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NONLINEAR STRUCTURES DETERMINED BY MEASURES ON BANACH SPACES

By K. David ELWORTHY

0. INTRODUCTION.

A. A Gaussian measure γ on a separable Banach space E, together with the topological vector space structure of E, determines a continuous linear injection $i : H \to E$, of a Hilbert space H, such that γ is induced by the canonical cylinder set measure of H. Although the image of H has measure zero, nevertheless H plays a dominant role in both linear and nonlinear analysis involving γ, [8], [9], [10]. The most direct approach to obtaining measures on a Banach manifold M, related to its differential structure, requires a lot of extra structure on the manifold : for example a linear map $i_x : H \to T_x M$ for each x in M, and even a subset M_H of M which has the structure of a Hilbert manifold, [6], [7]. In the manifold case it has not been clear how much of this additional structure is really required ; or, slightly reformulated : do certain measures on an infinite dimensional manifold M, together with the differential structure of M, determine any such additional structures ? As a special case of this we can ask whether every diffeomorphism of E which preserves the Gaussian measure γ necessarily maps i(H) to itself, or has derivatives which preserve i(H).

Along similar lines, it is plausible that the Hilbert manifold $L^{2,1}_{x_0}(x)$ of $L^{2,1}$ paths starting at x_0 on a Riemannian manifold X may play a central role for the Wiener measure on the manifold $C_{x_0}(X)$ of continuous paths in X, [6], [7]. If so it should be possible to characterise $L^{2,1}_{x_0}(X)$ in terms of that measure and the differentiable structure of $C_{x_0}(X)$.

Although we do not answer these questions, we show here that any strictly positive Radon measure on a smooth manifold determines some structure : namely a partition of M into subsets invariant under measure preserving diffeomorphisms, and subspaces in the tangent spaces to M invariant under the derivatives of such diffeomorphisms. For infinite dimensional M these are shown to be non-trivial in a class of important cases : and the partition may well be non trivial in general, in infinite dimensions. A concrete consequence is obtained in Corollary 4 A : *the group of diffeomorphisms of an infinite dimensional separable Banach space* E *preserving a given Gaussian measure does not act transitively on* E. This is false for the group of measure class preserving diffeomorphisms : Theorem 1B. Another consequence, concerning group actions, is given in § 3C.

The précise définitions of the invariants may seem rather unnatural : they have been chosen from a wide range of similar definitions simply in order to make the theorems true and to show that non-trivial invariants exist, not because of any obvious intrinsic geometric meaning. A particularly interesting point is that the interpolation K-functors, as described by PEETRE in [13], play an important role in several different places : especially Corollary 2 C and Proposition 4 B. Full proofs and a more detailed discussion will be made available elsewhere.

B. We are concerned with measures on topological spaces : by which we mean positive Borel measures, usually locally finite (or even finite) and strictly positive ; so every point has a neighbourhood of finite measure, and each open set has non-zero measure. Moreover in order that our constructions are non-trivial we shall often have to assume that the measures μ are *tight* i.e. for each Borel set B

$$\mu(B) = \sup\{\mu(K) : K \subset B,\ K \text{ compact}\}.$$

This follows automatically, when μ is finite, if the space is separable and admits a complete metric, see [12], [15]. Recall that a Borel measure is a *Radon* measure if it is locally finite and tight.

Two measures λ, μ on X are *equivalent*, $\lambda \approx \mu$, if they have the same sets of measure zero. If so the Radon-Nikodym derivatives $\frac{d\lambda}{d\mu}$, $\frac{d\mu}{d\lambda}$ are defined, almost everywhere, as measurable functions. This relation between measures seems to be to weak for our purposes (see Theorem 1B) : so for x in X we define λ and μ to be *pointwise equivalent at* x

$$\lambda \approx \mu \quad \text{pointwise at } x$$

if for all neighbourhood bases $\mathcal{U}$ at x, directed by inclusion, and all Borel sets B

$$\underline{\lim}_{U \in \mathcal{U}} \frac{\mu(U \cap B)}{\lambda(U \cap B)} > 0 \quad \text{and} \quad \underline{\lim}_{U \in \mathcal{U}} \frac{\lambda(U \cap B)}{\mu(U \cap B)} > 0$$

where, in the computation of the lower limits, we remplace $\frac{0}{0}$ by 1 and $\frac{r}{0}$ by ∞, if $r > 0$.

For strictly positive measures λ, μ, we see that if λ, μ are orthogonal they are not pointwise equivalent at any point, while, for X first countable, if λ and μ are equivalent they are pointwise equivalent at x iff both Radon-Nikodym derivatives are essentially bounded in some neighbourhood of x.

1. GAUSSIAN MEASURES.

A. Since Gaussian measures furnish our main test bed we quickly give the definition and most relevant properties. For simplicity we consider only strictly positive, mean zero measures.

Let E be a separable real Banach space. A measure γ on E is *Gaussian* if for all continuous linear surjections with finite dimensional range :

$$T : E \to F_T$$

the induced measure $T(\gamma)$ on F_T

$$T(\gamma)(B) = \gamma T^{-1}(B) \qquad B \in \mathrm{Borel}(F_T)$$

is given by

$$T(\gamma)(B) = (2\pi)^{-n/2} \int_B \exp(-\frac{|x|^2}{2})\,dx)$$

where $n = \dim F_T$, and the Lebesgue measure and norm come from some inner product $\langle\ ,\ \rangle_T$ on F_T.

For such a measure γ :

G1 : [4], there is a compact linear injective map $i : H \to E$, of a Hilbert space $(H, \langle\ ,\ \rangle)$ into E such that the inner product $\langle\ ,\ \rangle_T$ in the definition is the quotient inner product under the map $T \circ i : H \to F_T$.

G2 : the image of H, $i(H)$, has γ-measure 0.

G3 : translation, $T_z : E \to E$, by an element z of E preserves sets of measure zero, i.e. $T_z(\gamma) \approx \gamma$, iff z lies in the image of H.

G4 : if $j : E^* \to H$ denotes the adjoint of i then $T_z(\gamma) \approx \gamma$ pointwise at some point iff z lies in the image of E^* by $j \circ i$, in which case $T_z(\gamma) \approx \gamma$ pointwise at every point of E.

G5 : the image of H in E is the intersection of all measurable linear subspaces of E with non-zero measure. Such subspaces have measure 1 (see [2] for a short proof).

B. Given a Gaussian measure γ on E let $\varphi : U \to V$ be a C^1 diffeomorphism of open subsets of E, having the form $\varphi(x) = x + i \circ j \circ \alpha(x)$ where $\alpha : U \to E^*$ is C^1. Then H-H. KUO [10], proved that φ preserves sets of measure zero and its "jacobian"

$$\frac{d\varphi^{-1}(\gamma)}{d\gamma} = U \to R$$

is given by $x \longmapsto |\det D\varphi(x)| \exp\{\frac{1}{2}[-2\alpha(x)(x) - |j \circ \alpha(x)|^2]\}$ (the determinant refers to $D\varphi(x)|H : H \to H$, and is proved to exist).

It follows that

$$\varphi(\gamma|U) \approx \gamma|V \text{ pointwise on } V.$$

Ramer [14] has a stronger version of Kuo's theorem.

When E, and hence H, is infinite dimensional we can follow the construction used by BESSAGA [1], or [3], to show that $H - \{0\}$ is diffeomorphic to H, and for any r in E we can obtain a C^∞ diffeomorphism $\varphi : E \to E$ with $\varphi(0) = v$ and such that

$$\varphi|E - \{0\} \quad : E - \{0\} \to E - \{v\}$$

satisfies the conditions of Kuo's theorem. It follows that still $\varphi(\gamma) \approx \gamma$: although now the pointwise equivalence at v will be lost in general. This proves :

THEOREM 1B. - *Let γ be a strictly positive Gaussian measure on a separable Banach space* E. *Then the group of C^∞ diffeomorphisms preserving γ up to equivalence, acts transitively on* E.

We show below, Corollary 4A, that the theorem is false for infinite dimensional E when equivalence is replaced by pointwise equivalence at all points of E. In any case the theorem does not necessarily imply that measure class preserving diffeomorphisms can behave in a completely abandoned way : for example

Problem. - With the notation of the theorem : does there exist a diffeomorphism $\varphi : E \to E$ with

$$\varphi(\gamma) \approx \gamma$$
$$\text{and } \varphi(i(H)) \cap i(H) = \emptyset \ ?$$

2. TANGENT CONES AND INTERPOLATION FUNCTORS.

A. Let A be a subset of the real Banach space E.

For a point a in the closure $\bar{A}$ of A we shall define the *tangent cone* $TC_a(A)$ to A at a by

$$TC_a(A) = \{v \in E \text{ s.t. } d(a + sv, A) = O(s^2) \text{ as } s \to 0\}.$$

Note that the more natural definition would have $O(s)$ instead of our $O(s^2)$. This would have the advantage of being invariant under C^1 diffeomorphisms, but Corollary 2C below would not hold with that definition. Our construction is easily seen to have the following properties.

TC(i) : If $v \in TC_a(A)$ and $\lambda > 0$ then $\lambda v \in TC_a(A)$.

TC(ii) : If A is convex then so is $TC_a(A)$.

TC(iii) : If $\varphi : U \to V$ is a C^2 map of open sets of Banach spaces, and $\overline{A} \subset U$, then $D\varphi(a)(TC_a(A)) \subset TC_{\varphi(a)}(\varphi(A))$.

From TC(iii) it follows that tangent cones are defined for subsets A of C^2 Banach manifolds M. They then lie in the tangent spaces :

$$TC_a(A) \subset T_aM.$$

B. Let $\vec{E}$ denote a pair of Banach spaces (E_1, E) with a given continuous linear injection $i : E_1 \to E$. As in PEETRE [13], for $0 < t < \infty$ and $v \in E$ define

$$K(t,v) = \inf \{\|v-v_1\| + t\|v_1\|_1 : v_1 \in E_1\},$$

where $\| \ \|$ and $\| \ \|_1$ denote the norms of E and E_1 respectively, and elements of E_1 are identified with their image in E.

For $0 < \theta < 1$ define

$$\vec{E}_{\theta,\infty} = \{v \in E : K(t,v) = O(t^\theta)\}$$

and set $$\|v\|_{\theta,\infty} = \sup_{t > o} \frac{K(t,v)}{t^\theta} \quad \text{if } v \in \vec{E}_{\theta,\infty}.$$

This is a special case of the more general construction of K-functors described in [13]. The properties we need are

K1 : $\vec{E}_{\theta,\infty}, \ | \ |_{\theta,\infty}$ is a Banach space.

K2 : The map i factorizes by continuous linear maps

$$E_1 \xrightarrow{\alpha} \vec{E}_{\theta,\infty} \xrightarrow{\beta} E.$$

K3 : If i was compact then so are both α and β.

C. For $\vec{E}$ as above, let $B_1(x ; r)$ denote the closed ball

$$\{y : \|x-y\|_1 \leqslant r\}$$

about x, radius r, in E_1.

Define the *contact space* $\tau(E_1,E)$ of $\vec{E}$ by

$$\tau(E_1,E) = TC_o(i[B_1(0 ; 1)]).$$

PROPOSITION 2C. - *As subsets of* E

$$\tau(E_1, E) = \vec{E}_{\frac{1}{2}, \infty} .$$

From property K3 and the fact that any compact subset of E lies in the image of the unit ball of Banach space mapped into E by a compact linear map, the proposition yields :

COROLLARY 2C. - *If* K *is a compact subset of the infinite dimensional Banach space* E *then for all* $a \in K$

$$\text{Linear span } TC_a(K) \neq E.$$

[However $TC_a(K)$ can certainly be dense in E.]

3. INFINITESIMAL PROPERTIES OF MEASURES.

A. Let μ be a strictly positive, locally finite, measure on a metric space (M,d). We say that the Borel subset A of M *infinitesimally supports* μ *at the point* a of M, $A \in \operatorname{Supp}(\mu ; a)$, if for all $r > 0$ $\dfrac{\mu(B(a ; t) - A)}{\mu(B(a ; rt^2))} \to 0$ as $t \to 0$ where $B(a ; t)$ denotes the closed ball about a, radius t. It is easy to see that this definition depends only on the local Lipschitz class of the metric d, and on the pointwise equivalence class at a of μ.

PROPOSITION 3A. - *If* μ *is a Radon measure, (e.g. if* (M,d) *is complete and if* $A \in \operatorname{Supp}(\mu ; a)$*) then there is a compact set* K *with*

$$K \subset A \cup \{a\}$$

and $$K \in \operatorname{Supp}\{\mu ; a\}.$$

Now suppose that M is a separable C^2 Banach manifold and that the metric d is in the local Lipschitz class determined by the differentiable structure. For a in M define the *tangent cone*, $\tau c_a(\mu) = \cap \{TC_a(A) : A \in \operatorname{Supp}(\mu ; a)\}$.

Thus $\tau c_a(\mu) \subset T_aM$.

THEOREM 3A :

(i) For every strictly positive Radon measure μ *on the infinite dimensional metrizable* C^2 *Banach manifold* M *the tangent cone to* μ *at a general point a satisfies.*

$$\text{Linear span } \tau c_a(\mu) \neq T_aM.$$

(ii) Let $\varphi : M \to M$ *be a* C^2 *diffeomorphism such that*

$\varphi(\mu) \approx \mu$ *pointwise at* $\varphi(a)$. *Then*

$$T_a\varphi[\tau c_a(\mu)] = \tau c_{\varphi(a)}(\mu).$$

Part (i) follows from Corollary 2C and Proposition 3A, and (ii) is straightforward.

B. We have yet to show that $\tau c_a(\mu)$ can ever be larger than $\{0\}$ when E is infinite dimensional. This can be done usinge a more geometric differential invariant of measures : For a point a of a C^2 manifold M and a strictly positive measure μ on M define

$$Q_a(\mu) \subset T_aM$$

to consist of those tangent vectors v for which there exists a C^2 vector field ξ on M with $\xi(a) = v$ suchat that there is a neighbourhood V of a and positive constants ε, α satisfying

(i) the flow $\sigma : V \times (-\varepsilon,\varepsilon) \to M$ of ξ is defined.
(ii) there is a base $\mathcal{B}$ for the neighbourhood system of a in V with

$$\mu(\sigma_t(B)) \geqslant \alpha\mu(B) \text{ for all } 0 < t < \varepsilon,\ B \in \mathcal{B}.$$

THEOREM 3B. - $Q_a(\mu) \subset \tau c_a(\mu)$.

COROLLARY 3B1. - $Q_a(\mu)$ *does not span* T_aM, *if* M *is infinite dimensional and separable.*

COROLLARY 3B2. - *For a Gaussian measure* γ *on a separable Banach space* E,

$$\tau c_a(\gamma) \neq \{0\} \qquad \text{all } a \in E.$$

In 3B2 we have $i\circ j(E^*) \subset Q_a(\gamma) \subset \tau c_a(\gamma)$, each $a \in E$.

C. As an application of the above : *if* $G\times M \to M$ *is a* C^2 *action of a Banach Lie group* G *on a metrizable Banach manifold* M *which preserves, up to pointwise equivalence, some strictly positive, Radon measure on* M, *then for each* a *in* M *the derivative map at the identity*

$$T_eG \to T_aM$$

obtained from $g \longmapsto g.a$ *is compact.*

However it semms likely that the above is true for group actions which only preserve the measure up to equivalence. For the linear case with G a group of translations see [16].

4. ORDERING INDUCED BY A MEASURE.

A. Let μ be a strictly positive measure on a metric space (M,d). For x and y in M write

$$x < y, \quad \text{if} \quad \underline{\lim}_{s \to 0} \frac{\mu(B(y\,;\,s))}{\mu(B(x,rs))} = 0, \text{for all} \quad r > 0.$$

If neither $x < y$ nor $y < x$ write $x \sim y$. It is easy to see that this defines an equivalence relation on M.

PROPOSITION 4A. - *Let* $f : M \to M$ *be a homeomorphism which is locally bi-Lipschitz and satisfies* $f(\mu) \approx \mu$ *pointwise on* M. *Then* $x \sim f(x)$ *all* $x \in M$.

From the proposition, in order to show that the group of such homeomorphisms f of M does not act transitively on M it suffices to show that the equivalence relation $\sim$ is non-trivial. Possibly this is true for a general class of measures on M when M is infinite dimensional. The proof of the following theorem depends on the fact that Gaussian measures are convex in the sense of BORELL [2] : in fact the theorem is true for arbitrary convex measures.

THEOREM 4A. - *For a Gaussian measure* γ *on the Banach space* E *; if* $\overline{0}$ *denotes the equivalence class of* 0*, we have*

$$\overline{0} \subset \tau c_0(\gamma).$$

This combines with Theorem 3A to give the required non-triviality, whence :

COROLLARY 4A. - *Let* E *be a separable infinite dimensional Banach space and* γ *a Gaussian measure on* E. *Then the group of locally bi-Lipschitz homeomorphisms of* E *with* $f(\gamma) \approx \gamma$ *pointwise on* E *does not act transitively on* E.

B. The problem remains of characterizing the orbits of 0 under the group of diffeomorphisms of Corollary 4A, (or under the group of measure preserving diffeomorphisms) or perhaps a simpler problem is to characterize the equivalence class $\overline{0}$ of Theorem 4A. The following is suggestive, at least of the type of characterizations which may be true.

PROPOSITION 4B. - *For a Gaussian measure* γ *on* E*, with corresponding maps* $E^* \xrightarrow{j} H \xrightarrow{i} E$ *we have*

$$\tau(E^*, H) \subset \overline{0}$$

In fact for all $z \in H$ *and* $x \in \tau(E^*,H)$ *we have* $z \sim z+x$.

[We identify points of E^* and H with their images in E].

Proof. - Let $| \ |$ denote the norm of H, and $\| \ \|$ that of E.

Since $x \in \tau(E^*,H)$ there is a function $e : (0,1) \to E^*$ and a constant k with

$$|x-e(s)| < \frac{1}{2} s$$

and
$$\|e(s)\|_{E^*} \leqslant k/s.$$

By the change of variable formula, § 1B, for e = e(s)

$$\begin{aligned}\gamma(B(z\ ;\ s)) &= \int_{B(z+e\ ;\ s)} \exp(e(y) - \frac{1}{2}|e|^2)d\gamma(y) \\ &= \exp(e(z) + \frac{1}{2}|e|^2) \int_{B(z+e\ ;\ s)} \exp(e(y-e-z))d\gamma(y) \\ &\geqslant \exp(e(z) + \frac{1}{2}|e|^2) \exp\ (-s\|e\|_{E^*})\gamma(B(z+e\ ;\ s)) \\ &\geqslant \exp(<x,z> - <x-e,\ z> + \frac{1}{2}|e|^2 - k)\gamma(B(z+e\ ;\ s))\end{aligned}$$

Now $B(z+x\ ;\ \frac{1}{2} s) \subset B(z+e\ ;\ s)$, so we have

$$\underline{\lim}_{s \to 0} \frac{\gamma(B(z\ ;\ s))}{\gamma(B(z+x\ ;\ \frac{1}{2} s))} \geqslant \exp\ (<x,z> - k) > 0$$

whence $\quad z \leqslant z + x.$

substitution shows that also

$$z + x \leqslant (z+x) - x$$

giving $\quad z \sim z+x$ as required.

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