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Diagonal operators on spaces of measurable functions

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1. Introduction.

We denote by \( L \) the set of equivalence classes of real-valued measurable functions on a fixed measure space \( (X, \Sigma, \mu) \). \( L \) is an algebra with unit and a vector lattice with respect to almost everywhere pointwise operations. The space of essentially bounded real-valued functions \( L^\infty = L^\infty(\mu) \) is a normed subalgebra of \( L \) and \( L \) is a module over \( L^\infty \) with respect to almost everywhere pointwise multiplication. A subspace \( M \) of \( L \) is a solid sublattice of \( L \) if and only if \( M \) is an \( L^\infty \)-submodule of \( L \) \({}^4\). We will call an \( L^\infty \)-submodule \( M \) of \( L \) a locally convex \( L^\infty \)-module if \( M \) is a locally convex vector space whose topology is given by a family of seminorms \( p \) satisfying

\[
p(af) \leq ||a||_\infty p(f) \quad a \in L^\infty, f \in M.
\]

Such a seminorm is called a scalar \( L^\infty \)-seminorm \({}^7\). Since a scalar \( L^\infty \)-seminorm defined on a solid sublattice of \( L \) is a lattice seminorm and vice versa, \( M \) is a locally convex \( L^\infty \)-module if and only if it is a locally convex vector lattice and solid in \( L \) \({}^4\). The Banach spaces \( L^p(\mu), 1 \leq p \leq \infty \), and Köthe spaces equipped with Köthe topologies \({}^{12}\) are examples of locally convex \( L^\infty \)-modules.

A linear operator \( T \) mapping a subspace \( M \) of \( L \) into another subspace of \( L \) will be called diagonal if there is a locally measurable real-valued function \( g \) on \( X \) such that \( Tf = gf \) for every \( f \) in \( M \). A linear operator \( T \) mapping an \( L^\infty \)-submodule \( M \) into another \( L^\infty \)-submodule of \( L \) will be called \( L^\infty \)-linear if \( T(af) = aT(f) \) for every \( a \) in \( L^\infty \) and \( f \) in \( M \).

From now on \( M \) and \( N \) will denote locally convex \( L^\infty \)-modules (or equivalently, locally convex solid sublattices of \( L \)). Further \( N \) is assumed to be order complete. If \( A \) is a subset of \( L \), then \( A^+ \) denotes the set of positive elements of \( A \).

We present our results without proofs; a full account will appear elsewhere. Finally, we wish to express our gratitude to the Scientific and Technical Research Council of Turkey for their support.
2. $L^\infty$-linear operators.

Let $\mathcal{C}$ be the set of positive continuous linear operators from $M$ into $N$. Then $L(M,N) = \mathcal{C} - \mathcal{C}$ is a solid sublattice of the space $L^0(M,N)$ of order bounded linear operators from $M$ into $N$. By $\mathcal{H}(M,N)$ we denote the space of continuous $L^\infty$-linear operators from $M$ into $N$.

**Lemma.** $\mathcal{H}(M,N)$ is a sublattice of $L(M,N)$.

A locally convex $L^\infty$-module $M$ is said to have the dominated convergence property if for every sequence $(f_n)$ in $L$ with $|f_n| \leq g$ for some $g$ in $M$ and $\lim f_n(x) = f(x)$ on $X$, we have $\lim f_n = f$ in $M$.

**Proposition 1.** Let $A$ be a Köthe space, $T$ a Köthe topology on $A$ and $A^\alpha$ the $\alpha$-dual of $A$. Consider the following conditions:

- **a)** $T$ is compatible with the duality $(\cdot, \cdot^\alpha)$.
- **b)** If $f_n \in A$ and $f_n(x) \to 0$ on $X$ then $\lim f_n = 0$ in $\Lambda(T)$.
- **c)** $\Lambda(T)$ has the dominated convergence property.
- **d)** If $p$ is one of the scalar $L^\infty$-seminorms defining the topology $T$ on $A$ and $f \in A$, then for every $\epsilon > 0$ there exists $\delta > 0$ such that $\mu(E) < \delta$ implies $p(x \in f) < \epsilon$.

We have $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$.

We will also consider the following condition.

- **(A)** for every $f \in M^+$ there is an increasing sequence $(s_n)$ of positive simple functions of bounded support such that $s_n(x) + f_n(x)$ on $X$ and $\lim s_n = f$ in $M$.

The Banach spaces $L^p(\mu)$, $1 \leq p < \infty$, satisfy this condition.

**Proposition 2.** If a Köthe space $\Lambda(T)$ has the dominated convergence property, it satisfies (A).

From now on we assume $L^1(\mu)' = L^\infty(\mu)$.

A diagonal operator is certainly $L^\infty$-linear. Under certain assumptions the converse is also true.

**Proposition 3.**

- **a)** Let $M$ satisfy condition (A). If for every set of finite measure $B$, the characteristic function $\chi_B \in M$, then every element of $\mathcal{H}_0(M,N)$ is a diagonal operator.

- **b)** If $M$ is a Köthe space which has the dominated convergence property, then every element of $\mathcal{H}_0(M,N)$ is a diagonal operator.
Remark: The hypothesis of the proposition is satisfied by \( L^p(u) \), \( 1 \leq p < \infty \).

On the other hand, if \( T : L^\infty \to N \) is \( L^\infty \)-linear, since \( T(f) = T(1)f \) for every \( f \in L^\infty \), it is also diagonal.

The set of idempotents in \( L^\infty \) is denoted by \( I_\infty \) and non-negative finite linear combinations of elements of \( I_\infty \) are dense in \((L^\infty)^+\). If \( \chi \in I_\infty \), then \( \chi' = 1 - \chi \in I_\infty \) also.

**PROPOSITION 4.** There is a projection \( P \) of \( \mathcal{L}(M,N) \) onto \( \mathcal{N}_\infty(M,N) \) with \( 0 \leq P \leq I \).

The projection is constructed in successive steps. First, for \( T \in C \) and \( f \in M^+ \) we define an element of \( N \) by

\[
P(T)(f) = \sum_{I} \{ \chi T(\chi f) + \chi' T(\chi' f) \}.
\]

We prove that \( P(T) \) is additive on \( M^+ \) and then extend it to a positive linear operator on \( M \). In the next step \( P \) is proved to be additive on \( C \) and then extended to \( \mathcal{L}(M,N) \).

Remark 1. If we define an \( L^\infty \)-module structure on \( \mathcal{L}(M,N) \) by letting \( (a.T)(f) = T(af) \) for \( f \) in \( M \) and \( a \) in \( L^\infty \), then \( P \) is also \( L^\infty \)-linear.

Remark 2. If we take \( \mu \) to be the counting measure on the set of positive integers, a Köthe space becomes a solid sequence space \([5]\). Certain operators on sequence spaces can be represented by infinite-matrices \([8; p. 20]\). If \( (t_{ij}) \) is the matrix which represents the operators \( T \), then \( P(T) \) is the operator represented by the diagonal of the matrix \( (t_{ij}) \).

Let \( M \) and \( N \) be Banach sublattices of \( L \), and \( \mathcal{N}(M,N) \) the space of nuclear operators from \( M \) into \( N \) with the nuclear norm \( r( \cdot ) \). Every nuclear operator can be written as the difference of two positive nuclear operators. If \( u_i \in M^+ \) and \( g_i \in N \), \( i = 1, \ldots, n \), by \( \sum u_i \otimes g_i \) we denote the nuclear operator which sends each \( f \in M \) to \( \sum u_i(f) g_i \). We consider the following conditions on a Banach \( L^\infty \)-module \( Q \).

(B) Given \( f \in Q \) and \( \varepsilon > 0 \), there is \( \delta > 0 \) such that \( \mu(E) < \delta \) implies \( \|f_{|E^c}\| < \varepsilon \).

(C) The support of each \( f \in Q \) is \( \sigma \)-finite.

(D) \( Q \) has the dominated convergence property.

By \( \mathcal{N}_\infty(M,N) \) we will denote the space of nuclear \( L^\infty \)-linear operators from \( M \) into \( N \) with the nuclear norm.
PROPOSITION 5. - Let \( M \) and \( N \) the Banach \( L^\infty \)-modules. If \( M \) satisfies (B), \( N \) satisfies (C) and (D) and further for every finite family of atoms \( \{x_1, \ldots, x_n\}, \ u \in M' \) and \( g \in N \) we have
\[
(\forall n) \quad r(\sum_{k=1}^{n} u \otimes x_k g) \leq \|\sum_{k=1}^{n} u_{x_k g}\| \quad \|\sum_{k=1}^{n} u_{x_k g}_{x_k g}\|
\]
then the projection \( P \) maps \( \eta(M,N) \) onto \( \eta_\sigma(M,N) \) such that \( r(P(T)) \leq r(T) \) for each \( T \in \eta(M,N) \).

Remark: If \( M' \) has property (B) instead of \( M \), \( M \) has property (C) instead of \( N \) or if \( M' \) has property (D) instead of \( N \), the result still holds.

3. Diagonal and nuclear diagonal operators on \( L^p \)-spaces.

Let \( M \) and \( N \) be two normed \( L^\infty \)-modules and \( M \otimes N \) the complete projective tensor product as defined by Grothendieck [3]. Let \( K \) be the smallest closed subspace of \( M \otimes N \) containing all elements of the form \((af \otimes g) - (f \otimes ag)\) for every \( a \in L^\infty \), \( f \in M \) and \( g \in N \). The quotient space \( M \otimes N/K \) with the quotient norm is called the normed \( L^\infty \)-tensor product of \( M \) and \( N \), and denoted by \( M \otimes_\sigma N \).

If \( f \otimes g \) denotes \( f \otimes g \mod K \) for each \( f \in M \), \( g \in N \), then for \( u \in M \otimes_\sigma N \) the norm is given by \([4 \text{ and } 9]\)
\[
\gamma(u) = \inf \{ \sum_{i=1}^{m} \| f_i \| \| g_i \| : u = \sum_{i=1}^{m} f_i \otimes_\sigma g_i, f_i \in M, g_i \in N \}.
\]

With a measure space \((X, \Sigma, \mu)\) we associate for every real number \( s > 0 \) a weighted counting measure space as follows: \( \psi \) is the set of equivalence classes of atoms of \( \mu \) together with the equivalence class of sets of \( \mu \)-measure zero. We let \( \mu_a = \mu(A) \) for any \( A \in \alpha \), where \( a \in \psi \). For any subset \( S \) of \( \psi \) we define
\[
\psi^S(S) = \sum_{a \in S} \mu_a^S.
\]

PROPOSITION 6. - (Harte). Let \( 1/p + 1/q = 1/r \leq 1 \) where \( 1 \leq p, q \leq \infty \). Then \( L^p(\mu) \otimes_\sigma L^q(\mu) \) is isometrically \( L^r \)-isomorphic with \( L^r(\mu) \).

In the result complementary to this we have to use the weighted counting measure constructed above.

PROPOSITION 7. - Let \( s = 1/p + 1/q > 1 \) where \( 1 \leq p, q \leq \infty \). Then \( L^p(\mu) \otimes_\sigma L^q(\mu) \) is isometrically \( L^s \)-isomorphic with \( L^s(\mu^S) \).

This result can be found in [6]. Next we give characterizations of diagonal operators between \( L^p \)-spaces as another \( L^p \)-space. Again we have two cases, the
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first due to Harte [4] and the second to Orhon [6].

PROPOSITION 8. - Let 1/q - 1/p = 1/r where 1 ≤ p, q ≤ ∞. Then \( H^\infty(L^p, L^q) \) is isometrically \( L^\infty \)-isomorphic with \( L^r(\mu) \).

In the result complementary to this we again need the weighted counting measure.

PROPOSITION 9. - Let 1 ≤ p < q < ∞. Then \( H^\infty(L^p, L^q) \) is isometrically \( L^\infty \)-isomorphic with \( L^q(\mu) \).

Remark: Diagonal operators between \( L^p \)-spaces were characterized by A. Tong [11]. G. Crofts [1] has considered diagonal operators between sequence spaces.

Using the projection constructed in proposition 5 and its properties discussed in proposition 5, we can define a continuous linear operator from \( L^p(\mu) \otimes L^q(\mu) \) onto the space \( \mathcal{H}(L^p, L^q) \) of diagonal nuclear operators. This enables us to characterize \( \mathcal{H}(L^p, L^q) \) by using propositions 6 and 7.

PROPOSITION 10. - \( \mathcal{H}(L^p, L^q) \) is isometrically isomorphic with

(i) \( L^1(\mu^{1/r}) \), if \( 1 ≤ q < p < ∞ \) and \( 1/r = 1/q - 1/p \).

(ii) \( L^1(\psi_q) \), if \( 1 ≤ p = q < ∞ \) where \( \psi_q \) denotes the set of equivalence classes of atoms of \( \mu \).

(iii) \( L^s(\mu^{1-s}) \), if \( 1 ≤ p < ∞ \) and \( s = pq/pq - q + p \).

(iv) \( L^p(\mu^{1-p}) \), if \( 1 < p < ∞ \) and \( q = ∞ \).

Remark: In proposition 10 the cases \( \mathcal{H}(L^\infty, L^p) \), \( 1 ≤ p < ∞ \) and \( \mathcal{H}(L^1, L^\infty) \) are not covered. In the case \( \mathcal{H}(L^1, L^\infty) \) our method breaks down, since in this case the projection \( p \ (\text{cap.}) \) does not take nuclear operators to nuclear diagonal operators. Nuclear diagonal operators on \( L^p \)-spaces were characterized by A. Tong [11].

BIBLIOGRAPHIE


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