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ON OPERATORS FACTORIZABLE THROUGH L_p SPACE

by

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In this paper we give some necessary and sufficient conditions for an operator between Banach spaces be factorizable through L_p space, also conditions for factorizability through a subspace, a quotient and a subspace of a quotient of L_p. Hence, we obtain characterizations of Banach spaces isomorphic with complemented subspaces, with subspaces, with quotients and with subspaces of quotients of L_p. These conditions are given in terms of p-absolutely summing and p-integral operators. We use the general theory of ideals of operators, necessary definitions and facts of the theory given in § I. For more detailed treatment the reader is refered to the paper [3], by A. Grothendieck, where it is exposed in frame of tensor product theory, and also to papers of A. Pietsch. We end the paper with some applications.

§ I. Normed ideals of operators.

In the sequel L(E,F) will denote all bounded linear operators from Banach space E into Banach space F and $\|u\|$ the norm of an operator.

Let for each pair of Banach spaces E, F be given a linear subspace A(E,F) of L(E,F) and $\alpha_{E,F}$ a norm on A(E,F) such that

I. if $u \in A(E,F)$, $v \in L(X,E)$, $w \in L(E,Y)$ then $wuv \in A(X,Y)$ and $\alpha_{X,Y}(wuv) \leqslant \alpha_{E,F}(u)\ \|w\|\ \|v\|$

2. if $u \in A(E,F)$ then $\alpha_{E,F}(u) \geqslant \|u\|$

3. if $u \in L(E,F)$ is one dimensional then $u \in A(E,F)$ and $\alpha_{E,F}(u) = \|u\|$

Then we say that $|A, \alpha|$ is a normed linear ideal of operators.
In further we shall write $\alpha(u)$ instead of $\alpha_{E,F}(u)$.

A normed linear ideal $|A, \alpha|$ is defined to be maximal if it satysfies the following condition :

if for $u \in L(E,F)$ there exists a constant M such that for each finite dimensional Banach spaces X, Y and operators $v \in L(X,E)$, $w \in L(F,Y)$ it is $\alpha(wuv) \leqslant M\ \|w\|\ \|v\|$ then $u \in A(E,F)$ and $\alpha(u) \leqslant M$.

We say that $u \in A^{*}(E,F)$ if there exists a constant M such that for each finite dimensional Banach spaces X, Y and operators $v \in L(X,E)$, $w \in L(F,Y)$ and $z \in A(Y,X)$ there holds

$$|\text{trace } (wuvz)| \leqslant M \, \|w\| \, \|v\| \, \alpha(z).$$

The least such constant M is denoted by $\alpha^{*}(u)$.

It is easy to check that $|A^{*}, \alpha^{*}|$ is a maximal normed ideal of operators. We call it the dual ideal of $|A, \alpha|$. Moreover, given normed linear ideal $|A, \alpha|$ we define the following ideals :

right injective envelope of $|A, \alpha|$, denoted $|A\backslash, \alpha\backslash|$, as follows $u \in A\backslash(E,F)$ if for some Banach space G and isometric embedding i of F into G it is $iu \in A(E,G)$,
$\alpha\backslash(u) = \inf \alpha(iu)$, where infimum is taken over all such G and i,
left injective envelope of $|A, \alpha|$, denoted by $|/A, /\alpha|$, as follows $u \in /A(E,F)$ if for some Banach space H and normed surjection j of H on E (i. e. j maps the unite disck in H on the unite disck in E) $u\,j \in A(H,F)$ $/\alpha(u) = \inf\limits_{H,j} \alpha(uj)$,

right projective envelope of $|A, \alpha|$, denoted by $|A/, \alpha/|$, as follows $u \in A/(E,F)$ if for each Banach space H and a normed surjection j of H onto F there exists $v \in A(E,H'')$ such that $iu = {}^{tt}jv$, i is the cannonical injection of F in F'' and ${}^{tt}j$ is the second adjoint of j,

left projective envelope of $|A, \alpha|$, denoted by $|\backslash A, \backslash\alpha|$, as follows $u \in \backslash A(E,F)$ if for each Banach space G and isometric embedding i of E into G there exists $v \in A(G,F'')$ such that $ju = vi$, j is the cannonical injection of F in F''.

One can verify the following

$\boxed{\text{I.I.}}$ if $|A, \alpha|$ is maximal then each of the above defined ideals is maximal also,

$\boxed{\text{I.2.}}$ if $|A, \alpha|$ is maximal then $|(A^{*})^{*}, (\alpha^{*})^{*}|$ is equall to $|A, \alpha|$,

$\boxed{\text{I.3.}}$ $|(/A)^{*}, (/\alpha)^{*}|$ is equall to $|A^{*}/, \alpha^{*}/|$,

$\boxed{\text{I.4.}}$ $|(A\backslash)^{*}, (\alpha\backslash)^{*}|$ is equall to $|\backslash A^{*}, \backslash\alpha^{*}|$.

Example I. Ideal of p-absolutely summing operators, $|\Pi_p, \pi_p|$
$u \in \Pi_p(E,F)$ if for some constant M for each $x_I, \ldots x_n \in E$ there holds

$$\sum_{i=I}^{n} \|u(x_i)\|^p \leqslant M \sup_{x' \in E',\, \|x'\| \leqslant I} \sum_{i=I}^{n} |\langle x_i, x'\rangle|^p ,$$

$\pi_p(u)$ is the least such constant M.

Example 2. Ideal of p-integral operators, $|I_p, \iota_p|$

$u \in I_p(E,F)$ if there exists a probability measure space $(\Omega, \mathcal{M}, \mu)$ and operators $v \in L(E, L^\infty(\Omega, \mu))$ and $w \in L(L_p(\Omega, \mu), F'')$ such that $wjv = iu$, where j is the canonical injection of $L^\infty(\Omega, \mu)$ into $L_p(\Omega, \mu)$ and i the canonical injection of F into F'',

$\iota_p(u)$ is defined as $\inf \|v\| \|w\|$, infimum is taken over all such probability measure spaces $(\Omega, \mathcal{M}, \mu)$ and operators v and w.

It was proved by A. Pietsch that

$\boxed{I.5}$ $|I_p^\backslash, \iota_p^\backslash|$ is equall to $|\Pi_p, \pi_p|$,

$\boxed{I.6}$ $|I_p^*, \iota_p^*|$ is equall to $|\Pi_q, \pi_q|$ $(\frac{1}{p} + \frac{1}{q} = 1)$.

§ 2. Ideal of L_p factorizable operators

By L_p space we shall mean any Banach space isometric with the space $L_p(\Omega, \mu)$ for some measure space $(\Omega, \mathcal{M}, \mu)$.

We say that $u \in \Gamma_p(E,F)$ if for some L_p space there exist operators $v \in L(E,L_p)$ and $w \in L(L_p,F'')$ such that $iu = wv$, i is the cannonical injection of F into F''.

$\gamma_p(u)$ is defined as $\inf_{v,w} \|v\| \|w\|$, v and w are as in the definition of $\Gamma_p(E,F)$

Proposition I. Let $1 \leqslant p \leqslant \infty$. $|\Gamma_p, \gamma_p|$ is a maximal normed ideal of operators Proof. We shall make use of the following equallity

$$\boxed{2.1} \quad ab = \inf_{t>0} (p^{-1} t^p a^p + q^{-1} t^{-q} b^q)$$

which is valid for positive numbers a, b and q defined by $\frac{1}{p} + \frac{1}{q} = 1$.

Let for $k = 1,2$ $u_k \in \Gamma_p(E,F)$ and let $iu_k = w_k v_k$, where $v_k \in L(E, L_p(\Omega_k, \mu_k))$, $w_k \in L(L_p(\Omega_k, \mu_k), F'')$ and $\|v_k\| \|w_k\| \leqslant \gamma_p(u_k) + \varepsilon$ (cf. the definition of $|\Gamma_p, \gamma_p|$)

Let Ω_o be the disjoint sum of Ω_1 and Ω_2 and let $\mu_1 = \frac{1}{2}(\mu_1 + \mu_2)$.

We define $v_o \in L(E, L_p(\Omega_o, \mu_c))$ and $w_o \in L(L_p(\Omega_o, \mu_o), F'')$ as follows $v_o(x)$ is a function on Ω_o which coincides with $v_1(x)$ on Ω_1 and with $v_2(x)$ on Ω_2, $w_o(f) = w_1(f_1) + w_2(f_2)$, where $f_1 = f|\Omega_1$ and $f_2 = f|\Omega_2$.

Simple computations show that $i(u_1 + u_2) = w_o v_o$ and

$$\boxed{2.2} \quad \|v_o\| \leqslant (\tfrac{1}{2}\|v_1\|^p + \tfrac{1}{2}\|v_2\|^p)^{\frac{1}{p}}$$

$$\boxed{2.3} \quad \|w_o\| \leqslant (2^{\frac{q}{p}}\|w_1\|^q + 2^{\frac{q}{p}}\|w_2\|^q)^{\frac{1}{q}}$$

Applying 2.1 we obtain

$$\|v_o\|\ \|w_o\| \leqslant p^{-1}\|v_o\|^p + q^{-1}\|w_o\|^q.$$ Hence and by 2.2, 2.3

$$\|v_o\|\ \|w_o\| \leqslant \tfrac{1}{2}\|v_1\|^p p^{-1} + 2^{\frac{q}{p}}\|w_1\|^q q^{-1} + \tfrac{1}{2}p^{-1}\|v_2\|^p + 2^{\frac{q}{p}}q^{-1}\|w_2\|^q.$$

But we can replace v_1 by $t_1\ v_1$ and w_1 by $t_1^{-1}\ w_1$ and the same with v_2 and w_2. Taking the infimum with respect to t_1, t_2 the right side of the above inequallity is equall to $\|v_1\|\ \|w_1\| + \|v_2\|\ \|w_2\|$.

This proves that $u_1 + u_2 \in \Gamma_p(E,F)$ and $\gamma_p(u_1 + u_2) \leqslant \gamma_p(u_1) + \gamma_p(u_2)$.

If $u \in \Gamma_p(E,F)$ then tu also and $\gamma_p(tu) = |t|\gamma_p(u)$. Thus $\Gamma_p(E,F)$ is a linear space and γ_p a norm on it. Properties 1., 2., 3. are obvious.

The maximality of $|\Gamma_p, \gamma_p|$ may be obtained by the methods from the theory of ultraproducts of Banach spaces, developed by J. Krivine and D. Dacunha-Castelle, cf. [1].

<u>Proposition 2</u>. Let $1 \leqslant p \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then

$u \in \Gamma_p^*(E,F)$ if and only if there exist Banach space G and operators $v \in \Pi_q(E,G)$, ${}^t w \in \Pi_p(F', G')$ such that $u = wv$,

$\gamma_p^*(u) = \inf\ \pi_q(v)\ \pi_p(w)$, infimum is taken over all such G, v and w.

<u>Proof</u>. Suppose $u \in \Gamma_p^*(E,F)$. By the definition for each $h \in L(l_p^n,E)$, $g \in L(F,l_p^n)$ and i-identity operator in l_p^n there holds

$$|\mathrm{trace}(guhi)| \leqslant \gamma_p^*(u)\ \|g\|\ \|h\|\ \gamma_p(i)$$

Since $\gamma_p(i) = 1$ this is equivalent to : for each $x_1,..x_n \in E$, $y_1',..y_n' \in F'$

$$\sum_{i=1}^n < u(x_i),\ y_i' > \leqslant \gamma_p(u) \sup_{x' \in K_1} \left(\sum_{i=1}^n |<x_i,x'>|^q\right)^{\frac{1}{q}} \sup_{y \in K_2} \left(\sum_{i=1}^n |<y,y_i'>|^p\right)^{\frac{1}{p}},$$

where K_1 and K_2 are unite discks in E' and F'' correspondingly.

Applying 2.1 we get

$$\sum_{i=1}^n < u(x_i),\ y_i' > \leqslant \gamma_p^*(u) \sup_{x' \in K_1, y \in K_2} \sum_{i=1}^n (q^{-1}|<x_1,x'>|^q + p^{-1}|<y,y_i'>|^p).$$

By the theorem on separations of cones in locally convex spaces it is equivalent to the existence of a probability measure μ on K - the cartesian product of K_1 and K_2 such that for each $x \in E$ and $y' \in F$

$$|<u(x),y'>| \leqslant \gamma_p^*(u)(q^{-1} \int_K |<x,x'>|^q d\mu(x') + p^{-1} \int_K |<y,y'>|^p d\mu(y)).$$

Replacing x by tx and y'by $t^{-1}y'$ and taking infimum we have by 2.1

$$\boxed{2.4} \quad |<u(x),y'>| \leqslant \gamma_p^*(u) \ (\int_K |<x,x'>|^q d\mu(x'))^{\frac{1}{q}} \ (\int_K |<y,y'>|^p d\mu(y))^{\frac{1}{p}}.$$

Let $v \in L(E,L_q(K, \mu))$ be defined by $v(x)(x'y'')=<x,x'>$ on K ,

similary $w_o \in L(F', L_p(K,\mu))$ is defined by $w_o(y')(x'y'')=<y',y''>$ on K .

Let G denote the closure of $v(E)$ in $L_q(K, \mu)$ and H the closure of $w_o(F')$ in $L_p(K, \mu)$.

By Pietsch theorem 1.5 $v \in \Pi_q(E,G)$, $w_o \in \Pi_p(F', H)$ and $\pi_q(v)$, $\pi_p(W_o) \leqslant 1$

The inequality 2.4 implies the existence of an operator $z \in L(G,H')$ such that $\|z\| \leqslant \gamma_p^*(u)$ and ${}^t w_o z v = iu$, i being the canonical injection of F into F". The image of G by ${}^t w_o z$ is in F, so let $w = {}^t w_o z$ be considered as a member of $L(G,F)$. Then

$$\pi_p({}^t w) \leqslant \pi_p(w_o) \ \|z\| \ \leqslant \gamma_p^*(u).$$

Thus G, v and w satisfy the required conditions of Proposition 2, moreover $\pi_q(v) \ \pi_p({}^t w) \quad \leqslant \quad \gamma^*{}_p(u)$. This proves the necessity.

Now assume $u = wv$, where $v \in \Pi_q(E,G)$ and ${}^t w \in \Pi_p(F',G')$.

Let X and Y be finite dimensional Banach spaces, $h \in L(X,E)$, $g \in L(F,Y)$ and $z \in \Gamma_p(Y,X)$. We have to prove

$$|\text{trace}(zguh)| \leqslant \pi_q(v) \ \pi_p({}^t w) \ \gamma_p(z) \ \|g\| \ \|h\|.$$

Let $z = z_1 z_2$, $z_1 \in L(L_p,X)$, $z_2 \in L(Y,L_p)$ and $\|z_1\| \ \|z_2\| \leqslant \gamma_p(z) + \varepsilon$.

Then $vhz_1 \in \Pi_q(L_p,G)$ and ${}^t(z_2 gw) \in \Pi_p(L'_p,G')$. It was proved by A. Perrson [8], that if ${}^t r \in \Pi_p(L'_p,G')$ then $r \in I_p(G,L_p)$ and $\iota_p(r) \leqslant \Pi_p({}^t r)$. Applying this we obtain that

$$z_2 gw \in I_p(G,L_p) \quad \text{and} \quad \iota_p(z_2 gw) \leqslant \pi_p({}^t(z_2 gw)).$$

Since $|I_p, \iota_p|$ is the dual ideal of $|\Pi_q, \pi_q|$ we have

$$|\text{trace } (z_2 gwvhz_1)| \leqslant \iota_p(z_2 gw)\pi_q(vhz_1) \leqslant \pi_p({}^t w {}^t g {}^t z_2) \ \pi_q(vhz_1). \text{ Hence}$$

$$|\text{ trace } (zguh) \leqslant \pi_q(v)\pi_p({}^t w)\|g\|\,\|h\|\,\|z_1\|\,\|z_2\|.$$

Because $\|z_1\|\,\|z_2\| \leqslant \gamma_p(z) + \varepsilon$ and ε is arbitrary small this ends the proof.

Corollary 1. $u \in L(E,F)$ is factorizable through L_p space (i.e. $u \in \Gamma_p(E,F)$) if and only if for each Banach space G and $v \in \Pi_q(F,G)$ it is ${}^t(vu) \in I_Q(G',E')$.

Proof. Let $u \in \Gamma_p(E,F)$ and $v \in \Pi_q(F,G)$. By Proposition 2 if ${}^t w \in \Pi_p(E', G')$ then $wv \in \Gamma_p^*(F,E)$. From this we deduce that ${}^t(vu) \in \Pi_p^*(G',E')$ and hence ${}^t(vu) \in I_q(G',E')$.

Conversly, if u satisfies the condition of Corollary then u belongs to the dual ideal of $|\Gamma_p^*, \gamma_p^*|$. In view of the maximality of $|\Gamma_p, \gamma_p|$, by 1.2, u is its member.

Corollary 2. Let $1 \neq p \neq \infty$. E is isomorphic with a complemented subspace of L_p if and only if for each Banach space G and $v \in \Pi_q(E,G)$ it is ${}^t v \in I_q(G', E')$.

Proof. By Corollary 1 we obtain that the identity operator in E belongs to $\Gamma_p(E,E)$. This implies that E is reflexive and E isomorphic with a complemented subspace of L_p.

§ 3. Some related ideals.

By S_p space, resp. Q_p space, resp. SQ_p space, we shall mean any Banach space isometric with a subspace of L_p, resp. with a quotient of L_p, resp. with a subspace of a quotient of L_p.

We say that Banach space is of S_p type, resp. Q_p type, resp. SQ_p type, if it is isomorphic with S_p space, resp. Q_p space, resp. SQ_p space.

One can easy verify the following properties

$\boxed{3.1}$ $u \in \Gamma_p^\backslash(E,F)$ if and only if for some S_p space there exist $v \in L(E,S_p)$ and $w \in L(S_p,F)$ such that $u = wv$. Moreover $\gamma_p^\backslash(u) = \inf \|v\|\,\|w\|$, infimum is taken over all such S_p spaces, v and w.

$|\Gamma_p\backslash, \gamma_p\backslash|$ is denoted by $|\Sigma_p, \sigma_p|$,

$\boxed{3.2.}$ $u \in /\Gamma_p(E,F)$ if and only if for some Q_p space there exist $v \in L(E,Q_p)$ and $w \in L(Q_p,F'')$ such that $iu = wv$. Moreover $/\gamma_p(u) = \inf \|v\|\,\|w\|$, infimum is taken over all such Q_p spaces, v and w.

The ideal $|/\Gamma_p, /\gamma_p|$ is denoted by $|\Theta_p, \tau_p|$,

$\boxed{3.3}$ $u \in /\Gamma_p\backslash(E,F)$ if and only if for some SQ_p space there exist $v \in L(E,SQ_p)$ and $w \in L(SQ_p,F)$ such that $u = wv$. Moreover $/\gamma_p\backslash(u) = \inf \|v\|\,\|w\|$, inf is taken over all such SQ_p spaces, v and w.

The ideal $|/\Gamma_p\backslash, /\gamma_p\backslash|$ is denoted by $|\Sigma\Theta_p, \sigma\tau_p|$.

Taking into account the properties 1.3 - 1.6 and Proposition 2 we get

Proposition 3. $u \in \Sigma^{*}_{p_t}(E,F)$ if and only if there exist Banach space G and operators $v \in I_q(E,G)$ and ${}^t w \in \Pi_p(F''',G')$ such that $iu = wv$ i is the cannonical injection of F in F'' . $\sigma^{*}_p(u) = \inf\ \iota_q(v)\pi_p({}^t w)$, infimum is taken over all such G, v and w.

Similar arguments to those used in the proofs of Corollaries 1,2 give

Corollary 3. $u \in \Sigma_p(E,F)$, i.e. u is factorizable through S_p space, if and only if for each Banach space G and $v \in I_q(F,G)$ it is ${}^t(vu) \in I_q(G', E')$

Corollary 4. Let $1 \leqslant p \leqslant \infty$. E is of S_p type if and only if for each Banach space G and operator $v \in I_q(E,G)$ it is ${}^t v \in I_q(G', E')$.

The duall results to these are the following

Proposition 4. $u \in \Theta^{*}_p(E,F)$ if and only if there exist Banach space G and operators $v \in \Pi_q(E,G)$ and ${}^t w \in I_p(F', G')$ such that $u = wv$,
$\tau^{*}_p(u) = \inf\ \pi_q(v)\ \iota_p({}^t w)$, infimum is taken over all such G, v and w.

Corollary 5. $u \in \Theta_p(E,F)$, i.e. u is factorizable through Q_p space, if and only if for each Banach space G and $v \in \Pi_q(F,G)$ it is ${}^t(vu) \in \Pi_q(G', E')$

Corollary 6. Let $1 \leqslant p \leqslant \infty$. E is of Q_p type if and only if for each Banach space G and operator $v \in \Pi_q(E,G)$ it is ${}^t v \in \Pi_q(G',E')$.

Now, combining the above results and again the properties 1.3 - 1.6, we arrive at

Proposition 5. $u \in \Sigma\Theta_p^*(E,F)$ if and only if there exist Banach space G and operators $v \in I_q(E,G)$ and ${}^t w \in I_p(F'', G')$ such that $iu = wv$ i is the canonical injection of F into F'', $\sigma\tau_p^*(u) = \inf\ \iota_q(v)\ \iota_p({}^t w)$, infimum is taken over all such G, v and w.

Corollary 7. $u \in \Sigma\Theta_p(E,F)$, i.e. u is factorizable through SQ_p space, if and only if for each Banach space G and $v \in I_q(F,G)$ it is ${}^t(vu) \in \Pi_q(G',E')$

Corollary 8. E is of SQ_p type if and only if for each Banach space G and an operator $v \in I_q(E,G)$ it is ${}^t v \in \Pi_q(G', E')$.

§ 4. Applications, remarks and problems.

The following result is an answer to Problem 6 of [7]

Theorem 1. Let $1 \leqslant s \leqslant p \leqslant r \leqslant \infty$ and let $u \in L(L_r,L_s)$, then u is factorizable through L_p space.

Proof. By Corollary 2 it is enough to prove that ${}^t u\,{}^t v \in I_q(G',L_r')$ whenever $v \in \Pi_q(L_s,G)$. If $v \in \Pi_q(L_s,G)$ then $v \in \Pi_{s'}(L_s,G)$, because $q \leqslant s'$, where s' is defined by the equality $\frac{1}{s'} + \frac{1}{s} = 1$. By A. Persson theorem ${}^t v \in I_{s'}(G',L_s')$ and hence ${}^t u\,{}^t v \in I_{s'}(G',L_r')$. But for $s,p < r \leqslant 2$ $I_{s'}(F,L_r')$ is equall to $I_q(F,L_r')$ for each Banach space F.
This is obtained from the dual equality $\Pi_s(L_r',F) = \Pi_p(L_r',F)$ for $s,p < r \leqslant 2$, which is an easy consequence of Theorem 4 of [5], also cf. [10].

This proves the theorem in the case of $s,p < r \leqslant 2$. The case $2 \leqslant s,p \leqslant r$ is obtained by considering the adjoint operator ${}^t u$. The remaining case may be also derived from Corollary 2. Since this case was proved by J. Lindenstrauss and A. Pelczynski we omit it, cf. [7].

If $(\Omega, \mathcal{M}, \mu)$ is a measure space and E is Banach space then by $L_p(E, \Omega, \mu)$, briefly $L_p(E)$, we denote Banach space of all measurable vector valued in E functions on Ω which are strongly p-integrable.

Theorem 2. E is of SQ_p type if and only if for each operator $u \in L(L_p,L_p)$ there corresponds an operator $U \in L(L_p(E), L_p(E))$ such that

$$\langle U(f),x'\rangle = u(\langle f,x'\rangle) \text{ for each } x' \in E' \text{ and } f \in L_p(E).$$

Proof. Let us observe that Theorem holds for $E = L_p$ and that if it holds for any Banach space then for its subspaces and quotients also. These two observations prove the necessity, since SQ_p space is a subspace of a quotient of L_p space.

Let $p \neq 1,\infty$. By Corollary 8 it is enough to prove that if G is Banach space and $v \in I_q(E,G)$ then ${}^t v \in \Pi_q(G',E')$. By Theorem 1 of [5] E' separable ${}^t v \in \Pi_q(G',E')$ if and only if for each $w \in L(G,L_q)$ the operator wv is q - decomposable, cf. [5]. Let $iv = v_2 j v_1$, where $v_1 \in L(E,L_\infty)$, $v_2 \in L(L_q,G'')$ and j is the canonical injection of L_∞ into L_q, be a factorization of q-integral operator, cf. § 1. Let $w \in L(G,L_q)$ and let us denote by $\bar{w}$ the canonical extension of w to an element of $L(G'',L_q)$. The operator jv_1 may be represented in the form $\langle \cdot,f'\rangle$ for some fixed $f' \in L_q(E')$, i.e. $jv_1(x) = \langle x,f'\rangle$. Now, let $U \in L(L_p(E), L_p(E))$ denote the operator corresponding to the operator ${}^t(\bar{w}v_2) \in L(L_p,L_p)$, according to the assumption of Theorem. Then ${}^t U \in L(L_q(E'), L_q(E'))$ and it is seen that $wv = \bar{w}v_2 j v_1$ is represented by $\langle .,{}^t U(f')\rangle$ and this denotes that wv is q-decomposable operator. This ends the proof. for $p \neq 1, \infty$.

The case of $p = 1,\infty$ is much more simpler, and we omit it. Let us observe that in this case each Banach space is of SQ_p type.

The case when E' is not separable follows from the fact that if each adjoint separable quotient of E is of SQ_p type then E is of SQ_p type.

Remark 1. All the propositions and corollaries of § 3 remain true if we replace evrywhere in their formulations "Banach space 6" by "L_q space", resp. by "l_q space". We do not know if it is true with Proposition 2, cf. Problem 1. If we replace "Banach space G" by "l_q space" in Corollary 4 then it becomes a characterization of subspaces of L_p, given independently by J. Holub, cf. [4].

Remark 2. In this paper we started with the ideal $|\Gamma_p, \gamma_p|$ and then using the transformations of ideals defined in §1 some related ideals were introduced, cf. §3. It is possible to give a full list of ideals which may be obtained in this way. There is only finite number of them. In the case of $p = 1,2,\infty$ it was done by A. Grothendieck, cf. [3].

Remark 3. Another version of Theorem 2 is the following

Theorem 2'. E is of SQ_p type if and only if there exists a constant M such that for each matrix $(a_{i,j})$ defining an operator $u \in L(l_p, l_p)$ and each sequence (x_i) of elements from E there holds

$$\sum_{i=1}^{\infty} \| \sum_{j=1}^{\infty} a_{i,j} x_j \|^p \leqslant M \, \|u\| \sum_{i=1}^{\infty} \|x_i\|^p .$$

Remark 4. Theoreme 2' is especially interesting in the case of $p = 2$. Because spaces S_2, Q_2 and SQ_2 are Hilbert spaces we obtain a characterization of Banach spaces isomorphic with Hilbert space.
For $p = 2$ Corollary 4 coincides with a theorem proved by J. Cohen [2] and S. Kwapien [6].

Problem 1. Let $1 < p < \infty$. Is it true that Banach space of S_p type as well as of Q_p type is isomorphic with a complemented subspace of L_p ?

Problem 2. Is the space $L_2(L_r)$ of SQ_s type for $s < r < 2$ or $2 < r < s$?

Problem 3. Let $1 < p < \infty$, and let $u \in \Gamma_p(E,F)$, i.e. $iu = wv$ where $v \in L(E,L_p)$, $w \in L(L_p,F'')$ and i is the canonical injection of F into F''. Can u be represented in the form $u = w' v'$, where $v' \in L(E,L_p)$ and $w' \in L(L_p,F)$?

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