ELENA STROESCU

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A DILATATION THEOREM FOR OPERATORS ON BANACH SPACES

by

Elena STROESCU

Introduction.

Let $R^+$ be the set of all non-negative real numbers and $\mathcal{B}(\mathbb{X})$ the Banach algebra of all linear bounded operators on a Banach space $\mathbb{X}$. In this paper, we present a dilation theorem by which an object $\{\mathbb{Y}, \Gamma, U\}$ dilates into $\{\mathbb{Y}, \varphi, P, \tilde{\tau}, V\}$; where $\mathbb{Y}$ and $\tilde{\mathbb{Y}}$ are Banach spaces, $\varphi$ is a bicontinuous isomorphism of $\mathbb{Y}$ into $\tilde{\mathbb{Y}}$, $P$ a continuous projection of $\tilde{\mathbb{Y}}$ onto $\varphi(\mathbb{Y})$, $\Gamma = \{T_t\}_{t \in R^+} \subseteq \mathcal{B}(\mathbb{Y})$ and $\tilde{\Gamma} = \{\tilde{T}_t\}_{t \in R^+} \subseteq \mathcal{B}(\tilde{\mathbb{Y}})$ are operator semi-groups, $U$ is a $\mathcal{B}(\mathbb{X})$-valued linear map on an arbitrary algebra $\mathcal{A}$ estimated by a submultiplicative functional and $V$ a $\mathcal{B}(\tilde{\mathbb{Y}})$-valued representation on $\mathcal{A}$ such that $V_a\tilde{T}_t = \tilde{T}_t V_a$, for every $a \in \mathcal{A}$ and $t \in R^+$. This theorem is an extension of some previous results (see [8], [9]); it has arisen from the concern to characterize restrictions of spectral operators on invariant subspaces (or operators which dilate in spectral operators) by a map replacing the spectral representation.

Notations.

Throughout the following $\mathbb{C}$ denotes the complex plane; $\mathbb{N} = \{0,1,2,\ldots\}$; $\mathcal{A}$ an arbitrary algebra over $\mathbb{C}$ with unit element denoted by 1; $K$ a submultiplicative functional of $\mathcal{A}$ into $R^+$ (i.e. $K_{ab} \leq K_a K_b$ for any $a, b \in \mathcal{A}$) such that $K_1 = 1$; $\mathbb{X}$ a Banach space over $\mathbb{C}$; $\mathcal{B}(\mathbb{X})$ the Banach algebra of all linear bounded operators on $\mathbb{X}$ over $\mathbb{C}$; $I$ the identity operator. Let $T_1, T_2 \in \mathcal{B}(\mathbb{X})$ two commuting operators; then one says that $T_1$ is quasi-nilpotent equivalent with $T_2$ and denotes $T_1 \sim T_2$, if $\lim_{n \to \infty} \|(T_1 - T_2)^n\|^{1/n} = 0$. A family of operators $\{T_t\}_{t \in R^+} \subseteq \mathcal{B}(\mathbb{X})$ is called semi-group if $T_0 = I$ and $T_{t+s} = T_t T_s$ for any $t$ and $s \in R^+$.

THEOREM. - Let $\{T_t\}_{t \in R^+} \subseteq \mathcal{B}(\mathbb{X})$ be a semi-group of operators and $U : \mathcal{A} \to \mathcal{B}(\mathbb{X})$ a linear map such that $U_1 = I$, $\|U_a\| \leq K_a$, for any $a \in \mathcal{A}$.

Then, there exists a Banach space $\tilde{\mathbb{X}}$, an isometric isomorphism $\varphi$ of $\mathbb{X}$ into $\tilde{\mathbb{X}}$, a continuous projection $P$ of $\tilde{\mathbb{X}}$ onto $\varphi(\mathbb{X})$, a semi-group $\tilde{\Gamma} = \{\tilde{T}_t\}_{t \in R^+} \subseteq \mathcal{B}(\tilde{\mathbb{X}})$ and a representation $V : \mathcal{A} \to \mathcal{B}(\tilde{\mathbb{X}})$ such that:
(o) \( \| P \| = 1 \); \( \| T_t \| = \| T^{-1} \| \); for any \( t \in R^+ \); \( V_1 = \tilde{I} \) and 
\( \| V_{t} \| \leq K \); for any \( \alpha \in G \).

(ii) \( \tilde{T}_{\alpha} T_{\alpha} \tilde{T}_{\alpha}^{-1} = \tilde{T}_{\alpha} T_{\alpha} \tilde{V}_{\alpha} \), for any \( \alpha \in G \), \( \tau \in R^+ \).

(iii) \( \tilde{f} \) is the closed vector space spanned by \( \{ \tilde{T}_{\alpha} \tilde{V}_{\alpha} \phi(x); \alpha \in G, t \in R^+, x \in \mathfrak{X} \} \).

(iv) Let \( s \in R^+ \); then we have the following equivalences:

1° \( T_s \phi(x) = \phi(T_s x) \), for any \( x \in \mathfrak{X} \);

2° \( T_s \tilde{V}_a \phi(x) = \tilde{V}_a \phi(x) \), for any \( \alpha \in G \), \( x \in \mathfrak{X} \);

3° \( T_s \tilde{U}_{\alpha} = \tilde{U}_{\alpha} \), for any \( \alpha \in G \).

(v) Let \( b \in G \); then \( \tilde{V}_b \phi(x) = \phi(U_b x) \), for any \( x \in \mathfrak{X} \) is equivalent with \( U_{a b} = U_a U_b \), for any \( a \in G \).

(vi) Let \( \sigma \in R^+ \) and \( \beta \in G \) commuting with all the elements of \( G \) such that \( U_{a \beta} = U_a U_{\beta} \), \( T_{\alpha} U_{\beta} = U_a T_{\alpha} \), for any \( a \in G \); then \( \| (T_{\alpha} - U_{\beta})^n \| = \| (T_{\alpha} - U_{\beta})^n \| \), for every \( n \in N \).

**Proof:**

A) Let us consider the Cartesian product \( R^+ \times G = \prod_{(t,a) \in R^+ \times G} x_{(t,a)} \)

and the direct sum \( \mathfrak{X}(R^+ \times G) = \bigoplus_{(t,a) \in R^+ \times G} x_{(t,a)} \), where \( x_{(t,a)} = \mathfrak{X} \), for every \( t \in R^+, a \in G \).

An element \( y \in \mathfrak{X}(R^+ \times G) \) is a family \( (y_{(t,a)}(t,a)) \in R^+ \times G \) (many times we write \( y = (y_{(t,a)}(t,a)) \) of components \( y_{(t,a)} = y_{t,a} \in \mathfrak{X} \), for every \( t \in R^+, a \in G \). If \( y \in \mathfrak{X}(R^+ \times G) \), then \( (y_{(t,a)} = y_{t,a} \neq 0 \) for only a finite number of elements \( (t,a) \in R^+ \times G \).

Let us consider a map:

\( \Theta = (\Theta_{t,a}(t,a))_{t,a} \in R^+ \times G \) of \( \mathfrak{X}(R^+ \times G) \) into \( \mathfrak{X}(R^+ \times G) \)

defined by

\( \Theta y = (T_{t} \tilde{V}_{s,b} T_{s} U_{a} y_{s,b} T_{t,a} \), for every \( y \in \mathfrak{X}(R^+ \times G) \).

It is easy to see that \( \Theta \) is a well defined linear map. Then, we denote by \( \mathfrak{Y} \) the range of \( \Theta \) and by \( \tilde{\mathfrak{Y}} \) an arbitrary element of \( \mathfrak{Y} \).
For every \( \hat{y} \in \hat{X} \), we have:
\[
\sigma^{-1}({\hat{y}}) = \{ y \in (\mathbb{R}^+ \times \mathbb{A}) : \sigma y = \hat{y} \}.
\]
We define a function \( \omega : \hat{X} \to \mathbb{R}^+ \) by
\[
\omega(\hat{y}) = \inf_{\hat{y} \in \Theta^{-1}({\hat{y}})} \sum_{s,b} \| T_s \| K_b \| y_{s,b} \|,
\]
for every \( \hat{y} \in \hat{X} \); let us prove that \( \omega \) is a norm on \( \hat{X} \). Let \( \mu \in \mathbb{C} \) be non-zero, \( \hat{y} \in \hat{X} \) and \( \Delta(\mu \hat{y}) = \{ u \hat{y} : y \in \Theta^{-1}({\hat{y}}) \} \); then we show that \( \Theta^{-1}((\mu \hat{y})) = \Delta(\mu \hat{y}) \). Indeed, let \( u \hat{y} \in \Delta(\mu \hat{y}) \), i.e. \( y \in \Theta^{-1}({\hat{y}}) \), then \( \mu \hat{y} = (u \mu T_s) \in T_s u \hat{y} \), hence \( \mu y \in \Theta^{-1}((\mu \hat{y})) \). Let now \( z \in \Theta^{-1}((\mu \hat{y})) \), i.e. \( \Theta z = \mu \hat{y} \) or \( \Theta \frac{z}{\mu} = \hat{y} \), hence \( y' = \frac{z}{\mu} \in \Theta(\hat{y}) \) and \( z = \mu y' \in \Delta(\mu \hat{y}) \). Then \( \omega(\mu \hat{y}) = \inf_{y \in \Theta^{-1}((\mu \hat{y}))} \sum_{s,b} \| T_s \| K_b \| z_{s,b} \| = \inf_{z \in \Delta(\mu \hat{y})} \sum_{s,b} \| T_s \| K_b \| z_{s,b} \| = \inf_{z \in \Theta(\hat{y})} \sum_{s,b} \| T_s \| K_b \| z_{s,b} \| = |u| \inf_{y \in \Theta^{-1}({\hat{y}})} \sum_{s,b} \| T_s \| K_b \| y_{s,b} \| = |u| \omega(\hat{y}) \), i.e. \( \omega(\mu \hat{y}) = |u| \omega(\hat{y}) \), hence one deduces also that \( \omega(0) = 0 \). Then, for \( \mu = 0 \) we have \( \omega(0 \hat{y}) = 0 \) and \( \omega(\hat{y}) = 0 \), for any \( \hat{y} \in \hat{X} \). Hence \( \omega(\mu \hat{y}) = |u| \omega(\hat{y}) \), for any \( \hat{y} \in \hat{X} \), \( \mu \in \mathbb{C} \).

Let \( \hat{y}^1, \hat{y}^2 \in \hat{X} \) and
\[
\Delta(\hat{y}^1 + \hat{y}^2) = \{ y^1 + y^2 : y^1 \in \Theta^{-1}((\hat{y}^1)), y^2 \in \Theta^{-1}((\hat{y}^2)) \},
\]
then obviously we have \( \Delta(\hat{y}^1 + \hat{y}^2) \subseteq \Theta^{-1}((\hat{y}^1 + \hat{y}^2)) \) and
\[
\omega(\hat{y}^1 + \hat{y}^2) = \inf_{y \in \Theta^{-1}((\hat{y}^1 + \hat{y}^2))} \sum_{s,b} \| T_s \| K_b \| y_{s,b} \| \leq \inf_{z \in \Delta(\hat{y}^1 + \hat{y}^2)} \sum_{s,b} \| T_s \| K_b \| z_{s,b} \| = \inf_{y^1 \in \Theta^{-1}((\hat{y}^1))} \sum_{s,b} \| T_s \| K_b \| y^1_{s,b} + \inf_{y^2 \in \Theta^{-1}((\hat{y}^2))} \sum_{s,b} \| T_s \| K_b \| y^2_{s,b} \| \leq \inf_{y^1 \in \Theta^{-1}((\hat{y}^1))} \sum_{s,b} \| T_s \| K_b \| y^1_{s,b} + \inf_{y^2 \in \Theta^{-1}((\hat{y}^2))} \sum_{s,b} \| T_s \| K_b \| y^2_{s,b} \| \leq \inf_{y^1 \in \Theta^{-1}((\hat{y}^1))} \sum_{s,b} \| T_s \| K_b \| y^1_{s,b} + \inf_{y^2 \in \Theta^{-1}((\hat{y}^2))} \sum_{s,b} \| T_s \| K_b \| y^2_{s,b} \| = \omega(\hat{y}^1) + \omega(\hat{y}^2) \), for all \( \hat{y}^1, \hat{y}^2 \in \hat{X} \).

Then, from the definition of \( \omega \), for every \( \hat{y} \in \hat{X} \), we have:

1) \( \omega(\hat{y}) \leq \sum_{s,b} \| T_s \| K_b \| y_{s,b} \| \), for any \( y \in \Theta^{-1}((\hat{y})) \) and
2) \( \| \hat{T}_t(a) \| \leq \| T_a \| K_{\omega}(\hat{y}) \), for \( t \in \mathbb{R}^+ \), \( a \in \mathbb{A} \).

Hence \( \omega \) is a norm on \( \hat{X} \); we denote by \( \hat{X} \) the \( \omega \)-completion of \( \hat{X} \) and the norm on \( \hat{X} \) also by \( \omega \).
B) We define an isomorphism $\varphi$ of $x$ into $\mathbb{R}^+ \times a$ by $\varphi(x) = (T \cup x)_t, a = (T_s \cup_{s,b} a, f, a)$, for every $x \in x$.

Applying 1) and 2) we get

$$||x|| \leq \omega(\varphi(x)) \leq ||x||$$

for any $x \in x$.

Therefore $\varphi$ is an isometric isomorphism of $x$ into $\hat{x}$.

We define a projection $P$ of $\hat{x}$ onto $\varphi(x)$, by $P_{\hat{y}} = \varphi(\hat{y}_{0,1})$, for every $\hat{y} \in \hat{x}$.

Applying 3) and 2), we get $\omega(P_{\hat{y}}) = \omega(\varphi(\hat{y}_{0,1})) \leq ||\hat{y}_{0,1}|| \leq \omega(\hat{y})$, i.e.

$$\omega(P_{\hat{y}}) \leq \omega(\hat{y})$$

for any $\hat{y} \in \hat{x}$. Hence, $P$ can be extended by continuity to a continuous projection of $\hat{x}$ onto $\varphi(x)$, that will be denoted by the same symbol.

Let now $\tau \in \mathbb{R}^+$; then for every $\hat{y} \in \hat{x}$ we put

$$\tilde{T}_\tau \hat{y} = (T_t \cup_{s,b} T_{s+\tau} \cup_{s,b} y_{s,b}, b, a) = (T_t \cup_{s,b} y_{s-b, t, a} = (T_t \cup_{s,b} y_{s-b, t, a}$$

where we denote $s + \tau = \sigma$; $z_{\sigma,b} = y_{s-b, b}$ for $0 \geq \tau$ and $z_{\sigma,b} = 0$, for $0 \leq \sigma < \tau$, with $b \in a$.

We see easily that $\tilde{T}_\tau$ is a well defined linear map of $\hat{x}$ into $\hat{x}$.

Let us prove that also it is continuous.

For every $\hat{y} \in \hat{x}$, denoting $\Delta(\tau, \hat{y}) = \{\tilde{T}_\tau \hat{y} \in \hat{x} \mid \mathbb{R}^+ \times a; \tilde{T}_\tau \hat{y} = y_{\sigma-b, b} \text{ for } \sigma \geq \tau \text{ and } z_{\sigma,b} = 0 \text{ for } 0 \leq \sigma < \tau, b \in \mathbb{R}, \ y \in \mathbb{R} \} \}$, we see that

$\Delta(\tau, \hat{y}) \subset \mathbb{R}^+ \times a \{(T_t \cup_{s,b} y_{s-b, b}) \text{ for } 0 \leq \sigma < \tau, b \in \mathbb{R}, \ y \in \mathbb{R} \} \}$. Then, we have

$$\omega(\tilde{T}_\tau \hat{y}) = \inf_{\tilde{T}_\tau \hat{y} \in \Delta(\tau, \hat{y})} \sum_{\sigma, b} ||T_\sigma|| K_b ||\tilde{T}_\sigma \hat{y}|| = \inf_{\tilde{T}_\tau \hat{y} \in \mathbb{R}^+ \times a \{y \in \mathbb{R} \}} \sum_{\sigma, b} ||T_\sigma|| K_b ||y_{s-b, b}||$$

Thus, for every $\tau \in \mathbb{R}^+, \tilde{T}_\tau$ can be extended by continuity to an element of $\mathbb{R}(x)$, that will be denoted by the same symbol. Then, we see easily that

$$P_{\tilde{T}_\tau} \varphi(x) = \varphi(T_t \cup_{s,b} x)$$

for any $x \in x$.
Hence \[ \| T^x \| = \omega(\phi(T^x)) = \omega(P_T^x \phi(x)) \leq \omega(P_T^x \phi(x)) \leq \| T^\infty \| \omega(\phi(x)) = \| T^\infty \| \| x \| , \]
i.e.

6) \[ \| T^x \| \leq \| T^\infty \| \| x \| , \]
for any \( x \in \mathcal{X} \). At last, we see easily that \( \{ T^x \}_{x \in R^+} \) is a semi-group of operators, that we denote by \( \tau \).

C) Let us define a representation \( \mathcal{V} \). Let \( a \in \mathcal{Q} \); then for every \( \gamma \in \mathcal{X} \), we put

\[ \mathcal{V}_a \gamma = (T_t \sum_{s,b} T_s U^a_{s,b} y_{s,b}) t, a = (T_t \sum_{s,c} T_s U^a_{s,c} y_{s,b}) t, a = \]
\[ = (T_t \sum_{s,c} T_s U^a_{s,c}) t, a = \Theta u = \hat{u} \in \hat{\mathcal{X}} , \] where
\[ \mathcal{Q} = \{ b \in \mathcal{Q} \mid a b = c \} \quad \text{and} \quad u_{s,c} = \sum_{s,c} y_{s,b} , \]
for \( s \in R^+, c \in \mathcal{Q} \).

The map \( \mathcal{V}_a : \hat{\mathcal{X}} \to \hat{\mathcal{X}} \) is well defined. Indeed, let \( \hat{\gamma}^1 = \hat{\gamma}^2 \in \hat{\mathcal{X}} \); then there exists \( \gamma^1, \gamma^2 \in \mathcal{X}(R^2 \times \mathcal{X}) \) such that \( \hat{\gamma}^1 = \Theta \gamma^1 \) and \( \hat{\gamma}^2 = \Theta \gamma^2 \), hence

\[ T_t \sum_{s,b} T_s U^a_{s,b} y_{s,b} = T_t \sum_{s,b} T_s U^a_{s,b} y_{s,b} \]
for any \( t \in R^+, a \in \mathcal{Q} \).

Then, \( T_t \sum_{s,b} T_s U^a_{s,b} y_{s,b} = T_t \sum_{s,b} T_s U^a_{s,b} y_{s,b} \), for \( t \in R^+ \) and \( a^I = a \in \mathcal{Q} \) with \( a \in \mathcal{Q} \). We see easily that for every \( a \in \mathcal{Q} \), \( \mathcal{V}_a : \hat{\mathcal{X}} \to \hat{\mathcal{X}} \) is a linear map and \( \mathcal{V}_a \gamma = \gamma \), for any \( \gamma \in \mathcal{X} \). Moreover, \( \mathcal{V} : \mathcal{Q} \to \mathcal{L}(\hat{\mathcal{X}}) \) is a representation (see [4]); for a vector space \( \mathcal{X} \), \( \mathcal{L}(\mathcal{X}) \) denotes the algebra of all linear maps of \( \mathcal{X} \) into \( \mathcal{X} \). Now, we prove that, \( \mathcal{V}_a : \hat{\mathcal{X}} \to \hat{\mathcal{X}} \) is continuous, for every \( a \in \mathcal{Q} \). Let \( a \in \mathcal{Q} \), \( \gamma \in \mathcal{X} \) and \( \Delta(a, \gamma) = \{ u \in \mathcal{X}(R^2 \times \mathcal{Q}) \mid u_{s,c} = \sum_{s,c} y_{s,b} , y \in \Theta^{-1}(\{ \gamma \}) \} \), then we see

\[ \Delta(a, \gamma) = \Theta^{-1}(\{ \mathcal{V}_a \gamma \}) \]. Therefore, we have :

\[ \omega(\mathcal{V}_a \gamma) = \inf_{u \in \Delta(a, \gamma)} \sum_{s,c} T_s K_b u_{s,c} \leq \]
\[ \leq \inf_{u \in \Delta(a, \gamma)} \sum_{s,c} T_s K_b u_{s,c} = \inf_{y \in \Theta^{-1}(\{ \gamma \})} \sum_{s,c} T_s K_b y_{s,b} \]
\[ \leq \inf_{y \in \Theta^{-1}(\{ \gamma \})} \sum_{s,b} T_s K_a y_{s,b} \]
\[ \leq \inf_{y \in \Theta^{-1}(\{ \gamma \})} \sum_{s,b} T_s K_a y_{s,b} = K_a \inf_{y \in \Theta^{-1}(\{ \gamma \})} \sum_{s,b} T_s K_b y_{s,b} = K_a \omega(\gamma) ; \]
i.e. for every \( a \in \mathcal{Q} \) we get

7) \[ \omega(\mathcal{V}_a \gamma) \leq K_a \omega(\gamma) , \text{for any } \gamma \in \mathcal{X} . \] Hence, \( \mathcal{V}_a \) can be extended by continuity to an element of \( \mathcal{B}(\hat{\mathcal{X}}) \) that will be denoted by \( \mathcal{V}_a \), for every \( a \in \mathcal{Q} \).
Thus, (0) is completely proved. The property (i) is immediate, since for every $a \in \mathbb{C}$ and $t \in \mathbb{R}^+$, we have $\mathbb{T}_t V_\alpha \hat{y} = (T_t U_{a+b} V_{s,b} t, a) = V_{\alpha} \mathbb{T}_t \hat{y}$, for any $\hat{y} \in \mathbb{F}$. Using the definitions of $\phi$, $P$, $V_\alpha$ and $\mathbb{T}_t$, for $a \in \mathbb{C}$, $t \in \mathbb{R}^+$, we obtain immediately (ii), (iii) and (v).

D) Let us prove (iv). From $\mathbb{T}_s \phi(x) = (T_s U_{a} x)_{t,a}$ and $\phi(T_s x) = (T_s U_{a} T_s x)_{t,a}$, we see that $1^o$ and $3^o$ are equivalent.

Now, choosing $a = 1$ in $2^o$, and using $P \mathbb{T}_t \phi(x) = \phi(T_t x)$ for $t \in \mathbb{R}^+$, $x \in \mathbb{I}$ (see (ii)), we get $1^o$.

Conversely, taking into account of (ii) and writing $1^o$ with $U_{a} x$ instead of $x$, for $a \in \mathbb{C}$, we get $2^o$.

At last, we show (vi). Let $\sigma \in \mathbb{R}^+$, and $b \in \mathbb{C}$, as in the assumption, also let $n \in \mathbb{N}$ and $\mathbb{F} \in \mathbb{F}$; then, we write:

$$(\mathbb{T}_n - V_{\beta})^n \mathbb{F} = \sum_{k=0}^{n} (-1)^{n-k} (k) \mathbb{T}_n \mathbb{F} =$$

$$= \sum_{k=0}^{n} (-1)^{n-k} (k) \mathbb{F} (T_s U_{a+b} T_n U_{s,b} y, ab)_{t,a} = \mathbb{F} = \mathbb{F} \in \mathbb{F} ,$$

where $v$ is defined by

$$(T_n - V_{\beta})^n \mathbb{F} = \sum_{k=0}^{n} (-1)^{n-k} (k) T_n U_{s,b} y, ab)_{t,a} = \mathbb{F} = \mathbb{F} \in \mathbb{F} ,$$

and $b \in \mathbb{C}$.

Denoting by $\Delta(\sigma, \beta, n, \mathbb{F}) = \mathbb{F}^{-1}((\mathbb{T}_n - V_{\beta})^n \mathbb{F})$. Then, we have:

$$\omega((\mathbb{T}_n - V_{\beta})^n \mathbb{F}) = \inf_{v \in \mathbb{F}^{-1}((\mathbb{T}_n - V_{\beta})^n \mathbb{F})} \|T_s \| \|K_b\| \|v, b\| =$$

$$\leq \inf_{v \in \Delta(\sigma, \beta, n, \mathbb{F}), s, b} \|T_s \| \|K_b\| \|v, b\| =$$

$$= \inf_{v \in \mathbb{F}^{-1}((\mathbb{F}))} \|T_s \| \|K_b\| \|v, b\| =$$

$$\leq \|T_s \| \|K_b\| \|v, b\|.$$
A dilation theorem

\[ \omega((T^o - V_B)^n \gamma) \leq ||(T^o - U_B)^n|| \omega(\gamma) \] for any \( \gamma \in \mathcal{X} \); hence

\[ ||(T^o - V_B)^n|| \leq ||(T^o - U_B)^n||^n. \] Conversely, since \((T^o - V_B)^n \varphi(x) = \varphi((T^o - U_B)^n x)\), for any \( x \in \mathcal{X} \), we get easily

\[ ||(T^o - V_B)^n|| \leq ||(T^o - U_B)^n||. \]

DEFINITION. - Let \( \{ \mathcal{X}, \mathcal{P}, U \} \) be an object, where \( \mathcal{X} \) is a Banach space, \( \mathcal{P} = \{ T \} \) a semi-group of operators and \( U : \mathcal{G} \to \mathcal{B}(\mathcal{X}) \) a linear map as in the above theorem. Then, an object \( \{ \tilde{\mathcal{X}}, \varphi, \mathcal{P}, \tilde{T}, \tilde{V} \} \) where \( \tilde{\mathcal{X}} \) is a Banach space, \( \varphi \) a bicontinuous isomorphism of \( \mathcal{X} \) into \( \tilde{\mathcal{X}} \), \( \mathcal{P} \) a continuous projection of \( \tilde{\mathcal{X}} \) onto \( \varphi(\mathcal{X}) \), \( \tilde{T} = \{ \tilde{T}_t \}_{t \in R^+} \subset \mathcal{B}(\tilde{\mathcal{X}}) \) a semi-group of operators and \( \tilde{V} : \mathcal{G} \to \mathcal{B}(\tilde{\mathcal{X}}) \) a representation such that \( \tilde{V}_1 = I \), \( \tilde{V}_a \tilde{T}_t = \tilde{T}_t \tilde{V}_a \) for any \( a \in \mathcal{G} \), \( t \in R^+ \), is called an \( \mathcal{G} \)-spectral dilation of \( \{ \mathcal{X}, \mathcal{P}, U \} \) if the property (ii) is satisfied. An \( \mathcal{G} \)-spectral dilation is called minimal if also we have (iii).

Remark 1. - When \( \mathcal{G} \) is a Michael algebra and \( U : \mathcal{G} \to \mathcal{B}(\mathcal{X}) \) a linear continuous map, then \( K \) is the seminorm which estimates \( U \).

Remark 2. - Let \( \mathcal{T} \in \mathcal{B}(\mathcal{X}) \); then the above theorem is obviously true with \( \{ T^n \}_{n \in N} \) instead of \( \{ T_t \}_{t \in R^+} \).

Application. - Let \( \mathcal{U} \) be an admissible algebra in the sense of [1]. Then, an operator \( \mathcal{T} \in \mathcal{B}(\mathcal{X}) \) is called \( \mathcal{U} \)-subspectral (see [9]) if there is a Banach space containing \( \mathcal{X} \) as a closed subspace, a continuous projection \( P \) of \( \tilde{\mathcal{X}} \) onto \( \mathcal{X} \), a \( \mathcal{U} \)-spectral operator \( \mathcal{T} \in \mathcal{B}(\mathcal{X}) \) having a \( \mathcal{U} \)-spectral representation \( \mathcal{V} : \mathcal{G} \to \mathcal{B}(\tilde{\mathcal{X}}) \) with the properties \( \mathcal{V}_a \tilde{\mathcal{X}} \subset \mathcal{X} \) and \( \tilde{P} \mathcal{T} f x = \mathcal{T} \tilde{P} f x \), for any \( f \in \mathcal{U} \), \( x \in \mathcal{X} \), such that \( \tilde{T}_1 \mathcal{X} = \mathcal{T} \).

We have the following characterization for \( \mathcal{U} \)-subspectral operators: an operator \( \mathcal{T} \in \mathcal{B}(\mathcal{X}) \) is \( \mathcal{U} \)-subspectral if and only if there is a linear map \( U : \mathcal{U} \to \mathcal{B}(\mathcal{X}) \) with the properties:

1. \( U_1 = I \),
2. \( U_{tf} = U_t U_f \),
3. \( ||U_f|| \leq M L_f \) for any \( f \in \mathcal{U} \),

(where \( M \) is a positive constant and \( L : \mathcal{U} \to \mathcal{B}(\mathcal{X}) \), a linear map satisfying...
(j) \( \| L_{fg} \| \leq \| L_f \| \| L_g \| \), for any \( f, g \in \mathcal{U} \) and the function

(jj) \( \xi \rightarrow L_{f\xi} \) is analytic in \( \mathcal{U} \) sup \( f \), for every \( f \in \mathcal{U} \);

\( \mathcal{U} \) is a Banach space, such that \( TU_f = U_f T \), for any \( f \in \mathcal{U} \) and \( U_T \), (see [8] and [9]).

If \( \mathcal{U} \) is an admissible topologic algebra with the topology of Michael algebra, then the property (3) of \( U \) is replaced by its continuity.

For instance, let \( \gamma = \{ z \in \mathbb{C} ; |z| = 1 \} \); one denotes by \( L^p(\gamma)(p < \infty) \) the Banach space of the all complex-valued functions \( f \) on \( \gamma \) such that \( |f|^p \) is integrable with respect to the Lebesgue measure. (Thus a function \( f \in L^p(\gamma) \) if and only if the function \( f^{\sim} \) defined by \( f^{\sim}(\theta) = f(e^{i\theta}) \) for \( \theta \in [-\pi, +\pi] \) belongs to \( L^p\left(\frac{1}{2\pi} \, d\theta\right)\).

In the same way one considers the Banach algebra \( L^\infty(\gamma) \) of all complex-valued essential bounded functions with respect to the Lebesgue measure on \( \gamma \), (i.e. a function \( f \in L^\infty(\gamma) \) if and only if the function \( f^{\sim} \) defined by \( f^{\sim}(\theta) = f(e^{i\theta}) \) belongs to \( L^\infty\left(\frac{1}{2\pi} \, d\theta\right)\).

Let \( p \geq 1 \), as usual, the space \( H^p \) is the set of analytic functions in \( D = \{ z ; |z| < 1 \} \) such that \( f_r \) defined by \( f_r(\theta) = f(re^{i\theta}) \), for \( \theta \in [-\pi, +\pi] \), belongs to \( L^p\left(\frac{1}{2\pi} \, d\theta\right) \) for every \( 0 \leq r \leq 1 \), or with the other words, \( H^p \) is a closed subspace of functions \( f \) of \( L^p(\gamma) \) such that \( \int_{-\pi}^{\pi} e^{in\theta} f(e^{i\theta}) \, d\theta = 0 \), \( n = 1, 2, 3, ... \).

Taking \( \mathcal{U} = L^p(\gamma) \) and \( \mathcal{U} = L^\infty(\gamma) \), we define a representation \( V : \mathcal{U} \rightarrow \mathcal{B}(\mathcal{U}) \) by:

\[ V_{\varphi} f = \varphi f, \text{ for every } \varphi \in L^\infty(\gamma), f \in L^p(\gamma). \]

From the theorem of M. Riesz ([3], cap. IX) we have \( L^p(\gamma) = H^p \oplus \overline{H^p} \), \( 1 < p < \infty \), where \( \overline{H^p} \) is the space of complex-conjugate functions of \( H^p \) becoming zero at \( z = 0 \). Let \( P \) be the continuous projection of \( L^p(\gamma) \) onto \( H^p \). We define the continuous linear map \( U : L^\infty(\gamma) \rightarrow \mathcal{B}(H^p) \) by:

\[ U_{\varphi} f = P V_{\varphi} f, \text{ for every } \varphi \in L^\infty(\gamma), f \in H^p. \]

Obviously, \( U \) is a continuous linear map with the above properties (1) and (2). Then an operator \( T \in \mathcal{B}(H^p) \) such that \( U_{\varphi} T = T U_{\varphi} \), for \( \varphi \in L^\infty(\gamma) \) and \( T_{e^{i\theta} f} \) is a \( L^p(\gamma) \)-subspectral operator. For \( p = 2 \), \( V_{e^{i\theta}} \) is the bilateral shift and \( U_{e^{i\theta}} \) is the unilateral shift (see [2]).
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Academia R.S. Romania
Institutul de Matematică
Calea Grivitei 21
BUCHARESTI 12 (Roumanie)