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DIAGONAL OPERATORS ON SPACES OF MEASURABLE FUNCTIONS

by

M. ORHON and T. TERZIOGLU

1. Introduction.

We denote by $L$ the set of equivalence classes of real-valued measurable functions on a fixed measure space $(X, \mathcal{E}, \mu)$. $L$ is an algebra with unit and a vector lattice with respect to almost everywhere pointwise operations. The space of essentially bounded real-valued functions $L^\infty = L^\infty(\mu)$ is a normed subalgebra of $L$ and $L$ is a module over $L^\infty$ with respect to almost everywhere pointwise multiplication. A subspace $M$ of $L$ is a solid sublattice of $L$ if and only if $M$ is an $L^\infty$-submodule of $L$ [4]. We will call an $L^\infty$-submodule $M$ of $L$ a locally convex $L^\infty$-module if $M$ is a locally convex vector space whose topology is given by a family of seminorms $p$ satisfying

$$p(af) \leq ||a||_\infty p(f), \quad a \in L^\infty, f \in M.$$ 

Such a seminorm is called a scalar $L^\infty$-seminorm [7]. Since a scalar $L^\infty$-seminorm defined on a solid sublattice of $L$ is a lattice seminorm and vice versa, $M$ is a locally convex $L^\infty$-module if and only if it is a locally convex vector lattice and solid in $L$ [4]. The Banach spaces $L^p(\mu), 1 \leq p \leq \infty$, and Köthe spaces equipped with Köthe topologies [12] are examples of locally convex $L^\infty$-modules.

A linear operator $T$ mapping a subspace $M$ of $L$ into another subspace of $L$ will be called diagonal if there is a locally measurable real-valued function $g$ on $X$ such that $Tf = gf$ for every $f$ in $M$. A linear operator $T$ mapping an $L^\infty$-submodule $M$ into another $L^\infty$-submodule of $L$ will be called $L^\infty$-linear if $T(af) = aT(f)$ for every $a$ in $L^\infty$ and $f$ in $M$.

From now on $M$ and $N$ will denote locally convex $L^\infty$-modules (or equivalently, locally convex solid sublattices of $L$). Further, $N$ is assumed to be order complete. If $A$ is a subset of $L$, then $A^+$ denotes the set of positive elements of $A$.

We present our results without proofs; a full account will appear elsewhere. Finally, we wish to express our gratitude to the Scientific and Technical Research Council of Turkey for their support.
2. \( L^\infty \)-linear operators.

Let \( \mathcal{C} \) be the set of positive continuous linear operators from \( M \) into \( N \). Then \( \mathcal{L}(M,N) = \mathcal{C} \) is a solid sublattice of the space \( L^0(M,N) \) of order bounded linear operators from \( M \) into \( N \). By \( \mathcal{H}_\infty(M,N) \) we denote the space of continuous \( L^\infty \)-linear operators from \( M \) into \( N \).

**Lemma.** - \( \mathcal{H}_\infty(M,N) \) is a sublattice of \( \mathcal{L}(M,N) \).

A locally convex \( L^\infty \)-module \( M \) is said to have the dominated convergence property if for every sequence \( (f_n) \) in \( L \) with \( |f_n| \leq g \) for some \( g \) in \( M \) and \( \lim f_n(x) = f(x) \) on \( X \), we have \( \lim f_n = f \) in \( M \).

**Proposition 1.** - Let \( A \) be a Köthe space, \( T \) a Köthe topology on \( A \) and \( A^\times \) the \( \alpha \)-dual of \( A \). Consider the following conditions:

a) \( T \) is compatible with the duality \( (\alpha, \cdot^\times) \).

b) If \( f_n \in \alpha \) and \( f_n(x) \downarrow 0 \) on \( X \) then \( \lim f_n = 0 \) in \( A(T) \).

c) \( A(T) \) has the dominated convergence property.

d) If \( p \) is one of the scalar \( L^\infty \)-seminorms defining the topology \( T \) on \( A \) and \( f \in \alpha \), then for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( \mu(E) < \delta \) implies \( p(x_E f) < \epsilon \).

We have \( (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \).

We will also consider the following condition:

\((A)\) for every \( f \in M^+ \) there is an increasing sequence \( (s_n) \) of positive simple functions of bounded support such that \( s_n(x) \uparrow f(x) \) on \( X \) and \( \lim s_n = f \) in \( M \).

The Banach spaces \( L^p(\mu) \), \( 1 \leq p < \infty \), satisfy this condition.

**Proposition 2.** - If a Köthe space \( \Lambda(T) \) has the dominated convergence property, it satisfies \((A)\).

From now on we assume \( L^1(\mu)' = L^\infty(\mu) \).

A diagonal operator is certainly \( L^\infty \)-linear. Under certain assumptions the converse is also true.

**Proposition 3.** - a) Let \( M \) satisfy condition \((A)\). If for every set of finite measure \( B \), the characteristic function \( \chi_B \in M \), then every element of \( \mathcal{H}_\infty(M,N) \) is a diagonal operator.

b) If \( M \) is a Köthe space which has the dominated convergence property, then every element of \( \mathcal{H}_\infty(M,N) \) is a diagonal operator.
Remark: The hypothesis of the proposition is satisfied by $L^p(\mu), 1 \leq p < \infty$.

On the other hand, if $T : L^\infty \to N$ is $L^\infty$-linear, since $T(f) = T(1)f$ for every $f \in L^\infty$, it is also diagonal.

The set of idempotents in $L^\infty$ is denoted by $I_\infty$ and non-negative finite linear combinations of elements of $I_\infty$ are dense in $(L^\infty)^\ast$. If $\chi \in I_\infty$, then $\chi' = 1 - \chi \in I_\infty$ also.

**Proposition 4.** There is a projection $P$ of $\mathcal{L}(M,N)$ onto $\mathcal{N}_\infty(M,N)$ with $0 \leq P \leq I$.

The projection is constructed in successive steps. First, for $T \in C$ and $f \in M^\ast$ we define an element of $N$ by

$$P(T)(f) = \sum_{\chi \in I_\infty} \{\chi T(\chi f) + \chi' T(\chi' f)\}.$$ 

We prove that $P(T)$ is additive on $M^\ast$ and then extend it to a positive linear operator on $M$. In the next step $P$ is proved to be additive on $C$ and then extended to $\mathcal{L}(M,N)$.

Remark 1. If we define an $L^\infty$-module structure on $\mathcal{L}(M,N)$ by letting $(aT)(f) = T(af)$ for $f \in M$ and $a \in L$, then $P$ is also $L^\infty$-linear.

Remark 2. If we take $\mu$ to be the counting measure on the set of positive integers, a Köthe space becomes a solid sequence space [5]. Certain operators on sequence spaces can be represented by infinite-matrices [8; p. 20]. If $(t_{ij})$ is the matrix which represents the operators $T$, then $P(T)$ is the operator represented by the diagonal of the matrix $(t_{ij})$.

Let $M$ and $N$ be Banach sublattices of $L$, and $\mathcal{N}(M,N)$ the space of nuclear operators from $M$ into $N$ with the nuclear norm $r(\cdot)$. Every nuclear operator can be written as the difference of two positive nuclear operators. If $u_i \in M^\ast$ and $g_i \in N$, $i=1,\ldots,n$, by $\oplus u_i \otimes g_i$ we denote the nuclear operator which sends each $f \in M$ to $\sum_{i=1}^{n} u_i(f) g_i$. We consider the following conditions on a Banach $L^\infty$-module $Q$.

1. **(B)** Given $f \in Q$ and $\epsilon > 0$, there is $\delta > 0$ such that $\mu(E) < \delta$ implies $\|f|_E\| < \epsilon$.

2. **(C)** The support of each $f \in Q$ is $\sigma$-finite.

3. **(D)** $Q$ has the dominated convergence property.

By $\mathcal{N}_\infty(M,N)$ we will denote the space of nuclear $L^\infty$-linear operators from $M$ into $N$ with the nuclear norm.
PROPOSITION 5. - Let \( M \) and \( N \) the Banach \( L^\infty \) modules. If \( M \) satisfies (B), \( N \) satisfies (C) and (D) and further for every finite family of atoms \( \{x_1, \ldots, x_n\}, u \in M' \) and \( g \in N \) we have

\[
(\sum x_k u \otimes x_k g) \leq \|\sum x_k u\| \|\sum x_k g\|
\]

then the projection \( P \) maps \( \eta(M,N) \) onto \( \eta_{\infty}(M,N) \) such that \( r(P(T)) \leq r(T) \) for each \( T \in \eta(M,N) \).

Remark: If \( M' \) has property (B) instead of \( M \), \( M \) has property (C) instead of \( N \) or if \( M' \) has property (D) instead of \( N \), the result still holds.

3. Diagonal and nuclear diagonal operators on \( L^p \)-spaces.

Let \( M \) and \( N \) be two normed \( L^\infty \) modules and \( M \otimes N \) the complete projective tensor product as defined by Grothendieck [3]. Let \( K \) be the smallest closed subspace of \( M \otimes N \) containing all elements of the form \((af \otimes g) - (f \otimes ag)\) for every \( a \in L^\infty \), \( f \in M \) and \( g \in N \). The quotient space \( M \otimes N/K \) with the quotient norm is called the \( L^\infty \)-tensor product of \( M \) and \( N \), and denoted by \( M \otimes_{\infty} N \).

If \( f \otimes g \) denotes \( f \otimes g \) mod \( K \) for each \( f \in M \), \( g \in N \), then for \( u \in M \otimes_{\infty} N \) the norm is given by [4 and 9]

\[
\gamma_{\infty}(u) = \inf \left\{ \sum \|f_i\| \|g_i\| : u = \sum f_i \otimes_{\infty} g_i, f_i \in M, g_i \in N \right\}.
\]

With a measure space \((X, \Sigma, \mu)\) we associate for every real number \( s > 0 \) a weighted counting measure space as follows: \( \psi \) is the set of equivalence classes of atoms of \( \mu \) together with the equivalence class of sets of \( \mu \)-measure zero. We let \( \mu_a = \mu(A) \) for any \( A \in \alpha \), where \( a \in \psi \). For any subset \( S \) of \( \psi \) we define

\[
\psi^\infty(S) = \sum_{a \in S} \mu_a^S.
\]

PROPOSITION 6. - (Harte). Let \( 1/p + 1/q = 1/r \leq 1 \) where \( 1 \leq p, q \leq \infty \). Then \( L^p(\mu) \otimes_{\infty} L^q(\mu) \) is isometrically \( L^\infty \)-isomorphic with \( L^r(\mu) \).

In the result complementary to this we have to use the weighted counting measure constructed above.

PROPOSITION 7. - Let \( s = 1/p + 1/q > 1 \) where \( 1 \leq p, q \leq \infty \). Then \( L^p(\mu) \otimes_{\infty} L^q(\mu) \) is isometrically \( L^s \)-isomorphic with \( L^1(\mu) \).

This result can be found in [6]. Next we give characterizations of diagonal operators between \( L^p \)-spaces as another \( L^p \)-space. Again we have two cases, the
first due to Harte [4] and the second to Orhon [6].

PROPOSITION 8. - Let \( 1/q - 1/p = 1/r \) where \( 1 \leq p, q \leq \infty \). Then \( \mathcal{W}(L^p, L^q) \) is isometrically \( L^\infty \)-isomorphic with \( L^r \).

In the result complementary to this we again need the weighted counting measure.

PROPOSITION 9. - Let \( 1 \leq p < q \leq \infty \). Then \( \mathcal{W}(L^p, L^q) \) is isometrically \( L^\infty \)-isomorphic with \( L^\infty \).

Remark: Diagonal operators between \( L^p \)-spaces were characterized by A. Tong [11]. G. Crofts [1] has considered diagonal operators between sequence spaces.

Using the projection constructed in proposition 4 and its properties discussed in proposition 5, we can define a continuous linear operator from \( L^p(L^\infty) \otimes L^q(L^\infty) \) onto the space \( \mathcal{T}_\infty(L^p, L^q) \) of diagonal nuclear operators. This enables us to characterize \( \mathcal{T}_\infty(L^p, L^q) \) by using propositions 6 and 7.

PROPOSITION 10. - \( \mathcal{T}_\infty(L^p, L^q) \) is isometrically isomorphic with

(i) \( L^1(\psi_\infty) \), if \( 1 \leq q < p < \infty \) and \( 1/r = 1/q - 1/p \).

(ii) \( L^1(\psi_\infty) \), if \( 1 \leq p = q < \infty \) where \( \psi_\infty \) denotes the set of equivalence classes of atoms of \( \mu \).

(iii) \( L^\infty(\phi_{1-s}) \), if \( 1 \leq p < \infty \) and \( s = pq/pq - q + p \).

(iv) \( L^p(\phi_{1-p'}) \), if \( 1 \leq p < \infty \) and \( q = \infty \).

Remark: In proposition 10 the cases \( \mathcal{T}_\infty(L^\infty, L^p) \), \( 1 \leq p < \infty \) and \( \mathcal{T}_\infty(L^1, L^\infty) \) are not covered. In the case \( \mathcal{T}_\infty(L^1, L^\infty) \) our method breaks down, since in this case the projection \( p \) (cap.) does not take nuclear operators to nuclear diagonal operators. Nuclear diagonal operators on \( L^p \)-spaces were characterized by A. Tong [11].

BIBLIOGRAPHIE


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