ALEX.P. ROBERTSON

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# UNCONDITIONAL CONVERGENCE AND THE VITALI-HAHN-SAKS THEOREM

by

Alex. P. ROBERTSON

The notion of unconditional convergence appears in many contexts in functional analysis. One of these is the theory of vector measures, where some of the deeper results are largely transcriptions of properties of unconditionally convergent series. Since vector measures have recently been attracting some interest, it seems worth while to present a number of these properties in a simple and unified manner.

## 1. - Basic properties.

Let $E$ be a separated topological vector space. (Most of what follows continues to be valid for a separated additive abelian topological group.) Also let $(x_n)$ be a sequence of points of $E$. For each finite set $\phi$ of positive integers, put

$$s_\phi = \sum_{n \in \phi} x_n \ .$$

The sets $\phi$ are directed under inclusion, so that $(s_\phi)$ is a net in $E$ ; the series $\Sigma x_n$ is called unconditionally convergent to an element $s$ of $E$ iff $s_\phi \to s$, i.e. to each neighbourhood $U$ of the origin in $E$ corresponds a finite set $\phi_0$ such that $s_\phi \in s + U$ whenever $\phi_0 \subseteq \phi$.

It is useful to consider also the corresponding Cauchy condition when $(s_\phi)$ is a Cauchy net ; we shall then call $\Sigma x_n$ unconditionally Cauchy. This is equivalent to demanding that to each neighbourhood $U$ of the origin in $E$ corresponds a finite set $\phi_0$ such that $s_\phi \in U$ whenever $\phi$ and $\phi_0$ are disjoint.

There are various equivalent definitions. For example, $\Sigma x_n$ is unconditionally convergent to $s$ if and only if every rearrangement converges to $s$ ; and $\Sigma x_n$ is unconditionally Cauchy if and only if the partial sums of every subseries form a Cauchy sequence. Also $n$ may run through any index set, though here we shall stick to the set $N$ of positive integers for simplicity.

Investigation of the unconditional convergence of a series $\Sigma x_n$ involves the study of the map $\phi \to s_\phi$, from the set $H$ of all finite subsets of $N$, to $E$. The aim is to extend this map to the set $K$ of all subsets of $N$. Now $K$ can be identified with the product $2^N$ of copies of the discrete space $\{0, 1\}$ and, with the product topology, is a compact space, with $H$ a dense precompact subspace. (A concrete representation of $K$, with its compact uniform structure, as Cantor's ternary set, with the usual additive uniform structure, may be obtained by means of the correspondence

$$\sigma \to \sum_{n \in \sigma} 2.3^{-n}.)$$

Now suppose that $\phi \to s_\phi$ is continuous at one point $\psi$ of $H$. Then to each neighbourhood $U$ of the origin in $E$ corresponds $\phi_0$ in $H$ such that $s_\phi - s_\psi \in U$ whenever $\phi \cap \phi_0 = \psi \cap \phi_0$, so that $\Sigma x_n$ is unconditionally Cauchy. Moreover this last condition ensures that to $U$ corresponds a $\phi_0$ such that $s_\phi - s_\psi \in U$ for all $\phi$, $\psi$ containing $\phi_0$, i.e. that $\phi \to s_\phi$ is uniformly continuous on $H$. From this we deduce at once that the set $A$ of all $s_\phi$ is precompact in $E$. In a topological vector space, the converse holds (see [5]).

PROPOSITION 1. - <u>The series</u> $\Sigma x_n$ <u>of points of a topological vector space</u> $E$ <u>is unconditionally Cauchy if and only if the set</u> $A$ <u>of all</u> $s_\phi$ <u>for</u> $\phi \in H$ <u>is precompact in</u> $E$.

(This result can fail dramatically in a topological group ; for example $1 + 1 + 1 + \ldots$ has precompact set of partial sums in the discrete group $\{0, 1\}$.) Proposition 1 can be thought of as a generalisation of Riemann's theorem for conditionally convergent series of scalars.

Next, suppose that $\Sigma x_n$ is unconditionally Cauchy and that $A$ is contained in a complete subspace of $E$. Then $\phi \to s_\phi$ extends by uniform continuity to a map $\sigma \to s_\sigma$ of $K$ into $E$, and we have defined the sum of every infinite subseries of $\Sigma x_n$. Iff this can be done we call $\Sigma x_n$ <u>subseries convergent.</u> Since $K$ is compact and $H$ is dense in $K$ we have proved that if $\Sigma x_n$ is subseries convergent, the set of all infinite sums $s_\sigma$ is $\overline{A}$ and is compact in $E$. Along with Proposition 1 this gives a more precise result.

PROPOSITION 2. - <u>The series</u> $\Sigma x_n$ <u>of points of a topological vector space</u> $E$ <u>is subseries convergent if and only if</u> $A$ <u>is relatively compact in</u> $E$.

(This result remains valid in an additive topological group $E$ if and only if $E$ has no non-trivial compact subgroup. Unfortunately, the corresponding restriction does not rescue proposition 1 : there are additive topological groups with no non-trivial precompact subgroups in which proposition 1 fails. See [5].)

When $E$ is a topological vector space, the convex envelope $B$ of $A$ is readily identified to be the set of all sums of the form

$$\sum_{n \in \phi} \lambda_n x_n \quad , \text{ where } 0 \leqslant \lambda_n \leqslant 1 \quad \text{for each } n .$$

Suppose that $\Sigma x_n$ is unconditionally Cauchy, so that $A$ is precompact. In a locally convex space, the convex envelope of a precompact set is precompact, so that, for every bounded sequence of scalars $\lambda_n$ , $\Sigma \lambda_n x_n$ is also unconditionally Cauchy. It is interesting that this result continues to hold if $E$ is semiconvex (i.e. has a base of semiconvex neighbourhoods of the origin) even though in such a space the convex envelope of a precompact set need not be precompact. This shows the special nature of the precompact sets $A$ that can be obtained from unconditionally Cauchy series. There is a counter-example, due to Rolewicz and Ryll-Nardzewski, of an unconditionally convergent series $\Sigma x_n$ with a bounded sequence of scalars $\lambda_n$ , for which $\Sigma \lambda_n x_n$ is not unconditionally Cauchy, that shows that the above result cannot be extended to all topological vector spaces [6].

## 2. - Series of functions.

As before, let $E$ be a separated additive topological group. Also let $T$ be a subset of a metric sapce, with metric $d$ , and let $t_0$ be a point in the closure of $T$ . In this section we consider series of functions from $T$ to $E$ . Suppose that, for each $t$ in $T$, $\Sigma x_n(t)$ is subseries convergent, so that

$$s_\sigma(t) = \sum_{n \in \sigma} x_n(t)$$

is defined for each $t$ in $T$ and each $\sigma$ in $K$ . Suppose also that, for each $\sigma$ in $K$ , $s_\sigma(t)$ converges to a limit as $t \to t_0$ in $T$ , and denote this limit by $s_\sigma(t_0)$ . Thus for each $n$ , $x_n(t)$ converges to a limit, denoted by $x_n(t_0)$ , as $t \to t_0$ in $T$ , but it is not obvious that $s_\sigma(t_0)$ is the sum of the terms $x_n(t_0)$ with $n \in \sigma$ , until this is proved below in corollary 1.

LEMMA. - <u>Under the conditions described above, the convergence of</u> $s_\sigma(t)$ <u>to</u> $s_\sigma(t_0)$ <u>is uniform for</u> $\sigma \in K$ .

<u>Proof</u> : Let $U$ be any closed neighbourhood of the origin in $E$ . For each positive integer $m$ , let $K_m$ be the set of all $\sigma$ in $K$ such that $s_\sigma(t) - s_\sigma(t') \in U$ for all $t$ , $t' \in T$ with $d(t, t_0) < 1/m$ , $d(t', t_0) < 1/m$ . By hypothesis, each $\sigma$ belongs to $K_m$ for all sufficiently large $m$ , so that $K$ is the union of the sets $K_m$ . Now for each $t$ , $t'$, the map $\sigma \to s_\sigma(t) - s_\sigma(t')$ is continuous and $U$ is

closed ; thus

$$\{\sigma : s_\sigma(t) - s_\sigma(t') \in U\}$$

is closed, and so $K_m$ , an intersection of such sets, is also closed. Hence, by Baire's category theorem, there is an $m$ for which $K_m$ contains an open subset $W$ of $K$ .

There is therefore a $\phi_0$ in $H$ such that $\sigma \in W$ whenever $\phi_0 \subseteq \sigma \in K$ and for $d(t, t_0) < 1/m$ we have

$$s_\sigma(t) - s_\sigma(t_0) = \lim_{t' \to t_0} \{s_\sigma(t) - s_\sigma(t')\} \in \bar{U} = U .$$

Now there are only finitely many subsets of $\phi_0$ and so there is a $\delta$ with $0 < \delta \leqslant 1/m$ such that $s_\phi(t) - s_\phi(t_0) \in U$ for all $\phi \subseteq \phi_0$ and $d(t, t_0) < \delta$ . Also any $\sigma$ in $K$ is expressible in the form $\sigma = \tau \setminus \phi$ where $\phi \subseteq \phi_0 \subseteq \tau \in K$ , with the corresponding formulae

$$s_\sigma(t) = s_\tau(t) - s_\phi(t) \ , \ s_\sigma(t_0) = s_\tau(t_0) - s_\phi(t_0) \ . \quad \text{Thus}$$

$$s_\sigma(t) - s_\sigma(t_0) \in U + U$$

for $d(t, t_0) < \delta$ , which proves the uniform convergence.

Thus to each neighbourhood $U$ of the origin in $E$ correspond a $\phi_0$ in $H$ and a $\delta > 0$ such that $s_\phi(t) \in U$ for all $\phi$ disjoint from $\phi_0$ and all $t$ in $T$ distant less than $\delta$ from $t_0$ . This enables the next result to be proved easily.

COROLLARY 1. - Under the conditions of the lemma, $\Sigma x_n(t_0)$ is subseries convergent and, for each $\sigma$ in $K$ ,

$$\sum_{n \in \sigma} x_n(t_0) = s_\sigma(t_0) \quad .$$

We now vary the hypotheses slightly : instead of supposing that each $s_\sigma(t) \to s_\sigma(t_0)$ , we assume that for each $\sigma$ in $K$ the mapping $t \to s_\sigma(t)$ is continuous on $T$ . Then we easily verify that the mapping $(t, \sigma) \to s_\sigma(t)$ is continuous on $T \times K$ . For

$$s_\sigma(t) - s_{\sigma_0}(t_0) = \{s_\sigma(t) - s_\sigma(t_0)\} + \{s_\sigma(t_0) - s_{\sigma_0}(t_0)\} \ .$$

By the lemma the first term is small, uniformly in $\sigma$ , for $t$ near $t_0$ , and the second is small for $\sigma$ near $\sigma_0$ . It follows by a standard argument (or may be proved directly from the remark after the lemma) that the mappings $\sigma \to s_\sigma(t)$ are equicontinuous on every compact subset of $T$ . We note one consequence of this.

COROLLARY 2. - <u>With the above hypotheses, if</u> $T$ <u>is compact, to each neighbourhood</u> $U$ <u>of the origin corresponds a finite set</u> $\phi_0$ <u>such that</u> $s_\phi(t) \in U$ <u>for all</u> $t$ <u>in</u> $T$ <u>and all</u> $\phi$ <u>disjoint from</u> $\phi_0$ .

This result can be used to give a rapid proof of the Orlicz-Pettis theorem [3, 4].

THEOREM 1. - <u>Let</u> $F$ <u>be a locally convex space with dual</u> $F'$ . <u>If</u> $\Sigma x_n$ <u>is subseries convergent for the weak topology</u> $\sigma(F, F')$ , <u>it is also subseries convergent for the initial topology on</u> $F$ .

<u>Proof</u> : Without loss of generality, we may assume that the subspace generated by $\{x_n\}$ is dense in $F$ ; then if $V$ is any closed absolutely convex neighbourhood of the origin, its polar $V^0$ is compact metrisable for $\sigma(F', F)$ , and may be taken as $T$ in corollary 2. With $E$ the scalar field of $F$ , all the conditions are satisfied and so there is a finite set $\phi_0$ such that $|\langle s_\phi , x' \rangle| \leqslant 1$ for all $x' \in V^0$ and all $\phi$ disjoint from $\phi_0$ . Thus, for all such $\phi$ , $s_\phi \in V$ , and so $\Sigma x_n$ is unconditionally Cauchy for the initial topology. But $\{s_\sigma : \sigma \in K\}$ is $\sigma(F, F')$-compact and so complete for the initial topology, so that $\Sigma x_n$ is subseries convergent (see proposition 2).

## 3. - <u>Vector measures.</u>

Let $(S, \mathfrak{M})$ be a measurable space ; also let $E$ be a separated topological vector space (or topological group), as before. A vector measure on $S$ with values in $E$ is a mapping $\mu$ of $\mathfrak{M}$ into $E$ such that, for all sequences of disjoint sets $X_n$ in $\mathfrak{M}$ ,

$$\mu\left(\bigcup_{n=1}^{\infty} X_n\right) = \sum_{n=1}^{\infty} \mu(X_n) .$$

Clearly the convergence on the right is to be interpreted as unconditional convergence.

The theorems of Nikodým and Vitali-Hahn-Saks are concerned with a sequence $(\mu_k)$ of vector measures on $S$ to $E$ such that, for each $X$ in $\mathfrak{M}$ , $\mu_k(X)$ converges to a limit as $k \to \infty$ ; we denote this limit by $\mu(X)$ . See e.g. [1,2,7].

THEOREM 2. - Under the above conditions, $\mu$ is a vector measure and the countable additivity of the $\mu_k$ is uniform in $k$.

Proof : Take any disjoint sequence of sets $X_n$ in $\mathfrak{M}$ and put $x_n(1/k) = \mu_k(X_n)$, $T = \{1, 1/2, 1/3, \dots\}$, $t_0 = 0$, $\mu(X_n) = x_n(0)$. Then the theorem is a transcription of corollary 2 and the remark preceding it.

From this theorem we deduce that, if $(Y_n)$ is a decreasing sequence of sets of $\mathfrak{M}$ with empty intersection, then as $n \to \infty$ $\mu_k(Y_n) \to 0$ uniformly in $k$; we simply consider the disjoint sets $X_n = Y_n \setminus Y_{n+1}$.

Now suppose that each $\mu_k$ is absolutely continuous with respect to a positive measure $\nu$ on $(S, \mathfrak{M})$. (This means that as $\nu(X) \to 0$, $\mu_k(X) \to 0$ for each $k$). If $(Y_n)$ is decreasing as before but now $\nu(Y_0) = 0$ for the intersection $Y_0$ of the sets $Y_n$, then we still have $\mu_k(Y_n) \to 0$ as $n \to \infty$ uniformly in $k$. For $\mu_k(Y_0) = 0$ for each $k$ and so we can ignore $Y_0$ and apply the previous result. This takes us part way through the proof of the Vitali-Hahn-Saks theorem.

THEOREM 3. - If, in addition to the hypotheses of theorem 2, each $\mu_k$ is absolutely continuous with respect to a positive measure $\nu$, then the $\mu_k$ are equi-absolutely continuous (and $\mu$ is also absolutely continuous with respect to $\nu$).

Proof : The part in parentheses is an easy consequence of the rest, which we now prove by contradiction. Suppose this false ; then there exists a neighbourhood $U$ of the origin in $E$ such that, however small $\delta > 0$ is, there exist positive integer $k$ and set $Z \in \mathfrak{M}$ with $\mu_k(Z) \notin U$ but $\nu(Z) < \delta$. Let $V$ be a neighbourhood of the origin with $V + V \subseteq U$.

Starting off with $\delta_0 = 1$, $k(0) = 1$ we can now define sequences of sets $Z_r \in \mathfrak{M}$, of positive integers $k(r)$ and of positive numbers $\delta_r$ such that

$$\nu(Z_r) < \delta_{r-1}, \; k(r) > k(r-1), \; \mu_{k(r)}(Z_r) \notin U,$$

$$\delta_r < \tfrac{1}{2}\delta_{r-1} \text{ and } \mu_{k(r)}(Z) \in V \quad \text{whenever} \quad \nu(Z) < \delta_r .$$

Put $Y_n = \bigcup_{r \geqslant n} Z_r$. Then

$$\nu(Y_n \setminus Z_n) \leqslant \nu(Y_{n+1}) \leqslant \sum_{r>n} \nu(Z_r) < \sum_{r>n} \delta_r < \delta_n ,$$

so that $\mu_{k(n)}(Y_n \setminus Z_n) \in V$. But since

$$\mu_{k(n)}(Z_n) = \mu_{k(n)}(Y_n) - \mu_{k(n)}(Y_n \setminus Z_n) \notin U ,$$

we must have $\mu_{k(n)}(Y_n) \notin V$ .

Thus, although the sets $(Y_n)$ are decreasing and $\nu(Y_n) \to 0$ , we do not have $\mu_k(Y_n) \to 0$ uniformly in $k$ , which contradicts the result immediately preceding the statement of the theorem.

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University of Keele
KEELE, Staffs. ST5 5BG
(Grande-Bretagne)