MÉMOIRES DE LA S. M. F.

M. ORHON
T. TERZIOGLU

# Diagonal operators on spaces of measurable functions

*Mémoires de la S. M. F.*, tome 31-32 (1972), p. 265-270
<http://www.numdam.org/item?id=MSMF_1972__31-32__265_0>

Colloque Anal. fonctionn. [1971, Bordeaux]
Bull. Soc. math. France,
Mémoire 31-32, 1972, p. 265-270.

# DIAGONAL OPERATORS ON SPACES OF MEASURABLE FUNCTIONS

by

M. ORHON and T. TERZIOGLU

## 1. Introduction.

We denote by $L$ the set of equivalence classes of real-valued measurable functions on a fixed measure space $(X, \Sigma, \mu)$. $L$ is an algebra with unit and a vector lattice with respect to almost everywhere pointwise operations. The space of essentially bounded real-valued functions $L^\infty = L^\infty(\mu)$ is a normed subalgebra of $L$ and $L$ is a module over $L^\infty$ with respect to almost everywhere pointwise multiplication. A subspace $M$ of $L$ is a solid sublattice of $L$ if and only if $M$ is an $L^\infty$-submodule of $L$ [4]. We will can an $L^\infty$-submodule $M$ of $L$ a *locally convex* $L^\infty$-*module* if $M$ is a locally convex vector space whose topology is given by a family of seminorms $p$ satisfying

$$p(af) \leqslant \|a\|_\infty \, p(f) \qquad a \in L^\infty \, , \; f \in M \, .$$

Such a seminorm is called a *scalar* $L^\infty$-*seminorm* [7]. Since a scalar $L^\infty$-seminorm defined on a solid sublattice of $L$ is a lattice seminorm and vice versa, $M$ is a locally convex $L^\infty$-module if and only if it is a locally convex vector lattice and solid in $L$ [4]. The Banach spaces $L^p(\mu)$, $1 \leqslant p \leqslant \infty$, and Köthe spaces equipped with Köthe topologies [12] are examples of locally convex $L^\infty$-modules.

A linear operator $T$ mapping a subspace $M$ of $L$ into another subspace of $L$ will be called *diagonal* if there is a locally measurable real-valued function $g$ on $X$ such that $Tf = gf$ for every $f$ in $M$. A linear operator $T$ mapping an $L^\infty$-submodule $M$ into another $L^\infty$-submodule of $L$ will be called $L^\infty$-*linear* if $T(af) = a\,T(f)$ for every $a$ in $L^\infty$ and $f$ in $M$.

From now on $M$ and $N$ will denote locally convex $L^\infty$-modules (or equivalently, locally convex solid sublattices of $L$). Further $N$ is assumed to be order complete. If $A$ is a subset of $L$, then $A^+$ denotes the set of positive elements of $A$.

We present our results without proofs ; a full account will appear elsewhere. Finally, we wish to express our gratitude to the Scientific and Technical Research Council of Turkey for their support.

## 2. $L^\infty$-linear operators.

Let $\mathcal{C}$ be the set of positive continuous linear operators from $M$ into $N$. Then $\mathcal{L}(M,N) = \mathcal{C} - \mathcal{C}$ is a solid sublattice of the space $L^b(M,N)$ of order bounded linear operators from $M$ into $N$. By $\mathcal{H}_\infty(M,N)$ we denote the space of continuous $L^\infty$-linear operators from $M$ into $N$.

LEMMA. - $\mathcal{H}_\infty(M,N)$ is a sublattice of $\mathcal{L}(M,N)$.

A locally convex $L^\infty$-module $M$ is said to have the dominated convergence property if for every sequence $(f_n)$ in $L$ with $|f_n| \leqslant g$ for some $g$ in $M$ and $\lim f_n(x) = f(x)$ on $X$, we have $\lim f_n = f$ in $M$.

PROPOSITION 1. - *Let $\Lambda$ be a Köthe space, $T$ a Köthe topology on $\Lambda$ and $\Lambda^\times$ the $\alpha$-dual of $\Lambda$. Consider the following conditions:*

a) $T$ *is compatible with the duality* $(\Lambda, \Lambda^\times)$.

b) *If* $f_n \in \Lambda$ *and* $f_n(x) \downarrow 0$ *on* $X$ *then* $\lim f_n = 0$ *in* $\Lambda(T)$

c) $\Lambda(T)$ *has the dominated convergence property.*

d) *If* $p$ *is one of the scalar* $L^\infty$*-seminorms defining the topology* $T$ *on* $\Lambda$ *and* $f \in \Lambda$, *then for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $\mu(E) < \delta$ *implies* $p(\chi_E f) < \varepsilon$.

*We have* (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d).

*We will also consider the following condition.*

(A) for every $f \in M^+$ there is an increasing sequence $(s_n)$ of positive simple functions of bounded support such that $s_n(x) \uparrow f_n(x)$ on $X$ and $\lim s_n = f$ in $M$.

The Banach spaces $L^p(\mu)$, $1 \leqslant p < \infty$, satisfy this condition.

PROPOSITION 2. - *If a Köthe space* $\Lambda(T)$ *has the dominated convergence property, it satisfies* (A).

From now on we assume $L^1(\mu)' = L^\infty(\mu)$.

A diagonal operator is certainly $L^\infty$-linear. Under certain assumptions the converse is also true.

PROPOSITION 3. - a) *Let* $M$ *satisfy condition* (A). *If for every set of finite measure* $B$, *the characteristic function* $\chi_B \in M$, *then every element of* $\mathcal{H}_\infty(M, N)$ *is a diagonal operator.*

b) *If* $M$ *is a Köthe space which has the dominated convergence property, then every element of* $\mathcal{H}_\infty(M, N)$ *is a diagonal operator.*

<u>Remark :</u> The hypothesis of the proposition is satisfied by $L^p(\mu)$ , $1\leqslant p<\infty$ . On the other hand, if $T : L^\infty \to N$ is $L^\infty$-linear, since $T(f) = T(1)f$ for every $f\in L^\infty$ , it is also diagonal.

The set of idempotents in $L^\infty$ is denoted by $I_\infty$ and non-negative finite linear combinations of elements of $I_\infty$ are dense in $(L^\infty)^+$ . If $\chi\in I_\infty$ , then $\chi' = 1-\chi\in I_\infty$ also.

PROPOSITION 4. - <u>There is a projection</u> $P$ <u>of</u> $\mathcal{L}(M,N)$ <u>onto</u> $\mathcal{H}_\infty(M,N)$ <u>with</u> $0\leqslant P\leqslant I$ .

The projection is constructed in successive steps. First, for $T\in\mathcal{C}$ and $f\in M^+$ we define an element of $N$ by

$$P(T)(f) = \bigwedge_{I_\infty} \{\chi\, T(\chi f) + \chi' T(\chi' f)\} .$$

We prove that $P(T)$ is additive on $M^+$ and then extend it to a positive linear operator on $M$ . In the next step $P$ is proved to be additive on $\mathcal{C}$ and then extended to $\mathcal{L}(M,N)$ .

<u>Remark 1.</u> If we define an $L^\infty$-module structure on $\mathcal{L}(M,N)$ by letting $(a.T)(f)=T(af)$ for $f$ in $M$ and $a$ in $L^\infty$ , then $P$ is also $L^\infty$-linear.

<u>Remark 2.</u> If we take $\mu$ to be the counting measure on the set of positive integers, a Köthe space becomes a solid sequence space [5] . Certain operators on sequence spaces can be represented by infinite-matrices [8 ; p. 20] . If $(t_{ij})$ is the matrix which represents the operators $T$ , then $P(T)$ is the operator represented by the diagonal of the matrix $(t_{ij})$ .

Let $M$ and $N$ be Banach sublattices of $L$ , and $\mathcal{N}(M,N)$ the space of nuclear operators from $M$ into $N$ with the nuclear norm $r(\ .\ )$ . Every nuclear operator can be written as the difference of two positive nuclear operators. If $u_i\in M'$ and $g_i\in N$ , $i=1,\dots, n$ , by $\sum^n u_i\otimes g_i$ we denote the nuclear operator which sends each $f\in M$ to $\sum^n u_i(f)\, g_i$ . We consider the following conditions on a Banach $L^\infty$- module $Q$ .

(B) Given $f\in Q$ and $\varepsilon>0$ , there is $\delta>0$ such that $\mu(E)<\delta$ implies $\|f\chi_E\|<\varepsilon$ .

(C) The support of each $f\in Q$ is $\sigma$-finite.

(D) $Q$ has the dominated convergence property.

By $\mathcal{N}_\infty(M,N)$ we will denote the space of nuclear $L^\infty$-linear operators from $M$ into $N$ with the nuclear norm.

PROPOSITION 5. - Let $M$ and $N$ the Banach $L^\infty$- modules. If $M$ satisfies (B), $N$ satisfies (C) and (D) and further for every finite family of atoms $\{\chi_1, \ldots, \chi_n\}$, $u \in M'$ and $g \in N$ we have

$$(*) \qquad r\left(\sum^n \chi_k u \otimes \chi_k g\right) \leqslant \left\| \left(\sum^n \chi_k\right) u \right\| \quad \left\| \left(\sum^n \chi_k\right) g \right\|$$

then the projection $P$ maps $\mathcal{N}(M,N)$ onto $\mathcal{N}_\infty(M,N)$ such that $r(P(T)) \leqslant r(T)$ for each $T \in \mathcal{N}(M,N)$.

Remark : If $M'$ has property (B) instead of $M$, $M$ has property (C) instead of $N$ or if $M'$ has property (D) instead of $N$, the result still holds.

## 3. Diagonal and nuclear diagonal operators on $L^p$-spaces.

Let $M$ and $N$ be two normed $L^\infty$- modules and $M \otimes N$ the complete projective tensor product as defined by Grothendieck [3]. Let $K$ be the smallest closed subspace of $M \otimes N$ containing all elements of the form $(af \otimes g) - (f \otimes ag)$ for every $a \in L^\infty$, $f \in M$ and $g \in N$. The quotient space $M \otimes N/K$ with the quotient norm is called the normed $L^\infty$-tensor product of $M$ and $N$, and denoted by $M \otimes_\infty N$. If $f \otimes_\infty g$ denotes $f \otimes g$ mod $K$ for each $f \in M$, $g \in N$, then for $u \in M \otimes_\infty N$ the norm is given by [4 and 9]

$$\gamma_\infty(u) = \inf\left\{\sum^\infty \|f_i\| \quad \|g_i\| : u = \sum^\infty f_i \otimes_\infty g_i \ , \ f_i \in M \ , \ g_i \in N\right\} .$$

With a measure space $(X, \Sigma, \mu)$ we associate for every real number $s > 0$ a weighted counting measure space as follows : $\psi$ is the set of equivalence classes of atoms of $\mu$ together with the equivalence class of sets of $\mu$-measure zero. We let $\mu_\alpha = \mu(A)$ for any $A \in \alpha$, where $\alpha \in \psi$. For any subset $S$ of $\psi$ we define

$$\tilde{\mu}^s(S) = \sum_{\alpha \in S} \mu_\alpha^s .$$

PROPOSITION 6. - (Harte). Let $1/p + 1/q = 1/r \leqslant 1$ where $1 \leqslant p, q \leqslant \infty$. Then $L^p(\mu) \otimes_\infty L^q(\mu)$ is isometrically $L^\infty$-isomorphic with $L^r(\mu)$.

In the result complementary to this we have to use the weighted counting measure constructed above.

PROPOSITION 7. - Let $s = 1/p + 1/q > 1$ where $1 \leqslant p, q \leqslant \infty$. Then $L^p(\mu) \otimes_\infty L^q(\mu)$ is isometrically $L^\infty$-isomorphic with $L^1(\tilde{\mu}^s)$.

This result can be found in [6]. Next we give characterizations of diagonal operators between $L^p$-spaces as another $L^p$-space. Again we have two cases, the first due to Harte [4] and the second to Orhon [6].

PROPOSITION 8. - Let $1/q - 1/p = 1/r$ where $1 \leq p$ , $q \leq \infty$ . Then $\mathcal{H}_\infty(L^p(\mu), L^q(\mu))$ is isometrically $L^\infty$-isomorphic with $L^r(\mu)$ .

In the result complementary to this we again need the weighted counting measure.

PROPOSITION 9. - Let $1 \leq p < q \leq \infty$ . Then $\mathcal{H}_\infty(L^p(\mu) , L^q(\mu))$ is isometrically $L^\infty$-isomorphic with $L^\infty(\tilde{\mu})$ .

Remark : Diagonal operators between $l^p$-spaces were characterized by A. Tong [11]. G. Crofts [1] has considered diagonal operators between sequence spaces.

Using the projection constructed in proposition 4 and its properties discussed in proposition 5, we can define a continuous linear operator from $L^{p'}(\mu) \otimes_\infty L^q(\mu)$ onto the space $\mathcal{N}_\infty(L^p , L^q)$ of diagonal nuclear operators. This enables us to characterize $\mathcal{N}_\infty(L^p , L^q)$ by using propositions 6 and 7 .

PROPOSITION 10. - $\mathcal{N}_\infty(L^p(\mu) , L^q(\mu))$ is isometrically isomorphic with

(i) $L^1(\tilde{\mu}^{1/r})$ , if $1 \leq q < p < \infty$ and $1/r = 1/q - 1/p$ .

(ii) $l^1(\psi_o)$ , if $1 \leq p = q < \infty$ where $\psi_o$ denotes the set of equivalence classes of atoms of $\mu$ .

(iii) $L^s(\tilde{\mu}^{1-s})$ , if $1 \leq p < \infty$ and $s = pq/pq - q + p$ .

(iv) $L^{p'}(\tilde{\mu}^{1-p'})$ , if $1 < p < \infty$ and $q = \infty$ .

Remark : In proposition 10 the cases $\mathcal{N}_\infty(L^\infty, L^p)$ , $1 \leq p \leq \infty$ and $\mathcal{N}_\infty(L^1 , L^\infty)$ are not covered. In the case $\mathcal{N}_\infty(L^1 , L^\infty)$ our method breaks down, since in this case the projection p (cap.)does not take nuclear operators to nuclear diagonal operators. Nuclear diagonal operators on $l^p$-spaces were characterized by A. Tong [11].

## BIBLIOGRAPHIE

[1] CROFTS (G.). - Concerning perfect Fréchet spaces and diagonal transformations. Math. Ann. 182, 67-76, (1969).

[2] DIEUDONNÉ (J.). - Sur les espaces de Köthe. J. Analyse Math. 1, 81-115, (1951).

[3] GROTHENDIECK (A.). - Produits tensoriels topologiques et espaces nucléaires. Mém. Amer. Math. Soc., N° 16 (1955).

[4] HARTE (R. E.). - Modules over a Banach algebra. Ph. D. thesis. Cambridge, 1964.

[5] KOTHE (G.). - Topologische lineare Räume I. Springer-Verlag, 1960.

[6] ORHON (M.). - Vector semi-norms on modules. Ph. D. thesis, University of Wales, 1969.

[7] ORHON (M.). - On the Hahn-Banach theorem for modules over C(S). J. London Math. Soc. (2), 1, 363-368 (1969).

[8] PERESSINI (A. L.). - Ordered topological vector spaces. Harper and Row. 1967.

[9] RIEFFEL (M. A.). - Induced Banach representations of Banach algebras and locally compact groups. J. Func. Anal. 1, 443-491, (1967).

[10] SCHAEFER (H. H.). - Topological vector spaces. Macmillan 1966.

[11] TONG (A.). - Diagonal nuclear operators on $l_p$-spaces. Trans. Amer. Math. Soc., 143, 237-247, (1969).

[12] WELLAND (R.). - On Köthe spaces. Trans. Amer. Math. Soc., 112, 267-277, (1964).

[13] ZAANEN (A. C.). - Integration. North-Holland 1967.

Mathematics Department
Middle East Technical University
ANKARA (Turquie)