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TORSION OF THE WITT GROUP

by

M. KAROUBI

The purpose of this paper is to give an elementary proof of the following theorem (well known if A is a field).

Theorem. Let A be a commutative ring and let $\Gamma(A)$ be the subring (= subgroup) of the Witt ring $W(A)$ generated by the classes of projective modules of rank one. Then the torsion of $\Gamma(A)$ is 2-primary.

Proof. Let $L(A)$ be the Grothendieck group of the category of non degenerated bilinear A-modules. Let $x = [L_1 \oplus \ldots \oplus L_n] \in L(A)$ where L_i are projective of rank one and let us assume that the class of px in $W(A)$ is zero, p being an odd prime. We want to show that the class of x in $W(A)$ is equal to 0. We need the following lemma :

Lemma. Let Γ_o be the subring of $W(A)$ generated by the $\langle L_i \rangle$ and let $y \in \Gamma_o$ such that $py = 0$ in $W(A)$ with p an odd prime. Then $y \in p\,\Gamma_o$.

Proof of the lemma. Let $y = \langle R_1 \oplus \ldots \oplus R_m \rangle \in \Gamma_o$ where the R_i are projectives of rank one and monomial of the L_i and $\bar{L}_i$. Let $\bar{\Gamma}_o$ be the subring of $L(A)$ generated by the L_i and $\bar{L}_i$ and $\bar{y}$ be the class of $R = R_1 \oplus \ldots \oplus R_m$ in $L(A)$. Following Grothendieck we write

$$\lambda_t(\bar{y}) = 1 + t\,\lambda^1(\bar{v}) + \ldots + t^m \lambda^m(\bar{y}) \in L(A)\,[t] \quad \text{(note that } \lambda^i(\bar{y}) \in \bar{\Gamma}_o).$$

Since $\lambda_t(u + v) = \lambda_t(u)\,\lambda_t(v)$ according to the general properties of the exterior powers, we have

$$\lambda_t(p\bar{y}) = (\lambda_t(\bar{y}))^p = 1 + t^p \lambda^1(\bar{y})^p + \ldots + t^{mp} \lambda^m(\bar{y})^p \mod. p\,\bar{\Gamma}_o .$$

Moreover,

$$\lambda^1(\bar{y})^p = [R_1 \oplus \ldots \oplus R_m]^p = [R_1^p \oplus \ldots \oplus R_m^p] = [R_1 \oplus \ldots \oplus R_m] =$$
$$= \bar{y} \mod. p\,\bar{\Gamma}_o ,$$

because $[R_i]^2 = 1$. It follows from this computation that $\lambda^p(p\bar{y}) = \bar{y} \mod. p\,\bar{\Gamma}_o$. Since $p\bar{y}$ is stably metabolic and since p is odd, $\lambda^p(p\bar{y})$ is stably metabolic. Hence $y = 0 \mod. p\,\Gamma_o$.

Proof of the theorem (followed). Since $[L_i]^2 = 1$, Γ_o is a finitely generated $\mathbb{Z}$-module. From the lemma it follows that the p-torsion of is zero if p is odd. Hence the torsion of Γ_o is 2-primary which implies $x = 0$ as required.

Part of these considerations can be generalized for rings with involution. Of course we have not necessarily $[L]^2 = 1$ if L is projective of rank one (except if A is local). However, we can consider the subring $\Gamma^q(A)$ of $W(A)$ generated by the classes of projective modules of rank one such that $[L]^q = 1$ (see the example below). Then I claim that the torsion of $\Gamma^q(A)$ is 2q-primary (i.e. $px = 0$ implies $x = 0$ if p is prime to $2q$). The proof is along the same lines as the proof of the first theorem. If we write $x = \langle L_1 \oplus \dots \oplus L_n \rangle$ we can consider the subring Γ_o^q of $\Gamma^q(A)$ generated by the L_i and the subring $\overline{\Gamma}_o^q$ of $L(A)$ generated by the L_i and $\overline{L}_i$. Let α be an integer such that $p^\alpha - 1$ is divisible by q (for instance the Euler indicator). Then, with the notations of the lemma we have $\lambda_t(p^\alpha \overline{y}) = 1 + t^{p^\alpha} \lambda^1(\overline{y})^{p^\alpha} + \dots$ mod. $p\,\overline{\Gamma}_o^q$. Hence

$$\lambda^{p^\alpha}(p^\alpha \overline{y}) = \lambda^1(\overline{y})^{p^\alpha} = [R_1^{p^\alpha} \oplus \dots \oplus R_m^{p^\alpha}] = [R_1 \oplus \dots \oplus R_m] = \overline{y} \mod.\ p\,\overline{\Gamma}$$

(because $R_i^q = 1$). Therefore the p-torsion of Γ_o^q is p-divisible which implies $x = 0$.

<u>Example</u>. Let A be the ring of complexe continuous functions on the lens space $X = S^{2n+1}/\mathbb{Z}_q$ where S^{2n+1} is the $2n+1$-dimensional sphere imbedded in $\mathbb{C}^{n+1}$, $\mathbb{Z}_q$ acting by the action of q^{th} roots of the unity. If we provide A with the complex conjugation involution, the Witt ring $W(A)$ can be identified with the complex K-theory $K_C(X)$ of the space X (this is true for any compact space X). This complex K-theory is generated by the trivial bundles and by the line bundle $L = S^{2n-1} \times_{\mathbb{Z}_1} C$. If we put $t = \langle L \rangle$ we have in fact $W(A) = \mathbb{Z}[t]/I$ where I is the ideal generated by the polynomials $t^q - 1$ and $(t-1)^n$. Hence

$$W(A) = \Gamma^q(A) = \mathbb{Z} \oplus T$$

where T is a torsion group which is q-primary.

<u>Remark</u>. If we consider the ring B of real continuous functions on X, it is not hard to show that $W(B) \otimes \mathbb{Z}[\frac{1}{2}]$ is isomorphic to the invariant part of $W(A) \otimes \mathbb{Z}[\frac{1}{2}]$ by the action of $\mathbb{Z}_2$ acting by $t \to t^{-1} = t^{q-1}$. Hence $W(B)$ can have arbitrary torsion (not just 2-torsion).

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