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OPEN PROBLEMS IN THEORY OF METRIC LINEAR SPACES

by
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Let X be a metric linear space X with metric $\rho(x, y)$, i.e. a linear space with a metric in which the addition and multiplication by scalars are continuous. It is well known result of Kakutani [10] that in such the space X there is an invariant metric equivalent to the given one. Klee [11] has shown that if X is complete with respect to the given metric, then it is complete also with respect to an equivalent invariant metric $\tilde{\rho}$.

Let us put

$$\|x\| = \tilde{\rho}(x, 0) .$$

The function $\|x\|$ is called an F-norm. It is obvious that there is a one-to-one correspondance between F-norms and respective invariant metrics.

The first part of my talk will be connected with properties of F-norms.

The properties of metrics implies a following properties of F-norms :

1) $\|x\| = 0$ if and only if $x = 0$;
2) $\|-x\| = \|x\|$;
3) $\|x + y\| \leqslant \|x\| + \|y\|$.

A linear metric space with an F-norm is called F^{*}-space. If it is complete it is called F-space.

Each F-norm satisfies properties 1), 3), but my be interesting to find equivalent norms with additional properties. For example, the norm

$$\|x\|' = \sup_{|a| \leqslant 1} \|ax\|$$

is equivalent to the norm $\|x\|$ and it has additional properties

2') $\|ax\|' = \|x\|'$ for all $|a| = 1$

and

4) $\|tx\|'$ is a nondecreasing function for positive t .

Stronger result was given by Eidelhiet and Mazur [8] namely that there is always an equivalent norm such that

4') the function $\|tx\|$ is strictly increasing for positive t and all $x \neq 0$.

In the paper [3] it was shown that condition 4), can be replaced by a stronger one. Namely there is always an equivalent norm such that

4") the function tx is for positive t increasing concave and C^{∞}-class.

We say that a norm $\|x\|$ is concave (increasing, C^{∞}-class, analytic), if the function $\|tx\|$ is for positive t and $x \neq 0$ concave (respectivaly increasing, C^{∞}-class, analytic).

PROBLEM 1. - Let X be an F-space with a norm $\|x\|$. Does always an equivalent analytic norm exist ?

Let X be an F-space with the norm $\|x\|$. We say that a norm $\|x\|'$ is unbounded if $\sup_{x \in X} \|x\|' = +\infty$. In the paper [3] it was shown that there are spaces in which there is no equivalent unbounded norms. These space can characterized in a following way .

There is no unbounded norm in X if and only if for each neighbourhood of zero U there is a number n such that

$$X = \underbrace{U + \ldots + U}_{n \text{ fold}}$$

PROBLEM 2. - Does each F-space contains an infinite dimensional subspace with unbounded norm ? Does there exist a subspace such that in the quotient space there is an unbounded norm ?

We say that an F-space is locally bounded if there is a bounded neighbourhood of zero. It is known that in such a space there is a p-homogeneous F-norm equivalent to a given one (see [1] and [14]). (An F-norm $\|x\|$ is called to be p-homogeneous if $\|tx\| = |t|^p \|x\|$, $0 < p \leq 1$).

Let X and Y be two F-spaces over reals, with F-norms $\| \ \|_X$ and $\| \ \|_Y$ respectively. Since it does not lead to missunderstanding we denote both the norms in this same way $\| \ \|$. A mapping U non necessarily linear of the space X onto Y is called an isometry if $\|U(x) - U(y)\| = \|x - y\|$. An isometry U is called a rotation if additionally $U(0) = 0$. It is well known Mazur-Ulam (see [12]) therom that each rotation is a linear operator, provided X and Y being Banach spaces with homogeneous norms. Charzynski [5] has shown similar theorem for finite dimensional spaces with arbitrary F-norms. In paper [15] it was shown for locally bounded spaces with arbitrary concave norms.

PROBLEM 3. - Is each rotation a linear operator ?

A weaker problem is

PROBLEM 3'. - Is each rotation mapping a locally bounded space onto a locally bounded space linear without the assumption of concavity of norms ?

Let X be an F-space with an F-norm $\|x\|$. By $G(\| \|)$ we denote the group of rotations with respect to the norm $\| \|$. An F-norm is called maximal if for any equivalent F-norm $\|x\|_1$ such that $G(\| \|) \subset G(\| \|_1)$, $G(\| \|) = G(\| \|_1)$.

The notion of maximal norm for homogeneous norms in Banach space and majority of results were announced at Stockholm Congrees by A. Pelczynski and the author [13]. More exact informations about maximal norms will be contained in the book of the author [16].

We say that a norm $\|x\|$ is transitive if for each two elements x, y of norm one there is an isometry mapping x onto y. We say that a norm $\|x\|$ is almost transitive if for each x, y of norm one and for each $\varepsilon > 0$ there is anisometry U such that $\|Ux - y\| < \varepsilon$. One may show that each almost transitive norm is maximal. The standard norm in the spaces $L^p[0, 1]$ is almost transitive.

There is an old question of Banach : does each separable Banach space with transitive norm is a Hilbert space ? The answer is not known but there is a normed (non-complete) separable space with a transitive norm. There is also a Banach non-separable space with this same property.

A norm $\|x\|$ is called convex transitive if there is an element x of norm one such that $\operatorname{conv}\{Ux : U \in G(\| \|)\} = \{x : \|x\| \leq 1\}$. One may show that almost transitive norms are always maximal.

In the space C_o of periodic continuous complex valued functions the norm sup is convex transitive. If K is a Cantor set the norm sup in the space $C(K)$ of continuous real valued functions defined on K is convex transitive.

PROBLEM 4. - Is the norm sup in the space $C[0, 1]$ of continuous real on interval $[0, 1]$ maximal ?

Let X be an F-space with a basis. We say that an F-norm is symmetric if

$$\| \sum_{i=1}^{n} \ell_i e_i \| = \| \sum_{i=1}^{n} \varepsilon_i \ell_i e_{p_i} \|$$

for each $\varepsilon_i = \pm 1$ and for all permutations p_i .

It in F-space X the norm is symmetric then it is maximal (in particular the standard norms in the space ℓ^p , $0 < p \leqslant 1$, and c_o are maximal).

PROBLEM 5. - Let X be an F-space with an F-norm $\| x \|$. Does a Banach space Y with a homogeneous norm $\| x \|_1$ such that $G(\| \|) = G(\| \|_1)$ exist ?

Now problems connected with unconditional convergence and absolute convergence will be considered.

Let X be an F-space. A series $\sum_{n=1}^{\infty} x_n$ of elements of X is called unconditionally convergent if for any sequence $\varepsilon_n = \pm 1$ the series $\sum_{n=1}^{\infty} \varepsilon_n x_n$ is convergent. It is said to be stable convergent if for any bounded sequence of scalars a_n the series $\sum_{n=1}^{\infty} a_n x_n$ is convergent. Of course, each stable convergent series is unconditionally convergent. An exemple given in paper [19]shows that it does not hold in general. It is easy to verify that it holds in each locally pseudo-convex space i.e. in such spaces that the topology can be defined by a sequence of p-homogeneous (p can be different for different pseudonorms). P. Turpin [20] has shown that there are non-locally pseudo-convex spaces in which stable and unconditional convergence are equivalent.

The problem of equivalence of stable and unconditional convergence is strictly related to the problem of existence of integrals of scalar valued functions with respect to the vector valued measure. More exactly. Let X be an F-space. Let Ω be a set, Σ be a σ-field of subsets of the set Ω , μ be a countably additive function defined for $E \in \Sigma$ and with values in X . We can easily define an integrals of simple functions in a standard way. The problem is the existence of integrals with respect to of measurable bounded functions.

Integrals of this type for $X = S[0, 1]$ (space of all measurable functions defined on interval [0, 1] with the convergence with respect to the Lesbegues measure) play an important role in the theory of stochastic processes.

The example given in [19] shows that there is an F-space X and an atomic measure $\mu(E)$ and a real valued bounded measurable function $x(t)$ such that the integral $\int_{\Omega} x(t)\, d(t)$ does not exist.

PROBLEM 6. - Suppose that $\mu(E)$ is nonatomic. Does exists an integral $\int_{\Omega} x(t)\, d\mu(t)$ for each measurable bounded scalar valued function $x(t)$?

PROBLEM 7. - Let $X = S[0, 1]$. Does an integral $\int_{\Omega} x(t)\, d\mu$ exist for each measurable bounded scalar valued function $x(t)$ and each measure $\mu(E)$?

Let X be an F-space. We say that a serie $\sum_{n=1} x_n$ is absolutely metric convergent if $\sum_{n=1}^{\infty} \|x_n\| < +\infty$. Dvoretzky and Rogers [6] have shown that in each infinite dimensional Banach space with homogeneous norm there is an unconditionally convergent series which is not absolutely convergent. This result can be easily extended on all infinite dimensional F-spaces (see [17]).

Another notion of absolute convergence is following. Let U be a starlike open set containing 0 as its internal point. Let us put

$$[x]_U = \inf \{t\ 0 : \frac{x}{t} \in U\} .$$

We say that a series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent if for any U such as above the series $\sum_{n=1}^{\infty} [x_n]_U$.

An F-space is locally convex if and only if each absolutely convergent series is unconditionally convergent. The question when unconditional convergence implies absolute convergence is open in general case. Gröthendieck [9] has shown that in locally convex spaces both convergence are equivalent if and only if the spaces are nuclear.

Of course, the classical definition of nuclear spaces based on the notion of nuclear operators is not applicable for non-locally convex spaces. Thus we define nuclear spaces basing on the notion of diametral approximative dimension.

Let X be an F-space. We say that X is a nuclear space if

$$\lim_{n \to \infty} n\, \delta_n(K, U) = 0$$

for each compact set K and open set U , where

$$\delta_n(K, U) = \inf \{\varepsilon > 0 : K \subset L_n + \varepsilon U ,\ L_n \text{ runs over subspaces of dimension } n\} .$$

As follows from result Dynin and Mitiagin [7] in the case of locally convex spaces the above definition is equivalent to the classical one.

PROBLEM 8. - Let X be a nuclear F-space. Does each unconditionally convergent series is absolutely convergent ?

There are examplex of non-locally convex nuclear spaces but it is open.

PROBLEM 9. - Does a nuclear locally pseudo-convex non-locally convex F-space exist ? (1)

If there is a basis in nuclear locally p-convex F-space X (locally p-convex space it is a locally pseudo-convex space in which all p-homogeneous pseudo-norms can be choosen with this same p), then X is locally convex.

There are also other problems connected with nuclear non-locally convex spaces. For example :

PROBLEM 10. - Do in each nuclear F-space non trivial linear continuous functionals exist ?

In the paper [4] it was shown that each infinite dimensional locally convex space, which is not a Banach space contains infinite dimensional nuclear space.

PROBLEM 11. - Does each non-locally bounded infinite dimensional F-space contain an infinite dimensional nuclear space ?

If a nuclear F-space does not contain arbitrarily short lines (i.e. there is a 0 such that for all $x \neq 0$ $\sup_{-\infty<t<+\infty} \|tx\| > a$) then it is a Montel space (i.e. all bounded sets are compact).

PROBLEM 12. - Is each nuclear space a Montel space ?

The problem 12 is strictly related to a

PROBLEM 12'. - Let X be an F-space. Let K be a bounded set. Suppose that for each open starlike U containing 0 as an internal point

$$\delta_n(K, U) \to 0 \quad .$$

Is K a compact set ?

P. Turpin has shown that it holds for the space L^0 of all measurable functions with convergence with respect to the measure.

At the end, I want to present few question connected with the existence of nontrivial linear functionals. The main problem is :

PROBLEM 14. - Does each infinite dimensional F-space contains an infinite dimensional F-subspace with nontrivial linear continuous functionals ? Does it hold if we additionally assume that the space is either locally bounded or nuclear or without unbounded norms ?

We know only that if there are arbitrarily short lines in the space (i.e. for each $\varepsilon > 0$ there is an $x \neq 0$ such that $\sup\limits_{-\infty<t<+\infty} \|tx\| < \varepsilon$) then the space contains the space s of all sequences. Thus it contains an infinite dimensional subspace with nontrivial linear continuous functionals. It is not known does each space without unbounded norm contains the space s .

The following two problems are in certains sense dual to the problem 14.

PROBLEM 15. - Does each F-space contain a subspace such that there are nontrivial linear continuous functionals in the quotient space ?

PROBLEM 16. - Let X be a non-locally convex space. Does X contain a subspace such that there is no nontrivial linear continuous functionals in the quotient space ?

It is known [18] that in the space $S[0, 1]$ for any element x_1 there is a continuous differentiable function $x(t)$ defined on interval $[0, 1]$ with values in $S[0, 1]$ such that $x(0) = 0$, $x(1) = x(1)$ and $\frac{dx}{dt} = 0$.

PROBLEM 17. - Let X be an F-space without nontrivial linear continuous functional. Let $x_1 \in X$. Does a function $x(t)$ with vanishing derivative such that $x(0) = 0$ and $x(1) = x_1$ exist ?

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(1) After the Conference J.P. Ligaud has shown that each nuclear locally pseudoconvex space is locally convex (see J.P. Ligaud "Solution d'un problème de S. Rolewicz sur les espaces nucleaires" C.R. Acad. Sc. Sér. A T. 273, (1971), 113-114 and "Sur les rapports des topologies et bornologies dans les espaces nucleaires "Stud. Math. in print)

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