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THE REDUCED WITTINGRING

by

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These notes give a brief account on a joint work of L. Br"ucker and the author. Detailed proofs will appear in the Journal of Algebra.

Let $K$ be a real (= formally real) field, $X = X(K)$ the topological space of all orderings of $K$ [5, p.63], and $W(K)$ the Witt ring of the nondegenerated bilinearforms over $K$. By $W_t(K)$ we denote the torsion-subgroup of $W(K)$, which is known also to be its nilradical [6].

Let $C(X,Z)$ be the ring of all continuous functions $X \to Z$ (provided with the discrete topology). Then we get a homomorphism $\text{sign} : W(K) \to C(X,Z)$ defined by $(\text{sign}(p))(P) := \text{sign}_p(P) = \text{signature of } p$ at $P$. The following basic result is due to Pfister [6].

**Theorem 1**: The sequence $0 \to W_t(K) \to W(K) \to \text{sign} : C(X,Z)$ is exact.

So it must be considered as a main task in the theory of reduced Witt rings to characterize the elements of $\text{sign} W(K)$ among the functions $f \in C(X,Z)$. In order to state the main result the notion of a preorder has to be introduced. A subset $T \subset K$ is called a preorder of $K$ iff the following conditions are satisfied:

i) $T \subset K$, $T \cdot T \subset K$
ii) $K^2 \subset T$
iii) $T \cap -T = \{0\}$

A preorder $T$ is the intersection of all orderings in which it is contained. Given a preorder $T$, the subset $X_T := \{ P \supseteq T \mid P \text{ ordering of } K \}$ is a closed subspace of $X$. Clearly, we have the restriction-homomorphism $\text{Res} : C(X,Z) \to C(X_T,Z)$.

Denote by $W_T(K)$ the image of $\text{Res}$. By $\text{sign} : W(K) \to C(X_T,Z)$. Choose any ordering $P_o \supseteq T$. Set $P_o^x = P_o \setminus \{0\}$, $T^x = T \setminus \{0\}$; $P_o^x$ and $T^x$ are subgroups of $K^x$. As with $W(K)$, we find an epimorphism $\mathbb{Z}[P_o^x/T^x] \to W_T(K)$. Furthermore the mapping $X_T \to \text{char} (P_o^x/T^x)$, $P \to \text{sign}_p(a)$ is a topological embedding of $X_T$ into the Pontrjagin-character-group of $P_o^x/T^x$.

**Proposition**. For a preorder $T$ the following statements are equivalent:

i) $\mathbb{Z}[P_o^x/T^x] \to W_T(K)$ is an isomorphism.

ii) $X_T \to \text{char}(P_o^x/T^x)$ is a homeomorphism.

iii) $T + T_a = T \cup T_a$ for all $a \in K$, such that $a \notin T$. 
A preorder which satisfies the equivalent conditions of the last proposition, is called a fan (in French: éventail). Fans turn out to be of great importance in other contexts, too [1], [3].

**THEOREM 2.** A function \( f \in C(X, \mathbb{R}) \) lies in sign \( W(K) \) iff

\[
\sum_{T \in \Sigma} \tau(T) \equiv 0 \mod \frac{1}{2} (k^X : T^X)
\]

for all fans \( T \) with \((k^X : T^X) < \infty\).

The description of sign \( W(K) \) in \( C(X, \mathbb{R}) \) was also attacked by R. Brown [4] and settled for the case that \( K \) admits only finitely many real places. For the general case he was led to a conjecture which (in his terminology) states that all formally real fields are exact. From theorem 2 one can derive:

**THEOREM 3.** All formally real fields are exact.

The proof of theorem 2 heavily depends on two local-global principles for reduced quadratic forms, one of which has essentially been proved in [2]. Furthermore, the generalized theory of reduced Wittings [1] is extensively used, i.e. \( W(K) \) is factorized by forms \( \langle t, t \rangle \), where \( 0 \neq t \) and \( t \) belongs to an arbitrary but fixed preorder \( T \). This point of view turns out to be fundamental even for the study of ordinary reduced Wittings.

**REFERENCES**


