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Measures and Markov processes on function spaces

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MEASURES AND MARKOV PROCESSES ON FUNCTION SPACES

By Peter BAXENDALE

1. INTRODUCTION.

Let S be a compact metric space of finite dimension and M a smooth complete finite dimensional Riemannian manifold. We describe the construction of a Markov process with continuous sample paths in the manifold C(S,M). En particular if S and M are compact Riemannian manifolds then for each $p > \frac{1}{2}$ dim S and $q > \frac{1}{2}$ dim M + 1, we obtain a different Markov process on C(S,M). We hope that this concrete example will be the first stage of a more general theory for Markov processes on infinite-dimensional manifolds, in the same way that the close study of Brownian motion on the real line gave rise to the theory of Gaussian measures on abstract Banach and Fréchet spaces.

For proofs of the results, see [3].

2. GAUSSIAN MEASURES ON LINEAR FUNCTION SPACES.

Let E be a separable Fréchet space. A (mean-zero) Gaussian measure μ on E is a Borel probability measure on E such that for all $\xi \in E^*$, $\xi(\mu)$ is a Gaussian measure on $\mathbb{R}$ with mean 0 and variance $\sigma(\xi)$, where possibly $\sigma(\xi) = 0$. Henceforth, all our Gaussian measures will have mean 0.

Let $\mathfrak{A}(E)$ be the algebra of subsets of E generated by the continuous linear functionals $\xi \in E^*$, then the σ-algebra generated by $\mathfrak{A}(E)$ is the Borel σ-algebra of E. If H is a separable Hilbert space, there is a canonical additive set function γ on $\mathfrak{A}(H)$ characterised by the fact that $\xi(\gamma)$ is a Gaussian measure on $\mathbb{R}$ with mean 0 and variance $\|\xi\|^2$, for each $\xi \in H^*$. If dim H = ∞, γ is not σ-additive. However, if $i : H \to E$ is a continuous linear map, $i(\gamma)$ is an additive set function on $\mathfrak{A}(E)$ and we may ask whether $i(\gamma)$ is σ-additive. If i est injective and $i(\gamma)$ is σ-additive on $\mathfrak{A}(E)$, we say that (i,H,E) is an abstract Wiener space (AWS), and the extension of $i(\gamma)$ to the Borel σ-algebra of E is called the corresponding Wiener measure. We remark that some authors insist that i(H) be dense in E in the definition of AWS.

THEOREM. - *If* (i,H,E) *is a AWS, then the Wiener measure is a Gaussian measure. Conversely if* μ *is a Gaussian measure on* E, *there exist* H *and* $i : H \to_{\mu} E$ *such that* (i,H,E) *is an AWS with Wiener measure* μ.

For fuller details see [9], [11] and [12] in the case E = Banach space. The characterisation of Gaussian measures in the theorem is due to SATO [15] in the case E = Banach space. For the general case, see [6].

The following construction, which we shall need later, is due to GROSS. See [10] for the case E = Banach. (The hypothesis that $|\cdot|_E$ is in $L^2(\mu)$ is satisfied for all Gaussian measures μ, see [7].) Again the extension to the general case follows from results in [6]. For $t > 0$, define μ_t by $\mu_t(A) = \mu(t^{-\frac{1}{2}}A)$ for $A \in \mathrm{Borel}(E)$. Then $\mu_t * \mu_s = \mu_{t+s}$. If we put $P(t,x,A) = \mu_t(A-x)$, then the $P(t,x,A)$ form a set of transition probabilities for a Markov process in E. In fact we may construct a process $\{W_t : t \geqslant 0\}$ with the properties

(a) continuous sample paths
(b) independent increments
(c) the distribution of $W_{t+s} - W_t$ is independent of s
(d) $W_o = 0$
(e) the distribution of W_1 is μ.

$\{W_t : t \geqslant 0\}$ is characterised by (a) - (e), and is called the Wiener process in E generated by μ. We remark that if $\{Z_t : t \geqslant 0\}$ is any process in E satisfying (a) - (d), then $\exists X \in E$ such that $Z_t - tX$ is the Wiener process corresponding to some Gaussian measure μ on E.

Example 1. - $E = C(S,\mathbb{R})$, and μ a Gaussian measure on $C(S,\mathbb{R})$. For $s,t \in S$, let $K(s,t) = \int f(s)f(t)d\mu(f)$. Then K is called the covariance of μ and satisfies

(i) $K(s,t) = K(t,s)$
(ii) $\forall s_1,\dots,s_n \in S,\ \lambda_1,\dots,\lambda_n \in \mathbb{R},\ \sum_{i,j=1}^{n} \lambda_i\lambda_j K(s_i,s_j) \geqslant 0$.

Any function K satisfying (i) and (ii) is called a reproducing kernel for S, and there exists a corresponding Hilbert space H(K) consisting of functions $S \to \mathbb{R}$, characterised by

(i) $K_s \in H(K),\ \forall s \in S$, where $K_s(t) = K(s,t)$
(ii) $\langle K_s, f \rangle_{H(K)} = f(s),\quad \forall f \in H(K),\ s \in S.$

For details see [1] and [12]. For K defined as above, there is a continuous inclusion $H(K) \xhookrightarrow{i} C(S)$ and $(i, H(K), C(S))$ is an AWS with Wiener measure μ.

The existence of a large supply of Gaussian measures on C(S) is ensured by the following theorem due to DUDLEY [5].

THEOREM. - *If* S *is a compact metric space of finite dimension and* $i : H \to C^{\alpha}(S)$ *is a continuous injection for some* $\alpha > 0$*, then* $(i,H,C(S))$ *is an AWS.*

Example 2. - Let $\xi : V \to M$ be a smooth finite-dimensional vector bundle, and μ a Gaussian measure on $C(\xi)$. Let $\xi^* : V^* \to M$ be the dual bundle. For $x,y \in M$, define $P(x,y) \in L(V_x^*, V_y)$ by

$$(v,P(x,y)u) = \int_{C(\xi)} (u,f(x))(v,f(y))d\mu(f) \quad \text{for} \quad u \in V_x^*,\ v \in V_y^*.$$

P is called the covariance of μ, and satisfies

(i) $P(x,y)^* = P(y,x)$

(ii) $\forall n \in Z^+,\ x_1,\ldots,x_n \in M,\ u_1 \in V_{x_1}^*,\ldots,u_n \in V_{x_n}^*,\ \sum_{i,j=1}^{n} (u_j,P(x_i,x_j)u_i) \geq 0.$

Any P satisfying (i) and (ii) is called a reproducing kernel for ξ, and determines a Hilbert space H(P) of sections of ξ, characterised by

(i) $P_u \in H(P),\ \forall u \in V^*$, where $P_u(y) = P(\xi^*(u),y)du$

(ii) $(P_u,f)_{H(P)} = (u,f(\xi^*(u))),\ \forall f \in H(P),\ u \in V^*.$

(See [2].) For P defined as above, there is a continuous inclusion $H(P) \overset{i}{\hookrightarrow} C(\xi)$, and $(i,H(P), C(\xi))$ is an AWS with Wiener measure μ. We remark that since $P(x,x) \geq 0$ as an element of $L(V_x^*,V_x)$, it may be factored $P(x,x) = jj^*$ where j is a continuous linear injection $H_x \to V_x$ for some (finite-dimensional) Hilbert space H_x. Then the Gaussian measure $j(\gamma)$ on V_x is precisely the image of μ under the evaluation map $C(\xi) \to V_x$, $f \mapsto f(x)$.

THEOREM (DUDLEY-BAXENDALE, [2]). - *Suppose* $H \overset{i}{\longrightarrow} C^{k,\alpha}(V)$ *is a continuous linear injection. If either* $\ell < k$ *or* $\ell = k$ *and* $0 \leq \beta < \alpha$*, then* $(\bar{i},H,C^{\ell,\beta}(V))$ *is an AWS, where* $\bar{i}$ *denotes the composition* $H \overset{i}{\to} C^{k,\alpha}(V) \subset C^{\ell,\beta}(V)$.

Therefore the Sobolev inequalities [14] , [4], provide a large supply of Gaussian measures on the various function spaces $C^{\ell,\beta}(V)$.

3. MARKOV PROCESSES ON M.

We deal with case S = a point. Our initial information consists of a Gaussian measure μ of mean-zero on C(TM), and $X \in C(TM)$. We shall need to impose restrictions on μ and X later. To help the reader visualise the construction we proceed as follows. Let $\{W_t : t \geq 0\}$ be the Wiener process on C(TM) corresponding to μ, and let $Z_t = W_t + tX$. Fix $a \in M$ and $T > 0$. For each partition $\pi = \{0 = t_o < t_1 < \ldots < t_n = T\}$, define $Y_o = a$ and $Y_{r+1} = \exp_{Y_r}((Z_{t_{r+1}} - Z_{t_r})(Y_r))$ for $r = 0,\ldots,n-1$. Let λ_π be the distribution of Y_n in M. Under suitable conditions on μ,X and M, the measures λ_π will converge as mesh of π tends to 0 to some measure P(T,a). Then the P(T,a) are to be the transition probabilities for a Markov process on M. KUSHNER [13] gives conditions for a sequence of Markov chains with values in some Euclidean space to converge to the solution of a stochastic differential equation. Our method is to embed M in some Euclidean space V and to consider a stochastic differential equation in V. We are concerned not with the limiting procedure, but with the limit itself. Our conditions will ensure that the stochastic differential equation has a solution. They may or may not imply that the λ_π converge.

Let $M \subset V$ be a closed isometric embedding in some Euclidean space V, and h(x) the second fundamental form at $x \in M$. Define

$$Y(x) = \frac{1}{2} \int_{C(TM)} h(x)(f(x),f(x))d\mu(f) \quad \text{for} \quad x \in M,$$

then Y is a section of the normal bundle $T^{\perp}M \to M$. Using the natural inclusions $T_xM \subset V$ and $T_x^{\perp}M \subset V$, we think of μ as a measure on C(M,V), and of X,Y as taking values in C(M,V). In particular we think of the reproducing kernel P associated with μ as taking values in $L(V^*,V)$. We may choose a Gaussian measure $\bar{\mu}$ on C(V,V) such that $\bar{\mu}$ restricts on μ, and $\bar{X}, \bar{Y} \in C(V,V)$ which restrict to X,Y respectively. For $a \in V$ we consider the stochastic differential equation with values in V :

$$\left.\begin{aligned} d\xi(t) &= (\bar{X}+\bar{Y})(\xi(t))dt + d\bar{W}(t)(\xi(t)) \\ \xi(0) &= a \end{aligned}\right\} (*)$$

where $\bar{W}(t)$ is the Wiener process with values in C(V,V) generated by $\bar{\mu}$.

We need some extra conditions on μ, X and the embedding $M \subset V$

(A 1) $\forall N, \exists L_N$ such that $\|x\| \leqslant N, \ \|y\| \leqslant N, \ x,y \in M$

$$\Rightarrow tr(P(x,x) + P(y,y) - P(x,y) - P(y,x)) \leqslant L_N\|x-y\|^2$$

$$\text{and} \quad \|X(x) - X(y)\| \leqslant L_N\|x-y\|$$

(A 2) $\exists K > 0$ such that $\forall x \in M$,

$$tr(P(x,x)) \leqslant K(1 + \|x\|^2) \quad \text{and } \|X(x)\|^2 \leqslant K(1 + \|x\|^2)$$

(A 3) $\exists K > 0$ such that $x \in M$,

$$\|h(x)\| \ . \ tr(P(x,x)) \leqslant K(1 + \|x\|).$$

Notice that (A 1) is independent of the embedding. It is satisfied if $\mu\{f \in C(TM) : f \text{ is locally Lipschitz}\} = 1$ and X is locally Lipschitz. (A 2) and (A 3) are automatically satisfied if M is compact.

<u>PROPOSITION 1</u>. - *Under the assumptions (A1) - (A3), there exist extensions* $\overline{\mu}$, $\overline{X}$, $\overline{Y}$ *such that the problem (*) has a solution* $\xi(t)$ *which is continuous with probability one and has finite moments of all orders. If* ξ_1 *is any other solution of (*) with these properties, then* $P\{\xi(t) = \xi_1(t),\ \forall t \geq 0\} = 1$. *The solution* $\xi(t)$ *is a time homogeneous strong Markov process whose transition probabilities are given by* $P(t,x,A) = P\{\eta(t) \in A\}$ *where* η *is the solution to (*) with* $a = x$.

<u>PROPOSITION 2</u>. - *Under the assumptions of Proposition 1, if* $a \in M$, *then* $\xi(t) \in M$ *for all* $t \geq 0$ *with probability one.*

This result shows that we have obtained a Markov process in M, and hence that the process started at $a \in M$ is independent of the choices of extension $\overline{\mu}$, $\overline{X}$, $\overline{Y}$.

<u>PROPOSITION 3</u>. - *The Markov process has infinitesimal generator* A, *where*

$$(Ah)(x) = \frac{1}{2}\int \nabla^2 h(x)(f(x),\ f(x))d\mu(x) + \nabla h(x).(X(x))$$
$$= \frac{1}{2}\ \mathrm{tr}(\nabla^2 h(x)(P(x,x)) + \nabla h(x).(X(x))$$

for smooth $h : M \to \mathbb{R}$ *with compact support.*

Proposition 3 shows that the Markov process on M is independent of the embedding, so long as there exists an embedding satisfying (A1) - (A3).

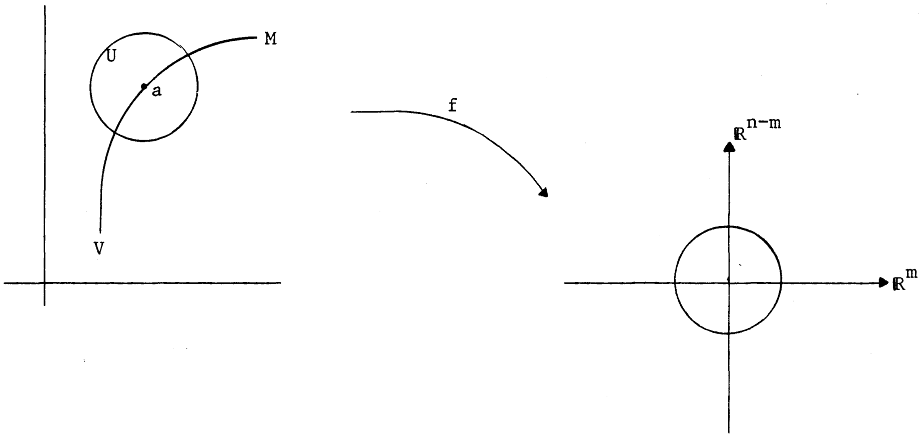
4. <u>DISCUSSION OF THE PROOFS AND EXAMPLES</u>.

Given assumptions (A1) - (A3) on μ, X and the embedding, we may choose extensions $\overline{\mu}$, $\overline{X}$ and $\overline{h}$ which satisfy similar conditions on all of V. This means that the stochastic differential equation (*) is a particular case of the general type

$$d\xi(t) = a(\xi(t))dt + b(\xi(t))d\overline{W}(t)$$

where $a : V \to V$ and $b : V \to L(C(V,V),\ V)$ satisfy local Lipschitz and global growth properties. The statement of Proposition 1 now follows from general results on existence and uniqueness for solutions of stochastic differential equations. See [8] for such results in the one-dimensional case.

The proof of Proposition 2 goes as follows



Let f be a diffeomorphism of a neighbourhood U of a in V onto a neighbourhood of 0 in $\mathbb{R}^n$ such that $f|_U$ maps into $\mathbb{R}^m \times \{0\}$. Let ξ be the solution of (*) in V started at a, and τ = time of first exit of ξ from U. Let $\eta(t) = f(\xi(t))$ for $t < \tau$. Then by Itô's lemma for the transformation of a stochastic integral, we see that

$$\begin{aligned} d\eta(t) &= Df(\xi(t))(\overline{X}(\xi(t)))dt \\ &+ \frac{1}{2}\operatorname{tr}\{[D^2f(\xi(t)) + Df(\xi(t))\overline{h}(\xi(t))]\,[\overline{P}(\xi(t),\xi(t))]\}dt \\ &+ Df(\xi(t))\,[d\overline{W}(t)(\xi(t))]. \end{aligned}$$

Since $D^2f(x)|_{T_xM \times T_xM} + Df(x) \circ h(x) = \nabla^2(f|_M)(x)$ for $x \in M$, then the above equation is of the form

$$d\eta(t) = a(\eta(t))dt + b(\eta(t))d\overline{W}(t)$$

where

$$a(y) \in \mathbb{R}^m \times \{0\} \quad\text{and}\quad b(y) \in L(L(V,V),\, \mathbb{R}^m \times \{0\}) \quad \text{for } y \in \mathbb{R}^m \times \{0\}.$$

Therefore, with probability one $\eta(t) \in \mathbb{R}^m \times \{0\}$ for $t < \tau$, so that $\xi(t) \in M \cap U$ up to time of first exit from U. The strong Markov property now shows that $\xi(t) \in M$ for all $t \geqslant 0$ with probability one.

To prove Proposition 3, suppose ξ is started at $x \in M$. By Itô's lemma,

$$h(\xi(t)) - h(x) = \int_0^t (Ah)(\xi(s))ds + \int_0^t Dh(\xi(t))d\overline{W}(t)(\xi(t)).$$

Take expectations and let $t \to 0$:

$$\frac{\mathbb{E}(h(\xi(t)) - h(x))}{t} = \mathbb{E}\,\frac{1}{t}\int_0^t (Ah)(\xi(s))ds \to (Ah)(x).$$

Example 3. – Let $M \subset V$ be an isometric embedding with induced inclusions $T_xM \subset V$, $x \in M$. Let P_x = orthogonal projection $V \to T_xM$, and define $P_{xy} = P_y \circ P_x^*$. There exists a continuous map $\theta : V \to C(TM)$ given by $(\theta v)(x) = P_x(v)$ for $x \in M$, $v \in V$, and the isometric image of V in C(TM) is precisely H(P). The important fact is that there exists a Gaussian measure with covariance P such that P(x,x) = natural isomorphism of T_x^*M with T_xM. It is easy to see that P varies smoothly, so it satisfies conditions (A1) and (A2). If we take X = 0, we obtain a Markov process on M with infinitesimal generator $\frac{1}{2}\Delta$ (where Δ = Laplace-Beltrami operator on M) under the sole condition that there exists an embedding of M with $\|h(x)\| \leqslant K(1 + \|x\|)$ for some $K > 0$.

Example 4. – Let G be a second order differential operator with C^2 coefficients, satisfying G(1) = 0, and $Gh(x) \leqslant 0$ whenever x is a local maximum of h. Then there exist a Gaussian measure μ on C(TM) and $X \in C(TM)$ satisfying condition (A1) such that

$$(Gh)(x) = \frac{1}{2}\int \nabla^2 h(x)(f(x),f(x))d\mu(x) + \nabla h(x)(X(x)).$$

Therefore, given suitable growth conditions on the coefficients of G, there exists a Markov process on M with infinitesimal generator G.

5. THE GENERAL CASE.

Let S be a compact metric space of finite metric dimension, and K a reproducing kernel on S. Let P be a reproducing kernel for $TM \to M$. Then we may define $Q(s,x,t,y) = K(s,t)P(x,y) \in L(T_x^*M, T_yM)$, a reproducing kernel for the product bundle $\phi : S \times TM \to S \times M$. We remark that H(Q) is naturally isomorphic to $H(K) \otimes H(P)$. Suppose also $X \in C(\phi)$.

As before let $M \subset V$ be a closed isometric embedding in some Euclidean space V, and let h(x) be the second fundamental form at $x \in M$. Think of elements of $C(\phi)$ as continuous functions $S \times M \to V$. We list some assumptions :

(A 4) $H(K) \subset C^\alpha(S)$ is a continuous inclusion, for some $\alpha > 0$

(A 5) $\exists C > 0$ such that $tr(P(x,x)) \leqslant C$ and
$$tr(P(x,x) + p(y,y) - P(x,y) - P(y,x)) \leqslant C\|x-y\|^2, \quad \forall x,y \in M$$

(A 6) $\exists C > 0$ such that $\|X(s,x)\| \leqslant C$ and

$$\|X(s,x) - X(t,y)\|^2 \leqslant C(\|x-y\|^2 + d(s,t)^{2\alpha}), \text{ some } \alpha > 0, \text{ for all } s,t \in S, \; x,y \in M.$$

(A 7) $\exists C > 0$ such that $\|h(x)\| \leqslant C$ and $\|h(x) - h(y)\| \leqslant C\|x-y\|$ for all $x,y \in M$.

We remark that (A 4) and (A 5) separately imply that K and P are covariances for Gaussian measures on C(S) and C(TM). Together, they imply that $Q = K \otimes P$ is the covariance for a Gaussian measure μ on $C(\phi)$. Choose extensions $\bar{\mu}$, $\bar{X}$ and $\bar{h}$ as before (so that, e.g. $\bar{\mu}$ is a measure on $C(S \times V, V)$) and define

$$\bar{Y}(s,x) = \frac{1}{2}\int \bar{h}(x)(f(s,x), f(s,x))d\bar{\mu}(f) \quad \text{for} \quad s \in S,\ x \in V.$$

For $\underline{s} = (s_1,\dots,s_k) \in S^k$ and $\underline{x} = (x_1,\dots,x_k) \in V^k$, consider the stochastic differential equation in V^k :

$$\left.\begin{aligned} d\xi_i(t) &= (\bar{X}+\bar{Y})(s_i,\xi_i(t))dt + d\bar{W}(t)(s_i,\xi_i(t)) \\ \xi_i(0) &= x_i \end{aligned}\right\} \quad i = 1,\dots,k \quad (**)$$

<u>PROPOSITION 4</u>. - *We assume (A 4) - (A 7). There exist entensions* $\bar{\mu}$, $\bar{X}$, $\bar{h}$ *such that (**) has a solution* $(\xi_1(t),\dots,\xi_k(t))$ *in* V^k *which is continuous with probability one, has finite moments of all orders and is unique in the sense of Proposition 1. For each fixed* $\underline{s} = (s_1,\dots,s_k)$ *the solution is a time homogeneous strong Markov process whose transition probabilities are given by*

$$P(t,\underline{y},A) = P\{\eta(t) \in A\}$$

where $\eta(t) = (\eta_1(t),\dots,\eta_k(t))$ *is the solution (**) with initial condition* $\eta_i(0) = y_i$, $i = 1,\dots,k$.

<u>PROPOSITION 5</u>. - *If* $\underline{a} \in M^k$, *then the solution* $(\xi_1(t),\dots,\xi_k(t))$ *is in* M^k *for all* $t \geqslant 0$ *with probability one.*

The proof of these results go just as in the previous case. We need the full strength of the assumptions in

<u>PROPOSITION 6</u>. - *Suppose we have (A 4) - (A 7) (with the same* α *in (A 4) and (A 6)). Then for each integer* $p > 0$, *and* $T > 0$ *there exists a constant* $C > 0$ *such that for all* $s_1,s_2 \in S$ *and* $x_1,x_2 \in M$, *then*

$$\mathbb{E}(\|\xi_1(T) - \xi_2(T)\|^{2p}) \leqslant C(\|x_1-x_2\|^{2p} + d(s,t)^{2p\alpha})$$

where $(\xi_1(t),\xi_2(t))$ *is the solution of (**) corresponding to* $\underline{s} = (s_1,s_2)$ *and* $\underline{x} = (x_1,x_2)$.

For each $\underline{s} = (s_1,\ldots,s_k) \in S^k$ and $\underline{x} = (x_1,\ldots,x_k) \in M^k$, let $\nu^t_{\underline{s},\underline{x}}$ be the distribution of $(\xi_1(t),\ldots,\xi_k(t))$ as in Proposition 4. For each fixed t, these form a consistent family of finite-dimensional distributions which yield a measure on $F(S \times M,M)$ = set of all functions $S \times M \to M$ with the σ-algebra generated by the evaluation maps. The estimate in Proposition 6 enables us to prove that $C(S\times M,M)$ has full outer measure, so that we obtain a measure ν_t on $C(S\times M,M)$. Now for $h \in C(S,M)$ define $P(t,h)$ = image of ν_t under the mapping $C(S\times M,M) \to C(S,M)$

$$f \longmapsto (s \longmapsto f(s,h(s))).$$

Then the Markov property of the finite-dimensional distributions ensures that the $P(t,h)$ form a set of transition probabilities for a Markov process with values in $C(S,M)$.

THEOREM. - K, P *and* X *satisfying (A 4) - (A 7) determine a Markov process on* $C(S,M)$ *with continuous sample paths. Suppose* $F : C(S,M) \to \mathbb{R}$ *is of the form* $F(h) = G(h(s_1),\ldots,h(s_k))$ *for some* $\underline{s} = (s_1,\ldots,s_k)$ *and* $G : M^k \to \mathbb{R}$ *smooth function with compact support. Then* F *is in the domain of the infinitesimal generator* A *of the Markov process, and*

$$(AF)(h) = \frac{1}{2}\,\mathrm{tr}\,[\nabla^2 G(h(s_1),\ldots, h(s_k)) \circ (K \otimes P)(\underline{s},h(\underline{s}))]$$
$$+ (\nabla G)(h(s_1),\ldots, h(s_k))[X(s_1,h(s_1)),\ldots,X(s_k,h(s_k))],$$

where $(K \otimes P)(\underline{s},h(\underline{s})) : \prod_{i=1}^{k} T^*_{h(s_i)}M \to \prod_{i=1}^{k} T_{h(s_i)}M$ *has matrix* $K(s_i,s_j)P(h(s_i), h(s_j))$, *and* ∇ *is covariant differentiation with respect to the product Riemannian structure on* M^k.

To show that there exists a version of the Markov process with continuous sample paths we use the same method to construct a measure ν on the space $C([0,\infty) \times S \times M, M)$ in such a way that the image of ν under evaluation at time t is ν_t.

Finally notice that $C(S\times M, M)$ is a topological semi-group with respect to $(f.g)(s,m) = f(s,g(s,m))$ for $f,g \in C(S \times M, M)$ and has a continuous left action on $C(S,M)$ by $(f.h)(s) = f(s,h(s))$ for $f \in C(S \times M, M)$, $h \in C(S,M)$. The Markov property of the $\nu^t_{\underline{s},\underline{x}}$ lifts to the equation $\nu_t * \nu_s = \nu_{t+s}$, and we obtain a random process Y_t in $C(S \times M, M)$ with continuous sample paths and independent increments on the left, in the sense that $P(Y_t \in A | Y_r, r \leq s) = \nu_{t-s}\{f : f.Y_s \in A\}$. For fixed $h \in C(S,M)$, let $Z_t = Y_t h$. Then Z_t is a Markov process with values in C(S,M) and is precisely the Markov process of the theorem. We claim this is the appropriate generalisation of the construction at the start of Section 2 to the non-linear case.

Example 5. – Suppose S and M are compact Riemannian manifolds and $p > \frac{1}{2}\dim S$, $q > \frac{1}{2}\dim M+1$. Then the Sobolev inequalities [14] give continuous inclusions

$$L^2_p(S) \subset C^{\alpha}(S),\ L^2_q(TM) \subset C^1(TM)$$

for some $\alpha > 0$. Then $L^2_p(S) = H(K)$ and $L^2_q(TM) = H(P)$ for reproducing kernels K on S and P on TM which satisfy (A 4) and (A 5). We take X=0 and note that (A 7) is satisfied, and we obtain a family of measures ν_t on $C(S \times M, M)$ and corresponding Markov process on C(S,M). For different pairs (p,q) we obtain different K and P, and therefore a different process.

Example 6. – Let S,p and K be as above. Suppose now M is a complete finite dimensional Riemannian manifold whose injective radius is bounded away from zero. Then for $q > \frac{1}{2}\dim M+1$, we have the continuous inclusion $L^2_q(TM) \subset C^1(TM)$ (see [4]). If there is an embedding $M \subset V$ satisfying (A 7) and such that the geodesic distance metric on M is uniformly equivalent to that induced from V, then the reproducing kernel P of $L^2_q(TM)$ satisfies (A 5). Take X=0 and apply the theorem.

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