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THE REDUCED WITTING

by

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These notes give a brief account on a joint work of L. Bröcker and the author. Detailed proofs will appear in the Journal of Algebra.

Let $K$ be a real (= formally real) field, $X = X(K)$ the topological space of all orderings of $K$ [5, p.61], and $W(K)$ the Witt ring of the nondegenerated bilinear forms over $K$. By $W_t(K)$ we denote the torsion-subgroup of $W(K)$, which is known also to be its nilradical [6].

Let $C(X,\mathbb{Z})$ be the ring of all continuous functions $X \to \mathbb{Z}$ (provided with the discrete topology). Then we get a homomorphism $\text{sign} : W(K) \to C(X,\mathbb{Z})$ defined by $(\text{sign}(\rho))(P) := \text{sign}_P(\rho) = \text{signature of } \rho \text{ at } P$. The following basic result is due to Pfister [6].

**THEOREM 1:** The sequence $0 \to W_t(K) \to W(K) \to \text{sign} C(X,\mathbb{Z})$ is exact.

So it must be considered as a main task in the theory of reduced Witt rings to characterize the elements of $\text{sign} W(K)$ among the functions $f \in C(X,\mathbb{Z})$. In order to state the main result the notion of a preorder has to be introduced. A subset $T \subset K$ is called a preorder of $K$ iff the following conditions are satisfied:

i) $T + T \subset K$, $T.T \subset K$
ii) $K^2 \subset T$
iii) $T \cap -T = \{0\}$.

A preorder $T$ is the intersection of all orderings in which it is contained. Given a preorder $T$, the subset $X_T := \{P \supset T \mid P \text{ ordering of } K\}$ is a closed subspace of $X$. Clearly, we have the restriction-homomorphism $\text{Res} : C(X,\mathbb{Z}) \to C(X_T,\mathbb{Z})$.

Denote by $W_T(K)$ the image of $\text{Res} : W(K) \to C(X_T,\mathbb{Z})$. Choose any ordering $P_0 \supset T$. Set $P_0^X = P_0 \setminus \{0\}$, $T^X = T \setminus \{0\}$; $P_0^X$ and $T^X$ are subgroups of $K^X$. As with $W(K)$, we find an epimorphism $\mathbb{Z}[P_0^X/T^X] \to W_T(K)$. Furthermore the mapping $X_T \to \text{char} (P_0^X/T^X), P \mapsto (aT^X \to \text{sgn}_P(a))$ is a topological embedding of $X_T$ into the (Pontrjagin-) character-group of $P_0^X/T^X$.

**PROPOSITION.** For a preorder $T$ the following statements are equivalent:

i) $\mathbb{Z}[P_0^X/T^X] \to W_T(K)$ is an isomorphism.

ii) $X_T \to \text{char}(P_0^X/T^X)$ is a homeomorphism.

iii) $T + T_\alpha = T \cup T_\alpha$ for all $\alpha \in K$, such that $\alpha \notin T$. 
A preorder which satisfies the equivalent conditions of the last proposition, is called a fan (in French : éventail). Fans turn out to be of great importance in other contexts, too [1], [3].

**THEOREM 2.** A function \( f \in C(X, \mathbb{Z}) \) lies in sign \( \text{W}(K) \) iff

\[
\sum_{T} f(T) \equiv 0 \text{ mod } \frac{1}{2} (K^X : T^X)
\]

for all fans \( T \) with \( (K^X : T^X) < \infty \).

The description of sign \( \text{W}(K) \) in \( C(X, \mathbb{Z}) \) was also attacked by R. Brown [4] and settled for the case that \( K \) admits only finitely many real places. For the general case he was led to a conjecture which (in his terminology) states that all formally real fields are exact. From theorem 2 one can derive:

**THEOREM 3.** All formally real fields are exact.

The proof of theorem 2 heavily depends on two local-global principles for reduced quadratic forms, one of which has essentially been proved in [2]. Furthermore, the generalized theory of reduced Wittings [1] is extensively used, i.e. \( \text{W}(K) \) is factorized by forms \( \langle 1, -t \rangle \), where \( 0 \neq t \) and \( t \) belongs to an arbitrary but fixed preorder \( T \). This point of view turns out to be fundamental even for the study of ordinary reduced Wittings.

**REFERENCES**