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The Milnor ring of a local ring

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Let $F$ be a field. Milnor defined a ring $k^*_e(F)$, and in the case that characteristic $(F) \neq 2$ he studied maps between $k^*_e(F)$, and groups or rings which play a role in the theory of quadratic forms. The aim of this talk is to extend some of his definitions and results to local rings. We do not suppose that $2$ is a unit of the local ring. The only restriction for the local rings is, that the residue field has more than $3$ elements.

Sections 1, 2, 3 give a survey of [3], though the definitions of [3] are a bit generalized. In section 4, the analogue of Milnor's map $s^*_e$ is given, and section 5 covers the example of a field of characteristic $2$.

1. We repeat some of the definitions given by Milnor [6]. Let $F$ be a field, denote $U(F) = \{x \in F | x \text{ is invertible}\}$. Let $M$ be the $Z$-module $U(F)$, and denote $\mathcal{T}(M)$ for the tensor algebra of $M$. We write $\mathcal{L} : M \to \mathcal{T}(M)$ for the imbedding of $M$ in $\mathcal{T}(M)$. $k^*_e(F)$ is defined as $\mathcal{T}(M) \mod I$, and $I$ is the two-sided ideal of $M$, generated by $\{\mathcal{L}(a) \mathcal{L}(1-a) | a, 1-a \in U(F)\}$. Remark that $\langle -a,1 \rangle \otimes \langle -(1-a),1 \rangle \cong \mathbb{R}$, as soon as $a,1-a \in U(F)$ and $2 \neq 0 \in F$. $K^*_e(F) = \mathbb{Z} \oplus K^1(F) \oplus K^2(F) \oplus ...$, and here $K^1_F = \mathcal{L}(M) \otimes ... \otimes \mathcal{L}(M) \mod \mathcal{L}(M) \otimes ... \otimes \mathcal{L}(M) \cap I$.

The elements of $K^n_F$ are again denoted as sums of terms $\mathcal{L}(a_1) ... \mathcal{L}(a_n)$. Finally, $K^*_e(F)$ is defined as $\mathbb{Z} \oplus K^1(F) \oplus 2K^1(F) \oplus K^2(F) \oplus 2K^2(F) \oplus ...$.

We remark that for $a \in U(A)$ and $x \in k^*_e(F)$, the element $\mathcal{L}(a^2)x = 2\mathcal{L}(a)x = 0 \in k^*_e(F)$. In fact, the defining relations for $k^*_e(F)$ are:

\[
\begin{align*}
\mathcal{L}(ab) &= \mathcal{L}(a) + \mathcal{L}(b) & a \in U(A), b \in U(A) \\
\mathcal{L}(a) \mathcal{L}(1-a) &= 0 & a,1-a \in U(A) \\
2\mathcal{L}(a) &= 0 & a \in U(A)
\end{align*}
\]

Suppose now that $\text{char}(F) \neq 2$. We write $\text{Quad}(F)$ for the Grothendieck monoid of finite-dimensional quadratic spaces over $F$. Milnor proved, that there exists a well-defined map

\[
SW : \text{Quad}(F) \to k^*_e(F) \text{ such that } SW \langle a_1, ..., a_n \rangle = (1 + \mathcal{L}(a_1)) ... (1 + \mathcal{L}(a_n))
\]

(1) Write $\mathcal{L}(a)$ for the class of $\mathcal{L}(a)$ in $K^1(F)$, etc.
We denote the Grothendieck-Witt ring of finite-dimensional quadratic spaces over $F$ by $W(F)$, and we write $I(F) \subset W(F)$ for the kernel of the dimension map $W(F) \rightarrow \mathbb{Z}/2\mathbb{Z}$. Milnor proved also, that there exists a homomorphism

$$s_n : k_n(F) \rightarrow \oplus I^n/F / I^{n+1}(F)$$

such that

$$s_n : K_n(F) \rightarrow \mathbb{Z}/2 \mathbb{Z}$$

and

$$s_n : I(a_1) \cdots I(a_n) = (\langle a_1 \rangle - 1) \cdots (\langle a_n \rangle - 1) + I^{n+1}(F).$$

2. Let $A$ be a local ring with maximal ideal $m$. Denote $U(A) = \{ a \in A \mid a \text{ has inverse in } A \}$. If $2 \in m$, then every nondegenerate quadratic form on $A$ of finite dimension has even dimension.

We denote $(a,b,c)$ for the form $q$ which has a basis $e,f$ satisfying $q(e) = a$, $q(f) = b$, $(e,f) = c$. The form $(a,b,c)$ is nondegenerate if and only if $4ab - c^2 \in U(A)$. If $|A \mod m| > 3$ then we may choose $a,b,1$ such that $a,b \in U(A)$. In that case $(a,b,1) \cong a(1,ab,1)$ and $ab$ determines an invariant of $(a,b,1)$ which we will describe now.

The following notions can be found in the notes of the 1968 Montpellier conference, Micali, Villamayor [4]. Let $A$ be an arbitrary ring, define $A^0 = \{ a \in A \mid 1-4a \in U(A) \}$. $A^0$ is a group under $\circ : A^0 \times A^0 \rightarrow A^0$, $a \circ b = a+b-4ab$.

The inverse of $a$ in $A^0$ is the element $\frac{1-4a}{2}$. Define $J(A) = \{x = x^2 \mid 1-2x \in U(A)\}$. If $a \in A^0$, then $a \circ a \in J(A)$. $J(A)$ is a subgroup of $A^0$, we denote $G(A) = A^0 \mod J(A)$. There exist homomorphisms $\sigma : A^0 \rightarrow U(A)$, $\sigma(a) = 1-4a$,

$$\bar{\sigma} : G(A) \rightarrow U(A) \mod U(A)^2, \bar{\sigma}(a \circ J) = (1-4a) \mod U(A)^2.$$

Examples. (1) If $2 \in U(A)$ then $\bar{\sigma}$ is an isomorphism.

(2) If $2 = 0$ then $A = A^0$ and $a \circ b = a+b$, $G(A) = A^0 \mod U(A)$.

Let $A$ be a local ring. The quadratic form $a(1,d,1)$ is nondegenerate if and only if $a \in U(A)$, $d \in A^0$. The class $d \circ J(A)$ is an invariant for the isometry class of $a(1,d,1)$, for the proof see [3].

In general, we have the following result: Suppose that $q$ is a nondegenerate quadratic form of dimension $2n$. Then

$$q \cong \oplus a_i(1,d_i,1)$$

$a_i \in U(A)$, $d_i \in A^0$, $1 \leq i \leq n$ and $d_1 \circ d_2 \circ \cdots \circ d_n \circ J(A)$ is an invariant for the isometry class of $q$.

Examples. (1) $F$ is a field of characteristic $\neq 2$, $q \cong \oplus a_i(1,d_i,1)$, then

$$\bar{\sigma}(d_1 \circ \cdots \circ d_n \circ J(A))$$

is the discriminant of $q$.

(2) $F$ is a field of characteristic 2, $q$ as before. Then $d_1 \circ \cdots \circ d_n \circ J(A)$ is the Arf invariant of $q$. 


3. Suppose again that \( A \) is a local ring, \(|A \mod m| > 3\). It is clear, that for the determination of the isometry class of \( a(1,d,1) \) a role is played by \( d \circ J(A) \) and by \( a \in \text{U}(A) \). So in the definition of the Milnor ring of \( A \), \( G(A) \) and \( \text{U}(A) \) should play a role. In the case of a fields of char \( \neq 2 \) it seemed important to remark that

\[
<-a,1> \otimes <-1-a,1> \cong 2 \mathbb{H}, \quad a, 1-a \in \text{U}(A).
\]

We translate that remark:

if \( a \in \text{U}(A) \cap \text{A}^0 \) then \((-a,0) \otimes (1,a,1) \cong 2 \mathbb{H} \).

Here \((-a,0) \otimes (1,a,1) \cong 2 \mathbb{H} \) denotes a symmetric bilinear form, and we use the tensor product which is defined for symmetric bilinear forms and quadratic forms by \( H, \) Bass [1].

We now give a construction of the ring \( g_e(A) \), which is almost equivalent to the construction of \( k_e(A) \).

We start with the \( \mathbb{Z} \)-module \( M = \text{A}^0 \oplus \text{U}(A) \), and we denote \( \omega(a) = (a,0) \) for \( a \in \text{A}^0 \) and \( \gamma(a) = (0,a) \) for \( a \in \text{U}(A) \). \( T(M) \) is again the tensor algebra of \( M \). \( J \) is the two-sided ideal of \( T(M) \) generated by

\[
[\omega(a) \gamma(a) \mid a \in \text{A}^0 \cap \text{U}(A)] \cup \{ \gamma(a) \omega(a) \mid a \in \text{A}^0 \cap \text{U}(A) \} \cup \{ \omega(a) \mid a \in J(A) \}.
\]

\( g_e(A) = T(M) \mod J \), \( g_e(A) \) is isomorphic with \( \mathbb{Z} \oplus g_1(A) \oplus g_2(A) \oplus \ldots \),

\( g_1(A) = M \otimes \ldots \otimes M / M \otimes \ldots \otimes M \cap J \).

We denote \( \bar{g}(a) \) for the image of \( \gamma(a) \) \( (a \in \text{U}(A)) \) in \( g_e(A) \). We write \( \bar{0}(A) \) for the image of \( \omega(a) \) \( (a \in \text{A}^0) \) in \( g_e(A) \). In fact, \( g_e(A) \) satisfies the following defining relations:

\[
\bar{g}(ab) = \bar{g}(a) + \bar{g}(b), \quad a \in \text{U}(A), b \in \text{U}(A)
\]

\[
\bar{0}(a \circ b) = \bar{0}(a) + \bar{0}(b), \quad a, b \in \text{A}^0
\]

\[
\bar{0}(a) \bar{0}(a) = \bar{0}(a) \bar{g}(a) = 0, \quad a \in \text{A}^0 \cap \text{U}(A)
\]

\[
\bar{0}(a) = 0, \quad a \in J(A)
\]

We would like to define a map \( \text{SW} : \text{Quad}(A) \rightarrow g_e(A) \).

The analogue of Milnor's definition is for even dimensional forms:

(DEF): \( \text{SW}(a_1(1,d_1,1) \oplus a_2(1,d_2,1) \oplus \ldots \oplus a_n(1,d_n,1)) = \)

\[
(1 + \bar{g}(-1) + \bar{0}(d_1) + \bar{g}(a_1) \bar{0}(d_1)) \ldots (1 + \bar{g}(-1) + \bar{0}(d_n) + \bar{g}(a_n) \bar{0}(d_n))
\]

This definition works for \( n = 1 \): if \( a(1,d,1) \cong a_1(1,d_1,1) \) then \( d \circ J(A) = d_1 \circ J(A) \), so \( \bar{0}(d) = \bar{0}(d_1) \), and it can easily be proved that \( \bar{g}(a_1) \bar{0}(d_1) = \bar{g}(a_1) \bar{0}(d_1) \).
For the proof that the definition works for $n = 2$, we have to impose some extra conditions. Some of these come from the commutativity of $\text{Quad}(A)$. The more important conditions are:

\begin{align*}
W_2 : \bar{g}(1-4a) - \bar{g}(a) \bar{g}(b) & \text{ should be equal to 0, as soon as } a \in A^0, \\
& b \in U(A) \cap A^0. \\
W_7 : \bar{g}(a) \bar{g}(a) \bar{g}(b) \bar{g}(d) - \bar{g}(a) \bar{g}(b) \bar{g}(d) \bar{g}(d) & \text{ should be equal to 0 for } \\
& a \in U(A), b \in A^0 \cap U(A), d \in A^0.
\end{align*}

So we consider the ring $g_e(A) \mod Cg_e(A)$, $Cg_e(A)$ being the ideal in $g_e(A)$ generated by the elements mentioned in $W_2$, $W_7$ and by some more elements. For an explicit and precise definition see [3].

Let us denote $\bar{g}(a)$ for $g(a) + Cg_e(A)$, $\bar{0}(a)$ for $0(a) + Cg_e(A)$.

Suppose that $2 \in \mathbb{m}$. Then one can prove that the map $SW : \text{Quad}(A) \to g_e(A) \mod Cg_e(A)$ as proposed in (DEF), is well-defined.

Suppose $2 \notin \mathbb{m}$. If $A$ is a field, then $g_e(A)$ and $k_e(A)$ are not isomorphic. We should have identified $A^0$ and $U(A)$. More precisely, choose $M = U(A) \oplus A^0 \mod \{\gamma(1-4a) - \omega(a) \mid a \in A^0\}$ and repeat the definition of $T(M)$ mod $J$, hence the defining relations for $T(M)$ mod $J$ are

\begin{align*}
\bar{g}(1-4a) = \bar{0}(a), & \quad a \in A^0 \\
\bar{g}(ab) = \bar{g}(a) + \bar{g}(b), & \quad a, b \in U(A) \\
\bar{g}(a) \bar{0}(a) = \bar{0}(a) \bar{g}(a), & \quad a \in A^0 \cap U(A) \\
\bar{g}(a) = 0, & \quad a \in U(A)^2
\end{align*}

In fact, this was the definition, proposed in [3] for any local ring $A$ with 2 unit in $A$.

It is then easily proved that $Cg_e(A) = 0$, and that $SW$ is defined on all of $\text{Quad}(A)$, such that

\((*) : SW \langle a_1, \ldots, a_n \rangle = (1 + \bar{g}(a_1)) \ldots (1 + \bar{g}(a_n))
\)

For isometry classes of even dimension, the definitions (*) and (DEF) coincide.

There are situations in which we have that $2 \notin U(A)$ and that we want to restrict ourselves to isometry classes of even-dimensional forms. It is possible to define $g_e(A)$ based on $M = U(A) \oplus A^0$. The map $SW$ can be defined as proposed in (DEF). For proving this, the proofs in [3] can completely be repeated. The map $SW$, as proposed in (*) cannot be defined, since $2 \bar{g}(a) (a \in U(A))$ is not necessarily equal to 0.

4. We give now the analogue for the map $s_e$. For convenience, we work with rings $g_e(A)$, based on $M = U(A) \oplus A^0$. 
\[ W_q(A) \] is the Witt-group of free finite-dimensional nondegenerate quadratic forms on \( A \). \( W(A) \) is the Witt ring of free finite-dimensional nondegenerate symmetric bilinear forms on \( A \), \( I(A) \subset W(A) \) is the ideal of forms of even dimension. We denote the class of a form in \( W_q(A) \), \( W(A) \) by square brackets. \( W^0_q(A) \subset W_q(A) \) is the Witt-group of forms of even dimension.

It is well known that \( W_q(A) \) can be considered as an \( W(A) \)-module. According to definitions given by Micali + Villamayor [5], we give \( W_q(A) \) a structure of ring by defining :

\[ q_1 \cdot q_2 = ( , ) q_1 \otimes q_2 . \]

This definition induces a structure of ring on \( \bigoplus \n^0_q(A) W^0_q(A) \mod \bigoplus \n^1_q(A) W^0_q(A) \mod \n^1_q(A) W^0_q(A) \).

In analogy with Milnor's definition, we would like to define a homomorphism of rings

\[ s_1 : g_\eta(A) \mod g_\eta(A) \rightarrow \bigoplus \n^0_q(A) W^0_q(A) \mod \bigoplus \n^1_q(A) W^0_q(A) \mod \n^1_q(A) W^0_q(A) \]

For \( a \in A^0 \), we propose to define \( s_1 g(a) = [-1,-a,1] + I(A) W^0_q(A) \).

If \( a \in U(A) \) we would like to define

\[ s_1 g(a) = \begin{bmatrix} -1 & 0 \\ 0 & a \end{bmatrix} + I^2(A) . \]

The map \( s_1 \) can be extended to a homomorphism of rings, if the image of \( s_1 \) satisfies the defining relations of \( g_\eta(A) \mod g_\eta(A) \). It is clear that the following results hold:

4.1. \( \begin{bmatrix} -1 & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & b \end{bmatrix} \in \begin{bmatrix} -1 & 0 \\ 0 & ab \end{bmatrix} + I^2(A) , \quad a,b \in U(A) \)

4.2. \( [-1,-a,1] + [-1,-b,1] \in [-1,-a\otimes b,1] + I(A) W^0_q(A) , \quad a,b \in A^0 \)

4.3. \( [-1,0] \cdot [-1,-a,1] = 0 , \quad a \in U(A) \cap A^0 \).

For proving the other relations, we derive some formulas.

4.4. Suppose \( 2 \in m \) . Let \( 1-pq \in U(A) , d \in U(A) \cap A^0 \). Then

\[ \begin{bmatrix} p \\ 1 \end{bmatrix} \cdot [-1,-d,1] = \begin{bmatrix} d(pq-1) & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} pd \\ 1-pq \end{bmatrix} , \quad \frac{q}{1-pq} , 1] . \]

Proof. Let \( e,f \) be a basis of \( V \), let \( ( , ) \) be a symmetric bilinear form on \( V \) such that \( (e,e) = p, (f,f) = q, (e,f) = 1 \). Let \( x,y \) be a basis of \( W \), \( q : W \rightarrow A \) a quadratic form and \( q(x) = -1, q(y) = -d, (x,y) = 1 \).

The bilinear form and the quadratic form are nondegenerate. Choose \( X = e \otimes (2dx+y), Y = (-qe+f) \otimes x, S = f \otimes y, T = (-e+pf) \otimes (x+y) \). Since \( 2 \in m \), we have that \( X, Y, S, T \) is a basis of \( V \otimes W \). Moreover, \( \langle X \rangle + \langle Y \rangle \perp \langle S \rangle + \langle T \rangle \). It is
clear that $\langle X \rangle + \langle Y \rangle \cong \left( \frac{pd}{1-4d}, \frac{q}{1-pq}, 1 \right)$ and that

$\langle S \rangle + \langle T \rangle \cong \left( -qd, \frac{-p}{(1-4d)(1-pq)}, 1 \right)$.

4.5. Lemma. $I(A) W^0_q(A)$ is generated as an additive group by elements of the form

$$\begin{bmatrix} a \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1, d, 1 \end{bmatrix}, a \in A^0, d \in A^0 \cap U(A).$$

4.6. Let $a \in A^0, d \in U(A) \cap A^0$. Then we have that

$$\begin{bmatrix} 1-4a \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1, -d, 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -2a \end{bmatrix} \begin{bmatrix} -1, -d, 1 \end{bmatrix} \in I^2(A) W^0_q(A).$$

Proof. If $2 \notin \mathfrak{m}$ then this statement is easily proved. So suppose $2 \in \mathfrak{m}$. Applying (4.4.) we find that

$$\begin{bmatrix} -2 \\ 1 \\ -2a \end{bmatrix} \begin{bmatrix} -1, -d, 1 \end{bmatrix} = \begin{bmatrix} d(4a-1) \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -2d \\ 1-4a \\ -2a \end{bmatrix} = \begin{bmatrix} p \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} d(4a-1) \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1, -d, 1 \end{bmatrix} \text{ for certain } p \in U(A).$$

Now we consider the form $\begin{bmatrix} 1-4a \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} -1, -d, 1 \end{bmatrix}$.

Let $e, f$ be a basis of $V$, and let $(\ , \ )$ be a symmetric bilinear form satisfying $(e, e) = 1-4a$, $(f, f) = -1$, $(e, f) = 0$.

Let $x, y$ be a basis of $W$, and let $q$ be a quadratic form such that $q(x) = -1$, $q(y) = -d$, $(x, y) = 1$. Denote $A = e \otimes y$, $B = (e+f) \otimes (x+2y)$, $C = f \otimes x$, $D = (e+(1-4a)f) \otimes (2dx+y)$.

$A, B, C, D$ is a basis for $V \otimes W$ and $\langle A \rangle + \langle B \rangle \perp \langle C \rangle + \langle D \rangle$.

$\langle A \rangle + \langle B \rangle \cong \left( \frac{-d}{1-4a}, \frac{-4a}{1-4d}, 1 \right) \cong -d(4a-1)(1-4d) \left( \frac{-4ad}{(1-4a)(1-4d)}, 1 \right)$

$\langle C \rangle + \langle D \rangle \cong \left( 1, \frac{4ad}{(1-4a)(1-4d)}, 1 \right) \cong -1 \left( \frac{-4ad}{(1-4a)(1-4d)}, 1 \right)$

Hence $V \otimes W \cong \left( -1, \frac{d(4a-1)}{1-4a}(1-4d), 1 \right)$.

Now it is easily proved that

$$\begin{bmatrix} 1-4a \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} -1, -d, 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -2a \end{bmatrix} \begin{bmatrix} -1, -d, 1 \end{bmatrix} = \begin{bmatrix} p \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} d(4a-1) \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1, -d, 1 \end{bmatrix} \text{ for certain } p \in U(A).$$
4.7. \[
\begin{pmatrix}
a & 0 \\
0 & -1
\end{pmatrix}
\cdot
\begin{pmatrix}
a & 0 \\
0 & -1
\end{pmatrix}
\cdot
\begin{pmatrix}
-2 & 1 \\
1 & -2b
\end{pmatrix}
\cdot
\begin{pmatrix}
-1, -c, 1 \\
0 & -1
\end{pmatrix}
\cdot
\begin{pmatrix}
a & 0 \\
0 & -1
\end{pmatrix}
\cdot
\begin{pmatrix}
-2 & 1 \\
1 & -2b
\end{pmatrix}.
\]
\[
\begin{pmatrix}
-2 & 1 \\
1 & -2b
\end{pmatrix}
\cdot
\begin{pmatrix}
-1, -c, 1 \\
0 & -1
\end{pmatrix}
= \begin{pmatrix}
1-4c & 0 \\
0 & -1
\end{pmatrix}
\cdot
\begin{pmatrix}
-1, -b, 1 \\
0 & -1
\end{pmatrix}.
\]
\[
c \in A^0.
\]
Proof. Since \( b \in U(A) \cap A^0 \) we have that
\[
\begin{pmatrix}
-2 & 1 \\
1 & -2b
\end{pmatrix}
\cdot
\begin{pmatrix}
-1, -c, 1 \\
0 & -1
\end{pmatrix}
= \begin{pmatrix}
-2 & 1 \\
1 & -2b
\end{pmatrix}
\cdot
\begin{pmatrix}
-1, -b, 1 \\
0 & -1
\end{pmatrix}.
\]
It is clear that
\[
\begin{pmatrix}
a & 0 \\
0 & -1
\end{pmatrix}
\cdot
\begin{pmatrix}
a & 0 \\
0 & -1
\end{pmatrix}
= \begin{pmatrix}
a & 0 \\
0 & -1
\end{pmatrix}
\cdot
\begin{pmatrix}
a & 0 \\
0 & -1
\end{pmatrix}.
\]
Now we calculate:
\[
\begin{pmatrix}
a & 0 \\
0 & -1
\end{pmatrix}
\cdot
\begin{pmatrix}
-1, -c, 1 \\
0 & -1
\end{pmatrix}
= \begin{pmatrix}
-2 & 1 \\
1 & -2b
\end{pmatrix}
\cdot
\begin{pmatrix}
-1, -c, 1 \\
0 & -1
\end{pmatrix}
\cdot
\begin{pmatrix}
-2 & 1 \\
1 & -2b
\end{pmatrix}.
\]
\[
\begin{pmatrix}
1-4c & 0 \\
0 & -1
\end{pmatrix}
\cdot
\begin{pmatrix}
-1, -b, 1 \\
0 & -1
\end{pmatrix}.
\]
The relations (4.1), (4.2), (4.3), (4.6), (4.7) are translations of relations which have been mentioned explicitly in the definition of \( g_e(A) \mod C_g(A) \). The other relations have to do with commutativity. Now,
\[
@ \ I^n(A) W_q(A) \mod I^{n+1}(A) W_q(A) @ \ I^n(A) \mod I^{n+1}(A) \]
is commutative with respect to multiplication. So we have verified that the defining relations for \( g_e(A) \mod C_g(A) \) also hold for the image of \( s_e \). Hence the following theorem is proved:

4.8. Theorem. There exists a well-defined homomorphism of rings \( s_e : g_e(A) \mod C_g(A) \to \bigoplus_{n \geq 0} I^n(A) W_q(A) \mod I^{n+1}(A) W_q(A) \).
Let \( g_n(A) \) denote the two-sided ideal generated by \( \{ \overline{g}(a) | a \in A^0 \} \). Let us write \( g_n(A) \) for the intersection of \( g(A) \) with \( g_n(A) \).

Denote \( s^o \) for the restriction of \( s^o \) to \( \sigma^n(A) \), and denote the restriction of \( s^o \) to \( \sigma(A) \) by \( s^o \).

4.9. Theorem. \( s^o : \sigma^n(A) \to \bigoplus_{n \geq 0} I^n(A) W^0_q(A) \) is a surjective homomorphism of rings.

Proof. The elements of \( \sigma^n(A) \) are of the form \( \sum_{i=1}^{n} x_i \overline{o}(a_i)y_i \), with \( a_i \in A^0 \), \( x_i, y_i \in g_n(A) \mod C_{g_n}(A) \).

So \( s^o \sigma^n(A) \subset \bigoplus_{n \geq 0} I^n(A) W^0_q(A) \).\( \sigma^n(A) \).

Lemma (4.5) proves that \( s^o \) maps \( \sigma^n(A) \) surjectively on \( \bigoplus_{n \geq 0} I^n(A) W^0_q(A) \).

We will now prove, that \( s^o \) is an injective map on \( \sigma^n(A) \).

4.10. There exists a homomorphism of groups \( \text{discr} : W^0_q(A) \to G(A) \), satisfying \( \text{discr}[a][1,d,1] = d \circ J(A) \), \( a \in U(A), d \in A^0 \).

The following sequence is exact:

\[
1 \to I(A) W^0_q(A) \to W^0_q(A) \xrightarrow{\text{discr}} G(A) \to 1
\]

Proof. The existence of the homomorphism \( \text{discr} \) follows from what is said in section 2. The map \( \text{discr} \) is surjective since \( \text{discr}[1,d,1] = d \circ J(A) \), \( d \in A^0 \). Lemma (4.5) shows that \( I(A) W^0_q(A) \) is generated by elements of the form \( \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \cdot [1,d,1] \), \( a \in U(A), d \in A^0 \). Hence \( I(A) W^0_q(A) \subset \ker(\text{discr}) \). Suppose that \( \bigoplus_{i=1}^{n} [a_i][1,d_i,1] \in W^0_q(A) \) and that \( d_1 \circ \ldots \circ d_n \circ J(A) = J(A) \). Then we have that

\[
\begin{align*}
\bigoplus_{i=1}^{n} [a_i][1,d_i,1] &= \bigoplus_{i=1}^{n} [a_i][1,d_i,1] \oplus [a_n][1,d_1 \circ \ldots \circ d_{n-1}] \\
&= I(A) W^0_q(A) = I(A) W^0_q(A)
\end{align*}
\]
Remark. Compare Knebusch [2], (7.10).

4.11. Theorem. $s_1 : \mathcal{O}_1(A) \to W_0^0(A) \mod I(A)W_0^0(A)$ is an isomorphism of additive groups.

Remark. We cannot repeat Milnor's proof for the injectivity of $s_2$, since we do not work with 1-dimensional quadratic forms.

5. Example. $F$ is a field of characteristic $2$, $F \neq F_2$.

We have $U(F) = \{a \in F \mid a \neq 0\}, F^0 = F$.

The most important defining relations for $g_n(F) \mod Cg_n(F)$ are

\[ g(ab) = g(a) + g(b), \quad a, b \neq 0 \]
\[ o(a+b) = o(a) + o(b), \]
\[ g(a) o(a) = 0, \quad a \neq 0 \]
\[ o(a) o(b) = 0. \]

The elements of $g_n(F) \mod Cg_n(F) \cap g_n(F)$ can be written as sums of elements of the type

\[ g(a_1) \ldots g(a_n) - g(a_1) \ldots g(a_{n-1}) o(b). \]

The elements of $\mathcal{O}_n(F) \mod Cg_n(F) \cap \mathcal{O}_n(F)$ are sums of terms $g(a_1) \ldots g(a_{n-1}) o(b)$.

Let $\oplus a_i (1, d_i, 1)$ be a quadratic form.

\[ SW(\oplus a_i (1, d_i, 1)) = 1 + \sum_{i=1}^{n} g(a_i) o(d_i). \]

Hence, $SW(\mathcal{O}(F)) = 0$, and we can extend $SW$ to a map

\[ SW : W_q(F) \to \mathcal{O}(F) \mod Cg_n(F) \cap \mathcal{O}(F). \]

We calculate the action of $SW$ on $I^n(F)W_q(F)$.

\[ SW [1, d, 1] = 1 + o(d) \]
\[ SW \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} [1, d, 1] = 1 + g(a) o(d). \]
\[ SW \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} [1, d, 1] = 0. \]

Hence, $SW$ acts trivially on $I^2(F)W_q(F)$.

We calculate $s_n : \mathcal{O}(F) \to \oplus I^n(F)W_q(F) \mod I^{n+1}(F)W_q(F)$.

\[ s_1 o(a) = [1, a, 1] + I(F)W_q(F). \]
\[ s_2 (\oplus g(a_i) o(d_i)) = \oplus \begin{bmatrix} a_i & 0 \\ 0 & 1 \end{bmatrix} [1, d_i, 1] + I^2(F)W_q(F). \]

It is easy to see, that:
SW \circ s_2(x) = 1 + x, \ x \in \mathcal{C}_2(F) / C_{g_2}(F) \cap \mathcal{C}_2(F).

This proves that \( s_2 \) is a monomorphism.

There are no results about the injectivity of \( s_1, i \geq 3 \).

\[ s_2 : \mathcal{C}_2(F) \mod \mathcal{C}_2(F) \cap C_{g_2}(F) \rightarrow I(F)W_q(F) \mod I^2(F)W_q(F) \]

is an isomorphism of additive groups.

We refer to another description of \( I(F)W_q(F) \mod I^2(F)W_q(F) \) by C.H. Sah, [7].

Let \( Cl[M,q] \) denote the class of the Clifford algebra of \((M,q)\) in the ungraded Brauer group of \( F \). \( Cl[M,q] \) is an element of \( \mathcal{Br}(F) \), the subgroup generated by the elements of order 2 of \( \mathcal{Br}(F) \). \( Cl \) induces a split exact sequence:

\[ 0 \rightarrow I^2(F)W_q(F) \rightarrow I(F)W_q(F) \xrightarrow{\phi} Br(F) \rightarrow 0 \]

Hence, \( Cl \) induces an isomorphism

\[ Cl : I(F)W_q(F) \mod I^2(F)W_q(F) \rightarrow 2Br(F) \]

In proving this theorem, C.H. Sah uses the following result:

Denote \((a,d)\) for the \( F \)-algebra \( H \) with \( F \)-basis \( 1, u, v, uv \) and with relations \( u^2 = a \neq 0 \), \( v^2 + v = d \), \( uv + vu = 1 \).

\( H \) is a quaternion algebra with norm form

\[ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, (1,d,1). \]

The class of the Clifford algebra of \( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, (1,d,1) \) is equal to the class \([H]\) of \( H \) in the Brauer group.

Combining these results, we find that

\[ Cl \circ s_2 : \mathcal{C}_2(F) \mod \mathcal{C}_2(F) \cap C_{g_2}(F) \rightarrow 2Br(F) \]

is an isomorphism of groups,

\[ \bigoplus_{i=1}^{n} \bigoplus_{(a_i,d_i)} Cl_0 s_2(\bar{g}(a_{i}) \bar{d}(d_{i})) = \bigoplus_{i=1}^{n} [(a_i,d_i)], \]

since tensor product induces multiplication in \( 2Br(F) \).

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