Przemysław Wojtaszczyk

Approximation properties and universal Banach spaces

Mémoires de la S. M. F., tome 31-32 (1972), p. 395-398

<http://www.numdam.org/item?id=MSMF_1972__31-32__395_0>
APPROXIMATION PROPERTIES AND UNIVERSAL BANACH SPACES

by

Przemyslaw WOJTASZCZYK

1. - In this talk we are dealing with two kinds of concepts.

The first one, approximation properties, has its origin in the concepts of Schauder basis [10] and metric approximation property of Grothendieck [2]. By "approximation property" we mean the answer to the question in what a way we can approximate the identity operator on a Banach space $X$ by finite dimensional operators. The second one, universality, has its origin in the classical Banach-Mazur theorem on the universality of the space of continuous functions on the Cantor discontinuum [1]. The question is, to find for a given class of Banach spaces a space which contains (in a nice way) any element of the given class.

2. - Definitions.

DEFINITION 1. - A Banach space $X$ has the bounded approximation property, shortly BAP (resp. unconditional bounded approximation property, shortly UBAP) iff there exists a sequence of finite dimensional operators $A_n : X \rightarrow X$, $n = 1, 2, \ldots$, such that for each $x \in X$ $x = \sum_{n=1}^{\infty} A_n (x)$ (and the series is unconditionally convergent).

Those concepts are modifications of the metric approximation property of Grothendieck [2].

DEFINITION 2. - A Banach space $X$ has a (unconditional) basis of finite dimensional subspaces iff there exists a sequence $(X_n)_{n=1}^{\infty}$ of finite dimensional subspaces of $X$ such that for each $x \in X$ we have a unique decomposition $x = \sum_{n=1}^{\infty} x_n$ where $x_n \in X_n$ (and the series is unconditionally convergent). If moreover $\dim X_n = 1$ for $n = 1, 2, \ldots$ the Banach space $X$ has the Schauder basis (resp. unconditional Schauder basis).

It is easily seen that those concepts are in fact "approximation properties".

DEFINITION 3. - Let us have a class $\mathcal{B}$ of Banach spaces. A space $X$ is complementably universal for the class $\mathcal{B}$ iff for each $Y \in \mathcal{B}$ there exists in $X$ a complemented subspace $Y_1$ which is isomorphic to $Y$. 
3. - About this concepts the following results are proved.

**THEOREM 1.** - [9] A Banach space $X$ has UBAP iff $X$ is isomorphic to a complemented subspace of a Banach space with the unconditional basis of finite dimensional subspaces. Moreover there exists one space $U_{fd}$ with the unconditional basis of finite dimensional subspaces such that any $X$ having UBAP is isomorphic to a complemented subspace of $U_{fd}$. The space $U_{fd}$ is unique up to isomorphism among spaces with UBAP.

**Remark 1.** - Some constructions used in the proof of theorem 1 has applications in simultaneous extensions of continuous functions (cf. [9]).

**THEOREM 2.** - A Banach space $X$ has BAP iff $X$ is isomorphic to a complemented subspace of a Banach space with the Schauder basis. Moreover there exists one space $B$ with the Schauder basis which is complementably universal for the class of all Banach spaces with BAP. The space $B$ is unique up to isomorphism among spaces with BAP.

This theorem was proved independently in [4] and [8] using results of [9] and [7].

**Remark 2.** - In [3] among others is proved the following fact.

There exists a family of separable Banach spaces $C_p$, $1 \leq p \leq \infty$ such that for any Banach space $X$ such that $X^*$ has BAP and any $p$ the space $X \oplus C_p$ has the Schauder basis.

**Remark 3.** - The space $B$ was constructed by different ways in [7], [9] and [5]. The proof that spaces constructed in this papers are exactly the same follows from [8]. The construction of Kadec [5] has interesting applications to the theory of preduals of $L_1$ (cf. [9] and [12]).

**Remark 4.** - Many interesting results concerning various approximation properties are proved in [4].

4. - To finish this talk we are going to state some unsolved problems.

**Problem 1.** - Is any Banach space with UBAP isomorphic to a complemented subspace of a Banach space with an unconditional basis?

**Problem 2.** - Find an example of a separable Banach space with UBAP not having an unconditional basis.
It is probably that such an example can be found among spaces constructed in [6].

Problem 3. - Prove that in a reflexive Banach space $X$ with BAP there exists a sequence of finite dimensional projections $P_n$ such that $P_n(X) \subseteq P_{n+1}(X)$ for $n = 1, 2, \ldots$, and $P_n(x) \to x$ for any $x \in X$.

Problem 4. - Does any reflexive Banach space with a basis of finite dimensional subspaces have a Schauder basis?


Problem 5. - [7]. Does there exist a separable Banach space complementably universal for the class of separable Banach spaces?

The solution of this problem would have important consequences connected with "basis problem" and "approximation problem" (cf. [11] p. 386).

BIBLIOGRAPHIE


Institute of Mathematics of the Polish Academy of Sciences
Sniadeckich 8
VARSOVIE 1
(Pologne)