ (14) \quad \hat{W}_n^{\theta_n t}(x) - \hat{W}_n^{\theta_{n+1} t}(x) \leq C_1 h_n \quad \text{with} \quad C_1 = C_0 T + 2KQ_0 \]

Hence if we denote:

\[ W^{-}(t,x) = \lim_{n} \sup_{\hat{W}_n^{\theta_n t}}(x) \]

we can show easily according to (14) that $\hat{W}_n^{\theta_n t}(x) \to W^{-}(t,x)$. We show exactly the same way that $\hat{W}_n^{\theta_n t}(x) \to W^{+}(t,x)$. The end of the proposition is a consequence of the following fact:

\[ \hat{W}_n^{k}(x) \leq \hat{W}_n^{k}(x) \quad \forall x, n, k \quad \text{Q.E.D.} \]

The following proposition is fundamental and is proved in FRIEDMAN [1], using the m.v. mappings:
Proposition 4

We have

\[ W(t,x) = W(t,x) \]

We write thus \( W(t,x) \) instead of \( W^-(t,x) \). This number is called the value of the game.

Proposition 5

For any \( u \in \mathbb{N} \) we have

\[
(15) \quad J_{t,x}(u, v) \leq W_n(x)
\]

(This means that any element of the left-hand side is smaller than the right-hand side.)

Proof

Using the notations (11) (changing \( v_n^k \) into \( \tilde{v}_n^k \)), we note that there exist \( q_n^k \in \tilde{v}_n(x_n^k) \) such that

\[
x_n^{k+1} = x_n^k + h_n(u_n^k(x_n^k) + q_n^k) \quad \forall k \geq \theta_n t
\]

Thus we have:

\[
\tilde{u}_n^k(x_n^k) = \tilde{u}_n^k(x_n^k, q_n^k) > \tilde{u}_n^{k+1}(x_n^k)
\]

Rewriting this inequality for \( k \) from \( \theta_n t \) to \( n \), we obtain (15). Q.E.D.

We have evidently also:

\[
(16) \quad J_{t,x}(\tilde{u}_n^k, v_n) \geq W_n(x)
\]
Let $n$ go to infinity in (15) and (16), we deduce immediately from the propositions 3 and 4 the following:

**Theorem**

For any admissible strategy $(u_n)$ and $(v_n)$, we have:

$$J_{t,x}((u_n)_n, (v_n)_n) \leq W(t,x) \leq J_{t,x}((\tilde{u}_n)_n, (v_n)_n)$$

Thus we have:

$$W(t,x) = \min \max J_{t,x}((u_n)_n, (v_n)_n) = \max \min J_{t,x}((u_n)_n, (v_n)_n)$$

And if $y$ is an accumulation point in $C^0$ of trajectories $y_n$ associated to $u_n$ and $\tilde{v}_n$ we have

$$W(t,x) = J_{t,x}((\tilde{u}_n)_n, (\tilde{v}_n)_n) = F(y(T))$$

**References:**


Remy S E N T I S
28 rue du Flea
92100 BOULOGNE