A dilation theorem for operators on Banach spaces

Elena Stroescu

Mémoires de la S. M. F., tome 31-32 (1972), p. 365-373

<http://www.numdam.org/item?id=MSMF_1972__31-32__365_0>
Introduction. -

Let \( R^+ \) be the set of all non-negative real numbers and \( \mathcal{B}(\mathcal{X}) \) the Banach algebra of all linear bounded operators on a Banach space \( \mathcal{X} \). In this paper, we present a dilation theorem by which an object \( \{\mathcal{X}, \Gamma, U\} \) dilates into \( \{\tilde{\mathcal{X}}, \varphi, P, \tilde{\Gamma}, V\} \); where \( \mathcal{X} \) and \( \tilde{\mathcal{X}} \) are Banach spaces, \( \varphi \) is a bicontinuous isomorphism of \( \mathcal{X} \) into \( \tilde{\mathcal{X}} \), \( P \) a continuous projection of \( \tilde{\mathcal{X}} \) onto \( \varphi(\mathcal{X}) \), 

\[ \Gamma = \{T_t\}_{t \in R^+} \subset \mathcal{B}(\mathcal{X}) \]

and \( \tilde{\Gamma} = \{\tilde{T}_t\}_{t \in R^+} \subset \mathcal{B}(\tilde{\mathcal{X}}) \) are operator semi-groups, \( U \) is a \( \mathcal{B}(\mathcal{X}) \)-valued linear map on an arbitrary algebra \( \mathcal{A} \) estimated by a submultiplicative functional and \( V \) a \( \mathcal{B}(\tilde{\mathcal{X}}) \)-valued representation on \( \mathcal{A} \) such that 

\[ V_a \tilde{T}_t = \tilde{T}_t V_a, \text{ for every } a \in \mathcal{A} \text{ and } t \in R^+. \]

This theorem is an extension of some previous results (see \([8], [9]\)) and has arisen from the concern to characterize restrictions of spectral operators on invariant subspaces (or operators which dilate in spectral operators) by a map replacing the spectral representation.

Notations. -

Throughout the following \( C \) denotes the complex plane; \( N = \{0,1,2,\ldots\} \); \( \mathcal{A} \) an arbitrary algebra over \( C \) with unit element denoted by \( 1 \); \( K \) a submultiplicative functional of \( \mathcal{A} \) into \( R^+ \) (i.e. \( K_{ab} \leq K_a K_b \) for any \( a, b \in \mathcal{O} \)) such that \( K_1 = 1 \); \( \mathcal{X} \) a Banach space over \( C \); \( \mathcal{B}(\mathcal{X}) \) the Banach algebra of all linear bounded operators on \( \mathcal{X} \) over \( C \); \( I \) the identity operator. Let \( T_1, T_2 \in \mathcal{B}(\mathcal{X}) \) two commuting operators; then one says that \( T_1 \) is quasi-nilpotent equivalent with \( T_2 \) and denotes \( T_1 \sim T_2 \), if \( \lim_{n \to \infty} \| (T_1 - T_2)^n \|^{1/n} = 0 \). A family of operators \( \{T_t\}_{t \in R^+} \subset \mathcal{B}(\mathcal{X}) \) is called semi-group if \( T_0 = I \) and \( T_{t+s} = T_t T_s \) for any \( t \) and \( s \in R^+ \).

THEOREM. - Let \( \{T_t\}_{t \in R^+} \subset \mathcal{B}(\mathcal{X}) \) be a semi-group of operators and \( U: \mathcal{A} \to \mathcal{B}(\mathcal{X}) \) a linear map such that \( U_1 = I \), \( \|U_a\| \leq K_a \), for any \( a \in \mathcal{A} \).

Then, there exists a Banach space \( \tilde{\mathcal{X}} \), an isometric isomorphism \( \varphi \) of \( \mathcal{X} \) into \( \tilde{\mathcal{X}} \), a continuous projection \( P \) of \( \tilde{\mathcal{X}} \) onto \( \varphi(\mathcal{X}) \), a semi-group 

\[ \tilde{\Gamma} = \{\tilde{T}_t\}_{t \in R^+} \subset \mathcal{B}(\tilde{\mathcal{X}}) \]

and a representation \( V: \mathcal{A} \to \mathcal{B}(\tilde{\mathcal{X}}) \) such that:
(o) \[ \|P\| = 1 ; \|\tilde{T}_t\| = \|T_t\| , \text{ for any } t \in \mathbb{R}^+ ; V_1 = \tilde{I} \text{ and } \|V_0\| \leq K_\alpha , \text{ for any } \alpha \in \mathbb{G} . \]

(i) \[ V_{t_a}^T = \tilde{T}_t V_a , \text{ for any } \alpha \in \mathbb{G} , \tau \in \mathbb{R}^+ . \]

(ii) \[ P_{t_a}^T V_{t_a} \varphi(x) = \varphi(T_{t_a} U x) , \text{ for any } \alpha \in \mathbb{G} , \tau \in \mathbb{R}^+ , x \in \mathcal{F} . \]

(iii) \( \mathfrak{f} \) is the closed vector space spanned by \( \{ \tilde{T}_t^\alpha \varphi(x) ; \alpha \in \mathbb{G} , t \in \mathbb{R}^+, x \in \mathcal{F} \} \).

(iv) Let \( s \in \mathbb{R}^+ \); then we have the following equivalences:

1° \[ T_s \varphi(x) = \varphi(T_s x) , \text{ for any } x \in \mathcal{F} ; \]

2° \[ \tilde{T}_s^\alpha V_{t_a} \varphi(x) = \tilde{T}_s^\alpha \varphi(x) , \text{ for any } \alpha \in \mathbb{G} , x \in \mathcal{F} ; \]

3° \[ U_{s_a} T_{t_a} = T_{s_a} U_{t_a} , \text{ for any } \alpha \in \mathbb{G} . \]

(v) Let \( b \in \mathbb{G} \); then \( V_b \varphi(x) = \varphi(U_b x) , \text{ for any } x \in \mathcal{F} \) is equivalent with \( U_{ab} = U_a U_b , \text{ for any } a \in \mathbb{G} \).

(vi) Let \( \sigma \in \mathbb{R}^+ \) and \( \beta \in \mathbb{G} \) commuting with all the elements of \( \mathbb{G} \) such that \( U_{a\beta} = U_a U_\beta , T_{a\beta} = U_a T_\beta , \text{ for any } a \in \mathbb{G} \); then \( \| (\tilde{T}_s^\sigma - V_\beta)^n \| = \| (T_{s_a} - U_{\beta a})^n \| , \text{ for every } n \in \mathbb{N} . \)

Proof : A) Let us consider the Cartesian product \( \mathcal{I}^{\mathbb{R}^+ \times \mathbb{G}} = \prod_{(t,a) \in \mathbb{R}^+ \times \mathbb{G}} \mathcal{I}^{(t,a)} \) and the direct sum \( \mathcal{I}^{\mathbb{R}^+ \times \mathbb{G}} = \bigoplus_{(t,a) \in \mathbb{R}^+ \times \mathbb{G}} \mathcal{I}^{(t,a)} \), where \( \mathcal{I}^{(t,a)} = \mathcal{I}^{(t,a)} \) for every \( t \in \mathbb{R}^+ , a \in \mathbb{G} \). An element \( y \in \mathcal{I}^{\mathbb{R}^+ \times \mathbb{G}} \) is a family \( (y_{t,a})_{(t,a) \in \mathbb{R}^+ \times \mathbb{G}} \) (many times we write \( y = (y_{t,a})_{(t,a)} \)) of components \( y_{(t,a)} = y_{t,a} \in \mathcal{I}^{(t,a)} \), for every \( t \in \mathbb{R}^+ , a \in \mathbb{G} \). If \( y \in \mathcal{I}^{(\mathbb{R}^+ \times \mathbb{G})} \subseteq \mathcal{I}^{\mathbb{R}^+ \times \mathbb{G}} \), then \( (y)_{t,a} = y_{t,a} \neq 0 \) for only a finite number of elements \( (t,a) \in \mathbb{R}^+ \times \mathbb{G} \).

Let us consider a map:

\[ \Theta = (\otimes t,a)_{(t,a) \in \mathbb{R}^+ \times \mathbb{G}} \text{ of } \mathcal{I}^{(\mathbb{R}^+ \times \mathbb{G})} \text{ into } \mathcal{I}^{\mathbb{R}^+ \times \mathbb{G}} \]

defined by:

\[ \Theta y = (T_{s,a} U_{t,a} y_{s,b})_{(t,a)} , \text{ for every } y \in \mathcal{I}^{(\mathbb{R}^+ \times \mathbb{G})} . \]

It is easy to see that \( \Theta \) is a well defined linear map. Then, we denote by \( \mathfrak{f} \) the range of \( \Theta \) and by \( x \) an arbitrary element of \( \mathfrak{f} \).
A dilation theorem

For every \( y \in \hat{X} \), we have:
\[
\sigma^{-1}(\{y\}) = \{ y \in (\mathbb{R}^+ \times \mathbb{A}) : \sigma y = y \}.
\]

We define a function \( \omega : \hat{X} \to \mathbb{R}^+ \) by
\[
\omega(y) = \inf \left\{ \sum_{s,b} \| T_s \| K_b \| y_{s,b} \| : y \in \sigma^{-1}(\{y\}) \right\},
\]
for every \( y \in \hat{X} \); let us prove that \( \omega \) is a norm on \( \hat{X} \). Let \( \mu \in \mathbb{C} \) be non-zero, \( y \in \hat{X} \) and \( \Delta(\mu y) = \{ \mu y : y \in \sigma^{-1}(\{y\}) \} \); then we show that \( \sigma^{-1}(\{\mu y\}) = \Delta(\mu y) \). Indeed, let \( \mu y \in \Delta(\mu y) \), i.e. \( y \in \sigma^{-1}(\{y\}) \), then \( \mu y = (\mu T_s)_{s,b} y_{s,b} \), hence \( \omega(y) = \inf_{y \in \sigma^{-1}(\{y\})} \sum_{s,b} \| T_s \| K_b \| y_{s,b} \| = \inf_{y \in \Delta(\mu y)} \sum_{s,b} \| T_s \| K_b \| y_{s,b} \| = \inf_{y \in \sigma^{-1}(\{y\})} \| \mu \| \omega(y) \), i.e. \( \omega(y) = |\mu| \omega(y) \), hence \( \omega(\hat{0}) = 0 \). Then, for \( \mu = 0 \) we have \( \omega(0y) = 0 \) and \( \omega(\hat{y}) = 0 \), for any \( y \in \hat{X} \). Hence \( \omega(y) = |\mu| \omega(y) \), for any \( y \in \hat{X} \), \( \mu \in \mathbb{C} \).

Let \( y_1, y_2 \in \hat{X} \) and
\[
\Delta(y_1 + y_2) = \{ y_1 + y_2 : y_1 \in \sigma^{-1}(\{y_1\}), y_2 \in \sigma^{-1}(\{y_2\}) \},
\]
then obviously we have \( \Delta(y_1 + y_2) \subset \sigma^{-1}(\{y_1 + y_2\}) \) and
\[
\omega(y_1 + y_2) = \inf_{z \in \sigma^{-1}(\{y_1 + y_2\})} \sum_{s,b} \| T_s \| K_b \| z_{s,b} \| \leq \inf_{z \in \Delta(y_1 + y_2)} \sum_{s,b} \| T_s \| K_b \| z_{s,b} \| \leq \inf_{y_1 \in \sigma^{-1}(\{y_1\}), y_2 \in \sigma^{-1}(\{y_2\})} \sum_{s,b} \| T_s \| K_b \| y_{s,b}^1 + y_{s,b}^2 \| \leq \inf_{y_1 \in \sigma^{-1}(\{y_1\}), y_2 \in \sigma^{-1}(\{y_2\})} \sum_{s,b} \| T_s \| K_b \| y_{s,b}^1 \| + \inf_{y_2 \in \sigma^{-1}(\{y_2\})} \sum_{s,b} \| T_s \| K_b \| y_{s,b}^2 \| \leq \omega(y_1) + \omega(y_2),\]
for all \( y_1, y_2 \in \hat{X} \).

Then, from the definition of \( \omega \), for every \( y \in \hat{X} \), we have:

1) \( \omega(y) = \sum_{s,b} \| T_s \| K_b \| y_{s,b} \| \), for any \( y \in \sigma^{-1}(\{y\}) \) and \( s,b \),

2) \( 1_{\mathbb{R}^+} \leq \sum_{s,b} \| T_s \| K_b \| y_{s,b} \| \omega(y) \), for \( t \in \mathbb{R}^+ \), \( a \in \mathbb{A} \).

Hence \( \omega \) is a norm on \( \hat{X} \); we denote by \( \hat{X} \) the \( \omega \)-completion of \( X \) and the norm on \( \hat{X} \) also by \( \omega \).
B) We define an isomorphism \( \varphi \) of \( \mathfrak{X} \) into \( \mathfrak{R} \times \mathfrak{A} \) by \( \varphi(x) = (T + x)\cdot_{s,b} a, \) for every \( x \in \mathfrak{X} \).

Applying 1) and 2) we get

3) \[ ||x|| \leq \omega(\varphi(x)) \leq ||x||, \] for any \( x \in \mathfrak{X} \).

Therefore \( \varphi \) is an isometric isomorphism of \( \mathfrak{X} \) into \( \mathfrak{X} \).

We define a projection \( P \) of \( \mathfrak{X} \) onto \( \varphi(\mathfrak{X}) \), by \( P\hat{\mathfrak{X}} = \varphi(\hat{\mathfrak{X}}) \), for all \( \hat{\mathfrak{X}} \in \hat{\mathfrak{X}} \). Applying 3) and 2), we get \( \omega(P) = \omega(\varphi(\hat{\mathfrak{X}})) \leq ||\hat{\mathfrak{X}}|| \leq \omega(\hat{\mathfrak{X}}) \), i.e.

\( \omega(P) \leq \omega(\hat{\mathfrak{X}}) \), for any \( \hat{\mathfrak{X}} \in \hat{\mathfrak{X}} \). Hence, \( P \) can be extended by continuity to a continuous projection of \( \mathfrak{X} \) onto \( \varphi(\mathfrak{X}) \), that will be denoted by the same symbol.

Let now \( \tau \in \mathfrak{R}^+ \); then for every \( \mathfrak{Y} \in \hat{\mathfrak{X}} \) we put

\[ \tilde{T}_\tau \mathfrak{Y} = (T_{s,b} + \tau) \cdot_{s,b} y_{s,b} = (T_{s,b} + \tau) \cdot_{s,b} y_{s,b} = \mathfrak{X}, \]

where we denote \( s + \tau = s; y_{s,b} = y_{s-b}, \) for \( \sigma \geq \tau \) and \( y_{s,b} = 0, \) for \( 0 \leq \sigma < \tau \), with \( b \in \mathfrak{A} \).

We see easily that \( \tilde{T}_\tau \) is a well defined linear map of \( \hat{\mathfrak{X}} \) into \( \hat{\mathfrak{X}} \). Let us prove that also it is continuous.

For every \( \mathfrak{Y} \in \hat{\mathfrak{X}} \), denoting \( \Delta(\tau, \mathfrak{Y}) = \{ \mathfrak{X} \in \hat{\mathfrak{X}}(\mathfrak{R}^+ \times \mathfrak{A}); \mathfrak{X}_{s,b} = y_{s-b}, \sigma \geq \tau, \sigma \geq \tau \}, \) we see that \( \Delta(\tau, \mathfrak{Y}) \) is \( \mathfrak{R}^+ \)-bounded. Then, we have \( \omega(\tilde{T}_\tau) = \inf_{\mathfrak{X} \in \Delta(\tau, \mathfrak{Y})} \| T_{s,b} \|_{s,b} \), \( \omega(\tilde{T}_\tau) \leq \omega(\hat{\mathfrak{Y}}) \), i.e.

\[ \omega(\tilde{T}_\tau) \leq \| T_{s,b} \|_{s,b} \), for any \( \mathfrak{Y} \in \hat{\mathfrak{X}} \).

Thus, for every \( \tau \in \mathfrak{R}^+ \), \( \tilde{T}_\tau \) can be extended by continuity to an element of \( \mathfrak{A}(\mathfrak{X}) \), that will be denoted by the same symbol. Then, we see easily that \( P\tilde{T}_\tau \varphi(x) = \varphi(T_{s,b} x), \) for any \( x \in \mathfrak{X} \).
A dilation theorem

Hence \( \| T^x \| = \omega(\varphi(T^x)) = \omega(P^\alpha \varphi(x)) \leq \omega(\varphi(x)) \leq \| T^x \| = \| T^x \| \), i.e.

6) \( \| T^x \| \leq \| T^x \| \), for any \( x \in X \). At last, we see easily that \( \{ T^T \}_{T \in R^+} \) is a semi-group of operators, that we denote by \( \hat{T} \).

C) Let us define a representation \( \hat{V} \). Let \( \alpha \in G \); then for every \( \hat{y} \in \hat{X} \), we put

\[
V_{\alpha} \hat{y} = (T_t \sum_{s,b \in \Omega} T_s U_{ab} y_{s,b} t_s y_{s,b} t_s)_{t \in R^+} =
\]

\[
= (T_t \sum_{s,b \in \Omega} T_s U_{ab} y_{s,b} t_s y_{s,b} t_s)_{t \in R^+}, \quad \text{where}
\]

\[\Omega = \{ (s,b) : \alpha = s \}, \quad \text{and} \quad y_{s,b} = \sum_{s,b \in \Omega} y_{s,b} t_s y_{s,b} t_s, \quad \text{for} \ s \in R^+, \ c \in C.\]

The map \( V_{\alpha} : \hat{X} \rightarrow \hat{X} \) is well defined. Indeed, let \( \hat{y}_1 = \hat{y}_2 \in \hat{X} \); then there exists \( y_1, y_2 \in X(R^+ \times G) \) such that \( \hat{y}_1 = \Theta y_1 \) and \( \hat{y}_2 = \Theta y_2 \), hence

\[
T_t \sum_{s,b \in \Omega} T_s U_{ab} y_{s,b} t_s y_{s,b} t_s = T_t \sum_{s,b \in \Omega} T_s U_{ab} y_{s,b} t_s y_{s,b} t_s, \quad \text{for any} \ t \in R^+, \ a \in G.\]

Then, \( T_t \sum_{s,b \in \Omega} T_s U_{ab} y_{s,b} t_s y_{s,b} t_s = T_t \sum_{s,b \in \Omega} T_s U_{ab} y_{s,b} t_s y_{s,b} t_s, \quad \text{for} \ t \in R^+ \) with \( a \in G \). We see easily that for every \( a \in G \), \( V_{\alpha} : \hat{X} \rightarrow \hat{X} \) is a linear map and \( V_{\alpha} \hat{y} = \hat{y} \), for any \( \hat{y} \in \hat{X} \). Moreover, \( V : G \rightarrow \mathcal{L}(\hat{X}) \) is a representation (see [4]); for a vector space \( X \), \( \mathcal{L}(X) \) denotes the algebra of all linear maps of \( X \) into \( X \). Now, we prove that \( V_{\alpha} : \hat{X} \rightarrow \hat{X} \) is continuous, for every \( a \in G \). Let \( a \in G \), \( \hat{y} \in \hat{X} \) and \( \Lambda(a, \hat{y}) = \{ u \in \mathcal{L}(R^+ \times G) : u_{s,c} = \sum_{s,b \in \Omega} y_{s,b} t_s y_{s,b} t_s, \ y_{s,b} \in \Theta^{-1}(\{ \hat{y} \}) \} \), then we see \( \Lambda(a, \hat{y}) \subset \Theta^{-1}(\{ V_{\alpha} \hat{y} \}) \). Therefore, we have:

\[
\omega(V_{\alpha} \hat{y}) = \inf_{u \in \Theta^{-1}(\{ V_{\alpha} \hat{y} \})} \sum_{s,c} \| T_s K_c u_{s,c} \| \leq \inf_{u \in \Theta^{-1}(\{ \hat{y} \})} \sum_{s,c} \| T_s K_c u_{s,c} \| = \inf_{u \in \Theta^{-1}(\{ \hat{y} \})} \sum_{s,c} \| T_s K_c u_{s,c} \| = K_{\alpha} \inf_{y \in \Theta^{-1}(\{ \hat{y} \})} \sum_{s,c} \| T_s K_c u_{s,c} \| = K_{\alpha} \omega(\hat{y});
\]

i.e. for every \( a \in G \) we get

7) \( \omega(V_{\alpha} \hat{y}) \leq K_{\alpha} \omega(\hat{y}) \), for any \( \hat{y} \in \hat{X} \). Hence, \( V_{\alpha} \) can be extended by continuity to an element of \( \mathcal{B}(\hat{X}) \) that will be denoted by \( V_{\alpha} \), for every \( a \in G \).
Thus, \((0)\) is completely proved. The property \((i)\) is immediate, since for every \(\alpha \in \mathcal{G}\) and \(\tau \in \mathbb{R}^+\), we have 

\[
\tilde{T}_\tau V_\alpha \tilde{y} = (T_\tau \begin{array}{c} s, b \\ s, b \end{array} U_{a+b} y_{s,b}, t,a = V_\alpha \tilde{T}_\tau \tilde{y},
\]

for any \(\tilde{y} \in \tilde{x}\). Using the definitions of \(\varphi\), \(P\), \(V_\alpha\) and \(\tilde{T}_\tau\), for \(\alpha \in \mathcal{G}\), \(\tau \in \mathbb{R}^+\), we obtain immediately \((ii)\), \((iii)\) and \((v)\).

D) Let us prove \((iv)\). From 

\[
\tilde{T}_s \varphi(x) = (T_s U_s x), t,a\quad \text{and} \quad \varphi(T_s x) = (T_s U_s T_s x), t,a,
\]

we see that \(1^0\) and \(3^0\) are equivalent.

Now choosing \(a = 1\) in \(2^0\), and using 

\[
P \tilde{T}_\tau \varphi(x) = \varphi(T_\tau x),
\]

for \(x \in \mathcal{I}\) (see \((ii)\)), we get \(1^0\).

Conversely, taking into account of \((ii)\) and writing \(1^0\) with \(U_s x\) instead of \(x\), for \(\alpha \in \mathcal{G}\), we get \(2^0\).

At last, we show \((vi)\). Let \(\sigma \in \mathbb{R}^+\), and \(\beta \in \mathcal{G}\), as in the assumption, also let 

\[
n \in \mathbb{N}\quad \text{and} \quad \tilde{y} \in \tilde{x};
\]

then, we write:

\[
(\tilde{T}_o - V_\beta)^n \tilde{y} = \sum_{k=0}^{n} (-1)^{n-k} (k) \tilde{T}_o U^n \tilde{y} =
\]

\[
= \sum_{k=0}^{n} (-1)^{n-k} (k) (T_s U_s T_s U_{a+b} y_{s,b}, t,a = 0 \quad \text{for} \quad \tilde{y} \in \tilde{x},
\]

where \(v\) is defined by

\[
v_{s,b} = \sum_{k=0}^{n} (-1)^{n-k} (k) T_s U_{a+b} y_{s,b},
\]

for \(y \in \tilde{x}\), \(s \in \mathbb{R}^+\), and \(b \in \mathcal{G}\).

Denoting by \(\Delta(\sigma,\beta,n,\tilde{y}) = \) the set of all element \(v\) so defined, we see that:

\[
\Delta(\sigma,\beta,n,\tilde{y}) \subset \Theta^{-1}(\tilde{T}_o - V_\beta)^n \tilde{y}.
\]

Then, we have:

\[
\omega((\tilde{T}_o - V_\beta)^n \tilde{y}) = \inf_{v \in \Delta(\sigma,\beta,n,\tilde{y})} \sum_{s,b} \|T_s \| \|K_b\| \|v_{s,b}\| =
\]

\[
\leq \inf_{v \in \Delta(\sigma,\beta,n,\tilde{y})} \sum_{s,b} \|T_s \| \|K_b\| \|v_{s,b}\| =
\]

\[
= \inf_{y \in \Theta^{-1}(\tilde{y})} \sum_{s,b} \|T_s \| \|K_b\| \sum_{k=0}^{n} (-1)^{n-k} (k) T_s U_{a+b} y_{s,b} =
\]

\[
\leq \|\tilde{x}\| \sum_{k=0}^{n} (-1)^{n-k} (k) T_s U_{a+b} \inf_{y \in \Theta^{-1}(\tilde{y})} \sum_{s,b} \|T_s \| \|K_b\| \|y_{s,b}\| =
\]
A dilation theorem

\[ n \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} u \beta^{n-k} \omega(y) \]. Therefore, for every \( n \in \mathbb{N} \),
we have \( \omega((T_0 - V_B)^n y) \leq \| (T_0 - U_B)^n \| \omega(y) \), for any \( y \in \mathcal{X} \); hence
\[ \| (T_0 - V_B)^n \| \leq \| (T_0 - U_B)^n \| . \]
Conversely, since \( (T_0 - V_B)^n \varphi(x) = \varphi((T_0 - U_B)^n x) \),
for any \( x \in \mathcal{X} \), we get easily \( \| (T_0 - V_B)^n \| \leq \| (T_0 - U_B)^n \| . \)

**DEFINITION.** Let \( \{ \mathcal{X}, \varphi, P, T, V \} \) be an object, where \( \mathcal{X} \) is a Banach space,
\( \varphi \) a bicontinuous isomorphism of \( \mathcal{X} \) into \( \mathcal{X} \), \( P \) a continuous projection of \( \mathcal{X} \) onto \( \varphi(\mathcal{X}) \), \( T = (T_t)^{+} \in \mathcal{B}(\mathcal{X}) \) a semi-group of operators and \( V : \mathcal{O} \to \mathcal{B}(\mathcal{X}) \) a representation such that \( V_\lambda = 1 \), \( V_\alpha T_t = T_t V_\alpha \), for any \( \alpha \in \mathcal{O} \), \( t \in \mathbb{R}^+ \), is called an \( \mathcal{O} \)-spectral dilation of \( \{ \mathcal{X}, \varphi, P, T, V \} \) if the property
(ii) is satisfied. An \( \mathcal{O} \)-spectral dilation is called minimal if also we have (iii).

**Remark 1.** When \( \mathcal{O} \) is a Michael algebra and \( U : \mathcal{O} \to \mathcal{B}(\mathcal{X}) \) a linear continuous map,
then \( K \) is the seminorm which estimates \( U \).

**Remark 2.** Let \( T \in \mathcal{B}(\mathcal{X}) \); then the above theorem is obviously true with
\( \{ T^n \}_{n \in \mathbb{N}} \) instead of \( \{ T_t \}_{t \in \mathbb{R}^+} \).

**Application.** Let \( \mathcal{U} \) be an admissible algebra in the sense of [1]. Then, an operator \( T \in \mathcal{B}(\mathcal{X}) \) is called \( \mathcal{U} \)-subspectral (see [9]) if there is a Banach space containing \( \mathcal{X} \) as a closed subspace, a continuous projection \( P \) of \( \mathcal{X} \) onto \( \mathcal{X} \),
a \( \mathcal{U} \)-spectral operator \( T \in \mathcal{B}(\mathcal{X}) \) having a \( \mathcal{U} \)-spectral representation \( V : \mathcal{O} \to \mathcal{B}(\mathcal{X}) \)
with the properties \( V_\lambda \mathcal{X} = \mathcal{X} \) and \( \mathcal{P} V_\lambda = \mathcal{P} V_\lambda \mathcal{X} \), for any \( \lambda \in \mathcal{U} \), \( x \in \mathcal{X} \), such that \( \mathcal{P} \mathcal{X} = T \).

We have the following characterization for \( \mathcal{U} \)-subspectral operators: an operator \( T \in \mathcal{B}(\mathcal{X}) \) is \( \mathcal{U} \)-subspectral if and only if there is a linear map
\( U : \mathcal{U} \to \mathcal{B}(\mathcal{X}) \) with the properties :

\[ \begin{align*}
1. & \quad U_1 = I , \\
2. & \quad U_{\mathcal{P}z} = U_\mathcal{P} U_z , \\
3. & \quad \| U_\mathcal{P} \| \leq M L_\mathcal{P} \quad \text{for any} \quad \mathcal{P} \in \mathcal{U} ,
\end{align*} \]
(where \( M \) is a positive constant and \( L : \mathcal{U} \to \mathcal{B}(\mathcal{X}) \), a linear map satisfying}
(j) \( \|L_{fg}\| \leq \|L_f\| \cdot \|L_g\| \), for any \( f, g \in \mathcal{U} \) and the function

(jj) \( \xi \rightarrow L_{f_{1\xi}} \) is analytic in \( \text{supp } f \), for every \( f \in \mathcal{U} \);

\( \mathcal{U} \) is a Banach space, such that \( T U \cdot f = U \cdot T f \), for any \( f \in \mathcal{U} \) and \( U \cdot T \), (see [8] and [9]).

If \( \mathcal{U} \) is an admissible topologic algebra with the topology of Michael algebra, then the property (3) of \( U \) is replaced by its continuity.

For instance, let \( \gamma = \{ z \in \mathbb{C} ; |z| = 1 \} \); one denotes by \( L^p(\gamma) \) the Banach space of the all complex-valued functions \( f \) on \( \gamma \) such that \( |f|^p \) is integrable with respect to the Lebesgue measure. (Thus a function \( f \in L^p(\gamma) \) if and only if the function \( \tilde{f} \) defined by \( \tilde{f}(\theta) = f(e^{i\theta}) \) for \( \theta \in [-\pi, +\pi] \) belongs to \( L^p\left( \frac{1}{2\pi} \, d\theta \right) \).

In the same way one considers the Banach algebra \( L^\infty(\gamma) \) of all complex-valued essential bounded functions with respect to the Lebesgue measure on \( \gamma \), (i.e. a function \( f \in L^\infty(\gamma) \) if and only if the function \( \tilde{f} \) defined by \( \tilde{f}(\theta) = f(e^{i\theta}) \) belongs to \( L^\infty\left( \frac{1}{2\pi} \, d\theta \right) \).

Let \( p \geq 1 \), as usual, the space \( H^p \) is the set of analytic functions in \( D = \{ z ; |z| < 1 \} \) such that \( f_n \) defined by \( f_n(\theta) = f(re^{i\theta}) \), for \( \theta \in [-\pi, +\pi] \), belongs to \( L^p\left( \frac{1}{2\pi} \, d\theta \right) \) for every \( 0 \leq r \leq 1 \), or with the other words, \( H^p \) is a closed subspace of functions \( f \) of \( L^p(\gamma) \) such that \( f^{(n)} e^{in\theta} f(e^{i\theta}) = d\theta = 0 \), \( n = 1, 2, 3, \ldots \).

Taking \( \mathfrak{X} = L^p(\gamma) \) and \( \mathcal{M} = L^\infty(\gamma) \), we define a representation \( V : \mathcal{M} \rightarrow \mathfrak{B}(\mathfrak{X}) \) by:

\[ V(\varphi) f = \varphi f, \text{ for every } \varphi \in L^\infty(\gamma), f \in L^p(\gamma). \]

From the theorem of M. Riesz ([3], cap. IX) we have \( L^p(\gamma) = H^p \oplus \overline{H^p} \), \( 1 < p < \infty \), where \( \overline{H^p} \) is the space of complex-conjugate functions of \( H^p \) becoming zero at \( z = 0 \). Let \( P \) be the continuous projection of \( L^p(\gamma) \) onto \( H^p \).

We define the continuous linear map \( U : L^\infty(\gamma) \rightarrow \mathfrak{B}(H^p) \) by:

\[ U(\varphi) f = P V(\varphi) f, \text{ for every } \varphi \in L^\infty(\gamma), f \in H^p. \]

Obviously, \( U \) is a continuous linear map with the above properties (1) and (2). Then an operator \( T \in \mathfrak{B}(H^p) \) such that \( U(\varphi) T = T U(\varphi) \), for \( \varphi \in L^\infty(\gamma) \) and \( T e^{i\theta} \) is a \( L^p(\gamma) \)-subspectral operator. For \( p = 2 \), \( V e^{i\theta} \) is the bilateral shift and \( U e^{i\theta} \) is the unilateral shift (see [2]).
A dilation theorem

BIBLIOGRAPHIE


Academia R.S. Romania
Institutul de Matematică
Calea Grivitei 21
BUCHAREST 12 (Romania)