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Remarks on random walks on semi simple Lie groups

Mémoires de la S. M. F., tome 54 (1977), p. 119-127

<http://www.numdam.org/item?id=MSMF_1977__54__119_0>
REMARKS ON RANDOM WALKS ON SEMI SIMPLE LIE GROUPS

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The purpose of this note is to discuss two problems concerning random walks and harmonic functions on Lie groups, and thereby to complete and enlarge upon the results of A. RAUGI announced in [6] and proved in detail in [7]. Both problems have their origin in the work of H. FURSTENBERG [2,3] as further extended by R. AZENCOTT [1] and A. VIRTSER [8]. For the sake of convenience in notation and bookkeeping, the results of this paper are stated for connected semi simple groups, which are clearly the most important case, but one could easily extend them as in [7] to general almost connected Lie groups (those with only finitely many connected components).

1. \(\mu\)-POSITIVE AND \(\mu\)-NEGATIVE COCYCLES

1.1.- Let \(G\) be a connected semisimple Lie group with finite center and with Iwasawa decomposition \(K\) \(\text{AN}\). Let \(a\) denote the Lie algebra of \(A\), let \(a^*\) be its real dual, and let \(\mathfrak{a}^*\) be the complexification of \(a^*\). In the discussion that follows, the term root will always refer to an element of \(a^*\) which is a restricted root of the pair \((G, A)\). The Weyl group is the finite group \(W\) of automorphisms of \(a\) or \(a^*\) induced by the conjugation action of the normalizer of \(A\) in \(K\); \(W\) permutes the roots. An ordering of the roots is canonically determined by the choice of \(N\). For \(x \in G\), write \(x = \text{kan}\) with \(k \in K\), \(a \in A\), and \(n \in N\), and let \(H(x) = \log (a) \in a\). Then the functions \(\phi\) on \(G \times K\) of the form

\[
\phi(x, k) = \lambda(H(xk)),
\]

where \(\lambda \in a^*, x \in G\), and \(k \in K\), are called \(A\)-cocycles (on \(K\)). (See [3, §6] — the present definition is not quite the same as Furstenberg's but is essentially equivalent). Every such function is associated with an elementary spherical function \(\psi\) on \(G\) defined by

\[
\psi(x) = \int_K \exp \{ \phi(x, k) \} \, dk.
\]

(Here \(dk\) denotes normalized Haar measure on the compact group \(K\)). In fact, all elementary spherical functions arise this way if we allow \(\lambda\) to be chosen from \(a^*_K\). For further discussion of spherical functions (about which there is an exten-

* Partially supported by the National Science Foundation.
Now suppose \( \mu \) is a probability measure on \( G \). If \( f \) is a function on \( G \), \( x \in G \), and \( X_1, X_2, \ldots \) are independent \( G \)-valued random variables each with distribution \( \mu \), it is of interest to determine the asymptotic behavior of 
\[
 f(X_1 \cdots X_n x) \quad \text{as} \quad n \to \infty.
\]
This, of course, is the subject of "laws of large numbers" and "central limit theorems" for the left \( \mu \)-walk on \( G \). (One could analogously consider \( f(xX_1 \cdots X_n) \) and the right \( \mu \)-walk on \( G \).

The function \( f \) is at our disposal here, and it is convenient to have \( f \) related to the group-theoretic structure at hand. Furstenberg showed in [3, 57] that if \( f \) is of the form \( \phi(\cdot, k) \) with \( \phi \) an \( A \)-cocycle as above and \( k \in K \), then 
\[
 n^{-1} f(X_1 \cdots X_n x) \quad \text{converges with probability} \quad 1 \quad \text{to a limit} \quad a_{\mu}(\phi) \quad \text{independent of} \quad x \quad \text{and} \quad k ,
\]
provided that \( \mu \) is regular enough. (Furstenberg takes \( \mu \) of class \( B_\preceq \)—absolutely continuous with a first moment—but it is clear from [1] that absolute continuity of \( \mu \) can be weakened to the condition that \( \mu \) be "étale". What is remarkable is that for certain choices of \( \phi \), this limit is strictly positive (or strictly negative) for all choices of \( \mu \). This was first shown by Furstenberg [3, Theorem 7.6] under the additional restriction that \( \mu \) be of class \( B_\preceq \) (determined by a bounded \( L^1 \) function of compact support) and then extended by Raugei [7, Cor. (7.15)] to the conclusion that \( a_{\mu}(\phi) > 0 \) for all \( \mu \) with a first moment, whenever the \( \lambda \in \hat{G} \) that determines \( \phi \) is a positive root. (The apparent change of sign from Raugei's statement is due to the fact that Raugei puts \( K \) on the right in the Iwasawa decomposition, whereas we are following Furstenberg and putting \( K \) on the left). The purpose of this section is to point out that, with \( \mu \) restricted to being of class \( B_\preceq \), one can deduce this fact directly from [3, Theorem 7.4] and elementary facts about semisimple groups. This provides another proof of the conjecture on p. 417 of [3].

1.2. Proposition. - With notation as above, if \( \lambda \in \mathfrak{a}^* \) is a positive root and \( \phi \) is the \( A \)-cocycle defined by (1), then \( a_{\mu}(\phi) > 0 \) for all probability measures \( \mu \) of class \( B_\preceq \).

Proof. - By [3, Theorem 7.4], we will have \( a_{\mu}(\phi) < 0 \) whenever the elementary spherical function \( \Psi \) defined by (2) satisfies \( \| \Psi \|_{L^\infty(G)} \leq 1 \). By the easy direction of the Helgason-Johnson Theorem [4, p. 66], this will be the case if \( \lambda = i\nu - \rho \) with \( \rho \) half the sum of the positive roots and \( \nu \in \mathfrak{a}^* + i\mathfrak{c}^\rho \), where \( \mathfrak{c}^\rho \) is the (closed) convex hull of the orbit of \( \rho \) under the Weyl group \( W \). This is equivalent (since \( \lambda \) is real-valued) to having \( \lambda \in -(\rho + \mathfrak{c}^\rho) \). But if \( \sigma_0 \in W \) is the element taking every positive root to a negative root, if \( \alpha \) is a
simple root ("with multiplicity"), and if \( s_\alpha \in W \) is the reflection in the hyperplane normal to \( \alpha \), then \( s_\alpha \rho^\# = s_\alpha (-\rho) = \alpha - \rho \), and thus \( \alpha + \rho + (\alpha - \rho) \in \rho + C_\rho \). This shows that each \( A \)-cocycle associated to the negative of a simple root is \( \mu \)-negative for all \( \mu \) of class \( B_\mu \). Since the \( \mu \)-positive and \( \mu \)-negative cocycles constitute convex cones which are negatives of one another, and since each positive root is a sum of simple roots (possibly with repetitions), the conclusion of the proposition follows.

1.3. Remark. - It is interesting to observe that the convex cone generated by \( \rho + C_\rho \) is exactly the same as that generated by the positive roots (since for \( s \in W \), \( \rho + s \rho = \rho + (ss_\rho) = \rho - (ss_\rho) \rho \) is a sum of positive roots by dominance of \( \rho \)). Furthermore, the Helgason-Johnson Theorem also shows that \( \| \|_\rho \leq 1 \) only when \( \lambda \in \Delta_\rho - (\rho + C_\rho) \). Thus the present method produces no more cocycles positive for all \( \mu \) than does RAUGI's. If \( G \) is of real rank 1, every non zero \( A \)-cocycle is either \( \mu \)-positive for all \( \mu \) or \( \mu \)-negative for all \( \mu \), since every element of \( \Delta_\rho \) is a scalar multiple of a root. But when the real rank of \( G \) is \( > 1 \), it is possible for a \( \lambda \in \Delta_\rho \) to be neither in the cone \( C_\rho \) generated by the positive roots nor in the cone \( C^- \) generated by the negative roots. It seems likely that in this case the corresponding \( A \)-cocycle \( \phi \) would be \( \mu \)-positive for certain \( \mu \) and \( \mu \)-negative for certain other \( \mu \).

When \( \mu \) is the class \( B_\mu \) probability measure defined by a \( K \)-biinvariant positive continuous function \( g \) of total mass 1, it is easy to check that

\[
\phi_\mu (\phi) = \int_G g(x) \int_K \varphi(x, k) dk \ dx
\]

where \( \Delta^+ = \{ \lambda \in \Delta \mid \lambda(X) > 0 \} \) for all positive roots \( \lambda \}, \ \Delta \) is a certain positive continuous function on \( \Delta^+ \), \( \Delta \) is Lebesgue measure, and \( T: \Delta \rightarrow \Delta \) is given by

\[
T(X) = \int_K H([\exp X] k) dk.
\]

Since \( g|_{\Delta^+} \) can have support in an arbitrarily small neighborhood of a given point \( \lambda \in \Delta^+ \), Proposition 1.2 implies that \( \lambda(T(X)) > 0 \) for all positive roots \( \lambda \).

In other words, we have the following (purely group-theoretic) result:

1.4. Corollary. - With \( T \) defined as in (3), \( T(\Delta^+) \subseteq \Delta^+ \). If one could show that in fact \( T(\Delta^+) = \Delta^+ \), then for \( \lambda \notin (\Delta^+ \cup \Delta^-) \), \( \lambda \circ T \) would change sign in \( \Delta^+ \), and one could choose \( g \) so that \( \phi_\mu (\phi) \) was either positive or negative.
2. SEMISIMPLE GROUPS WITH INFINITE CENTER

2.1. The purpose of this section is to examine random walks on semisimple Lie groups with infinite center, and in particular to extend some of the results of [8]. Our principal result, Theorem 2.10, is exactly what is needed to make it possible to drop the condition on the center of $G/R$ in the proof of theorem (8.4) of [7]. Although semisimple groups with infinite center may not often arise in practice, it is nice to have a unified treatment of random walks and harmonic functions valid for all connected Lie groups (without side conditions on the center).

Let $G$ be a connected semisimple Lie group with center $Z$, let $\mu$ be an étale probability measure on $G$, and let $\pi_\mu$ be the Poisson space of $\mu$. Then if $Z$ is infinite, it may happen [5] that $G$ is not transitive on $\pi_\mu$. More precisely, let $T_\mu$ be the closed semigroup generated by the support of $\mu$ and let $S_\mu$ be the open semigroup of $[1$, Définition IV.2]. Then by $[1$, Proposition IV.5], $G$ is transitive on $\pi_\mu$ if and only if $[Z : Z \cap S_\mu^{-1}] < \infty$. Since $\pi_\mu$ depends only on $T_\mu$ by $[1$, Théorème II.4], one expects to be able to phrase this condition in terms of $T_\mu$, and in fact, $Z \cap S_\mu^{-1} = Z \cap T_\mu^{-1}$.

2.2. Lemma. - Let $G$ be a group with center $Z$, and let $S$ and $T$ be non-empty sub-semigroups of $G$ satisfying $S \subset T$ and $T \subset S$. Then $Z \cap SS^{-1} = Z \cap TT^{-1}$.

Proof. - Since $S \subset T$, $Z \cap SS^{-1} \subset Z \cap TT^{-1}$. But if $z \in Z \cap TT^{-1}$, we can write $z = t_1 t_2^{-1}$ with $t_1, t_2 \in T$. Let $s \in S$. Then $z = (t_1 s)^{-1} z (t_1 s)$ (since $z$ is central) $= s^{-1} t_1^{-1} t_2^{-1} t_1 s = (t_2)^{-1} (t_1 s) \in (T)^{-1} (TS)^{-1} (TS) \subset S^{-1} S$, so $Z \cap TT^{-1} \subset Z \cap S^{-1} S = Z \cap SS^{-1}$.

Corollary. - In the setting of 2.1 above, $Z \cap SS^{-1} = Z \cap TTT^{-1}$.

Proof. - By $[1$, p. 76], we can take $S = S_\mu$ and $T = T_\mu$ in the lemma.

2.3. Let $KAN$ be an Iwasawa decomposition of $G$ and let $M$ be the centralizer of $A$ in $K$. We begin by analyzing the asymptotic behavior of the left $\mu$-walk on $G/AN \supseteq K$ (equivalently, we could deal with the right $\mu$-walk on $AN \setminus G$) in the case where $G$ is transitive on $\pi_\mu$. The results are basically as in the finite center case, except for the fact that since $K$ is no longer assumed compact, it is necessary to consider infinite measures.
2.4. Lemma. - Let $G = K \mathcal{A} N$ and $M$ be as in 2.3, and let $M_1$ be an open subgroup of finite index in $M$. Then $M AN$ leaves fixed on $G/AN \cong K$ exactly one Radon measure $m$ (up to scalar multiples), namely, the image in $K$ of a Haar measure on $M$. $M_1 AN$ leaves fixed on $G/AN \cong K$ exactly (up to scalar multiples) the convex combinations of the restrictions of this measure to the right $M_1$-cosets in $M$. In particular, the cone of $M_1 AN$-invariant (positive) Radon measures on $G/AN$ is generated by its extremal rays, and there are exactly $[M : M_1]$ of these.

Proof. - $G/ZAN = K/Z$ is compact, so every Radon measure on it is a multiple of a probability measure. By [3, Theorem 2.6], there is exactly one probability measure $\lambda$ on $G/ZAN$ which is $MAN/Z$-invariant, and $\lambda$ is obviously normalized Haar measure on the image of $M/Z$.

Let $m$ be an $MAN$-invariant Radon measure on $G/AN \cong K$. Since $Z \subseteq M$, $m$ is $Z$-invariant and projects to an $MAN/Z$-invariant measure on $K/M$. Hence $m$ is of the form

$$m = \int_K f \, dm = \int_K \int_Z f(zk) \, dz \, d\lambda(k)$$

for some Haar measure $dz$ on $Z$, and this proves the first assertion.

Now if $\gamma \in M$, $(M_1 AN)(\gamma AN) = AN(M_1 \gamma AN)$ (since $M_1$ normalizes $AN$), so the restrictions of $m$ to right $M_1$-cosets in $M$ (or rather, their images in $G/AN$) are $M_1 AN$-invariant. Conversely, suppose $\sigma$ is an $M_1 AN$-invariant measure on $G/AN$. Let $r = [M : M_1]$ and choose representatives $\gamma_1, \ldots, \gamma_r$ for $M/M_1$ in $M$. Then $\Sigma_{\gamma_1} \sigma$ is $MAN$-invariant, hence is a multiple of $m$, and is therefore supported on $MAN$. It follows that $\sigma$ is supported on $MAN$, or on $M$ if we identify $G/AN$ with $K$. But an $M_1$-invariant measure on $M$ is a sum of translates of Haar measure on $M_1$. This proves the lemma.

2.5. Lemma. - Let $G = K \mathcal{A} N$ and $M$ be as in 2.3. Let $\mu$ be an étale probability measure on $G$ such that $G$ is transitive on $\pi_\mu$. We take $\pi_\mu = G/\mathcal{A}N$ with $M_\mu$ an open subgroup of $M$; since $\pi_\mu$ is compact, $[M : M_\mu] < \infty$. Let $\nu$ denote the Poisson kernel for $\mu$ on $\pi_\mu$, and let $X$ be a locally compact (left) $G$-space. Then if $\lambda$ is an $M_\mu AN$-invariant Radon measure on $X$, we can define a $\mu$-stationary measure $\nu \star \lambda$ on $X$ by

$$\int_X f \, d(\nu \star \lambda) = \int_X \int_{\pi_\mu} f(xy) \, d\nu(x) \, d\lambda(y)$$

(the integral converges since $\nu$ has compact support). Conversely, every $\mu$-stationary Radon measure on $X$ is of the form $\nu \star \lambda$ with $\lambda$ as above.

Proof. - This follows from trivial modifications in the proof of [3, Lemma 2.1].
2.6. Theorem. - Assume the hypotheses of 2.5. Then $G/AN \simeq K$ admits (up to scalar multiples) exactly $[M : H]$ ergodic $\mu$-stationary Radon measures, and every $\mu$-stationary measure on $K$ is (up to scalar multiples) a convex combination of these. With probability 1, any element of $K$ enters the support of one of the ergodic measures after finitely many steps of the left $\mu$-walk on $G/AN$, and then stays there thereafter.

Proof. - Note that ergodic $\mu$-stationary measures on $K$, considered up to scalar multiples, are the same as extremal rays in the cone of $\mu$-stationary measures on $K$. By Lemmas 2.4 and 2.5, this cone is generated by its extremal rays, and there are exactly $[M : H]$ of these. Also, we see that the ergodic measures have disjoint supports whose union $E$ is the inverse image in $K$ (under the natural map $p: K \to K/M$) of the support of $\nu$. Hence, if $k \in K$ and $X_1, \ldots, X_n, \ldots$ are independent $G$-valued random variables with distribution $\mu$, $X_1 \ldots X_n$ eventually lands in $E$ if and only if $X_1 \ldots X_n p_1(k)AN$ eventually lands in $p_1(\text{supp } \nu)$, where $p_1$ denotes projection onto $K/M$. But we know that this last assertion is true either by the ergodic theorem (cf. [3, p. 397]) or else by the Doeblin condition ([6, Lemma 1]) applied to the unique stationary $\mu$-process on the maximal boundary $G/MAN \simeq K/M$ of $G$.

Now we go on to the general case (in which $G$ is not necessarily transitive on $\pi_\mu$).

2.7. Lemma. - Let $G$ be a connected semisimple Lie group with (not necessarily finite) center $Z$, and let $S$ be a non-empty open sub-semigroup of $G$. Then one can choose an Iwasawa decomposition $KAN$ of $G$ so that $S$ intersects $ZA$ and so that the projection of $S \cap ZA$ onto $A$ contains a set of the form $C \cap B^0$, where $C$ is an open subcone of the (open) positive Weyl chamber in $A$ and $B^0$ is the complement of some relatively compact neighborhood $B$ of the identity element $e$.

Proof. - Since the set of regular elements in $G$ is dense, and since $S$ is non-empty and open, $S$ meets some Cartan subgroup $H$ of $G$. Choose a Cartan decomposition $\mathfrak{k} + \mathfrak{g}$ of the Lie algebra of $G$ so that $H = H_{\mathfrak{k}}. H_{\mathfrak{g}}$, where $H_{\mathfrak{k}}$ lies in the connected subgroup $K$ of $G$ with Lie algebra $\mathfrak{k}$, and where $H_{\mathfrak{g}}$ is a vector group contained in $\exp (\mathfrak{g})$. We have $Z \subseteq H_{\mathfrak{k}}$, and $H_{\mathfrak{k}}/Z$ is compact. (For all this, see [9, §1.4]). Hence the projection onto $H_{\mathfrak{k}}/Z$ of the image in $H/Z$ of $S \cap H$ is a non-empty sub-semigroup of a compact group, and is therefore [10] a group. In particular, the identity element of $H_{\mathfrak{k}}/Z$ lies in the image of $S \cap H$, which means that $S$ meets $ZH$. Now we can choose an Iwasawa
decomposition KAN of G with K as above and with H, C A , so S meets ZA . Since S ∩ ZA is open, it must contain an element of the form za with z ∈ Z and with a regular (that is, with a contained in one of the open Weyl chambers of A), and assuming a suitable ordering of the roots (and thus a suitable choice of N), a will be in the positive Weyl chamber. The last statement now follows from the structure of open sub-semigroups of vector groups.

2.8. Lemma. - Let G, Z, and S be as in the hypotheses of 2.7, and let K, A, and N be chosen as in that lemma. Let M be the centralizer of A in K, and let Z' be a subgroup of Z such that Z' ∩ S⁻¹ S = {e}. Let p : G → G/Z' be the canonical map. Then for any m ∈ M, p is injective on SmAN.

Proof. - Assume this is false. Then for some m ∈ M, z ≠ e, in Z', a₁ and a₂ in A, n₁ and n₂ in N, and s₁ and s₂ in S, we have

\[ z s m a n₁ = s m a n₂ \] and \[ s⁻¹ s₂ = (z m a n₁) (m a n₂)⁻¹ = z m a n₁ n₂⁻¹ a₂⁻¹ m⁻¹ \], and zan ∈ S⁻¹ S for suitable a ∈ A, n ∈ N (recall M normalizes AN). Let C and B₀ be as in Lemma 2.7, and choose h₁ ∈ C ∩ B₀. Also choose z ∈ Z with z₁ h₁ ∈ S (this is possible by the conclusion of 2.7). Then for any integer q, we have \[ z₁⁻¹ h₁⁻¹ q ∈ S⁻¹ \] and \[ z₁ h₁⁻¹ q ∈ S \], so that \[ (z₁⁻¹ h₁⁻¹ q) (z₁ h₁⁻¹ q) = z h₁⁻¹ h₁ q ∈ S⁻¹ S \].

(Note that z₁, z₁⁻¹, and a commute with h₁). But as q → ∞, h₁⁻¹ h₁ q → e (cf. [1: Lemme III.1]), so za ∈ S⁻¹ S. One can now choose h₂ ∈ C ∩ B₀ so that h₂ = ah₂ ∈ C ∩ B₀, and if z₂, z₃ ∈ Z are such that z₃ h₂, z₃ h₂ ∈ S, we get \[ (z₃ h₃⁻¹) (za) (z₃⁻¹ z₂ z) = z₃⁻¹ z₂ z ∈ S⁻¹ S \]. In fact, it is easy to see that we can take z₂ = z₃ if h₂ is chosen far enough away from the identity element of A, so z ∈ S⁻¹ S. This actually implies that z ∈ S⁻¹ S, for we can choose a neighborhood U of e in G and an element s ∈ S with Us ⊆ S (since S is open) and with \[ z u ∈ S⁻¹ S \] for some \[ u ∈ U⁻¹ \], and then \[ z ∈ z(s⁻¹ U⁻¹) = (Us)⁻¹ (Us) ⊆ S⁻¹ S \]. But this is a contradiction, since we assumed z ≠ e and \[ Z' ∩ S⁻¹ S = {e} \].

2.9. Theorem. - Let G be a connected semisimple Lie group with possibly infinite center Z, and let µ be an étale probability measure on G. Then we can choose an Iwasawa decomposition KAN of G such that with respect to the left µ-walk on G/AN × K, K decomposes into a transient set and a countable disjoint union of ergodic invariant sets. With probability 1, any element of K enters one of the ergodic sets after finitely many steps of the left µ-walk on G/AN.
Proof. - If \([Z : Z \cap S^{-1}_u S^u] < \infty\), Theorem 2.6 applies. If not, it follows from the structure theorem for finitely generated abelian groups (as applied to \(Z\)) that we can choose a subgroup \(Z'\) of \(Z\) with \(Z' \cap S^{-1}_u S^u = \{e\}\) and \([Z : Z'] < \infty\). Apply 2.7 with \(S = S^u\) to get an Iwasawa decomposition of \(G\), and let \(M\) be the centralizer of \(A\) in \(K\). The set \(S^u MAN\) is open and invariant for the left \(\mu\)-walk on \(G/AN = K\), and with probability 1, any element of \(K\) enters \(S^u MAN\) after finitely many steps of the \(\mu\)-walk on \(G/AN\), because of the ergodicity of the \(\mu\)-walk on \(G/AN\) (the maximal boundary of \(G\)). (The \(\mu\)-walk on \(G/AN\) may be viewed as a walk for the image of \(\mu\) in \(G/Z\) on the same space, so [8, p.671] and [3] apply.) Hence it is enough to decompose \(S^u MAN\) into a transient set and countably many ergodic sets. Let \(m_1, m_2, \ldots\) be a sequence dense in \(M\). Then \(S^u MAN\) is the (not necessarily disjoint) union of the invariant sets \(S^u m_i AN\). Let \(\mu : G \rightarrow G/Z'\) be the canonical map. Then \(\mu\) is clearly equivariant for the \(\mu\)-walk on \(G/AN\) and the \(\mu(\mu)\)-walk on \(G/Z'AN\), and by 2.8, \(\mu\) is injective on each \(S^u m_i AN\). But by construction of \(Z'\), the \(\mu(\mu)\)-walk on \(G/Z'AN\) satisfies the hypotheses of Theorem 2.6. Therefore each \(S^u m_i AN\) decomposes into a transient set and finitely many ergodic sets. Since ergodic sets (up to sets of measure zero) either coincide or are disjoint, we conclude that \(S^u MAN\) can be decomposed into a transient set (the union of the transient parts of the \(S^u m_i AN\)) and countably many ergodic sets. This proves the theorem.

2.10. Theorem. - Let \(G\) be a connected semisimple Lie group with Iwasawa decomposition \(KAN\) and with possibly infinite center \(Z\), and let \(\mu\) be an étale probability measure on \(G\). Then with respect to the left \(\mu\)-walk on \(G/AN = K\), \(K\) decomposes into a transient set and a countable disjoint union of ergodic invariant sets. With probability 1, any element of \(K\) enters one of the ergodic sets after finitely many steps of the left \(\mu\)-walk on \(G/AN\).

Proof. - (-kindly suggested by A. RAUGI). The only difference between this theorem and 2.9 is that here our Iwasawa decomposition is specified in advance. Let \(G = K^1 A^1 N_1\) be a decomposition as provided by 2.9. Then for some \(k \in K\), \(A_1 N_1 = k AN_1\). Let \(G/A_1 N_1 = T \cup E_{1k}\) be a decomposition of \(G/A_1 N_1\) into a transient set \(T\) and countably many ergodic sets \(E_{1k}\). Then \(Tk\) and the \(E_{1k}\) are evidently invariant for the left \(\mu\)-walk and give a partition of \(G/AN\); they are also clearly transient and ergodic, respectively. Finally, if \(x \in K\), and if \(X_1, X_2, \ldots\) are independent \(G\)-valued random variables each with distribution \(\mu\), then by 2.9 we know that with probability 1, \(X_n \ldots X_1 (xk^{-1}) A_1 N_1 \in E_{1k}\) for some \(i\) and some \(n\). This says that \(X_n \ldots X_1 xAN \in E_{1k}\), so that \(x\) enters one
of the ergodic sets for the $\mu$-walk on $G\backslash\mathbb{AN}$ after finitely many steps.

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