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ON OPERATORS FACTORIZABLE THROUGH $L_p$ SPACE

by

Stanislaw KWAPIEN

In this paper we give some necessary and sufficient conditions for an operator between Banach spaces be factorizable through $L_p$ space, also conditions for factorizability through a subspace, a quotient and a subspace of a quotient of $L_p$. Hence, we obtain characterizations of Banach spaces isomorphic with complemented subspaces, with subspaces, with quotients and with subspaces of quotients of $L_p$. These conditions are given in terms of $p$-absolutely summing and $p$-integral operators. We use the general theory of ideals of operators, necessary definitions and facts of the theory given in § I. For more detailed treatment the reader is referred to the paper [3], by A. Grothendieck, where it is exposed in frame of tensor product theory, and also to papers of A. Pietsch. We end the paper with some applications.

§ I. Normed ideals of operators.

In the sequel $L(E,F)$ will denote all bounded linear operators from Banach space $E$ into Banach space $F$ and $\|u\|$ the norm of an operator.

Let for each pair of Banach spaces $E$, $F$ be given a linear subspace $A(E,F)$ of $L(E,F)$ and $\alpha_{E,F}$ a norm on $A(E,F)$ such that

1. if $u \in A(E,F)$, $v \in L(X,E)$, $w \in L(E,Y)$ then $wuv \in A(X,Y)$ and $\alpha_{X,Y}(wuv) \leq \alpha_{E,F}(u) \|w\| \|v\|

2. if $u \in A(E,F)$ then $\alpha_{E,F}(u) \geq \|u\|

3. if $u \in L(E,F)$ is one dimensional then $u \in A(E,F)$ and $\alpha_{E,F}(u) = \|u\|

Then we say that $|A, \alpha|$ is a normed linear ideal of operators.

In further we shall write $\alpha(u)$ instead of $\alpha_{E,F}(u)$.

A normed linear ideal $|A, \alpha|$ is defined to be maximal if it satisfies the following condition:

if for $u \in L(E,F)$ there exists a constant $M$ such that for each finite dimensional Banach spaces $X$, $Y$ and operators $v \in L(X,E)$, $w \in L(F,Y)$ it is $\alpha(wuv) \leq M \|w\| \|v\|$ then $u \in A(E,F)$ and $\alpha(u) \leq M.$
We say that \( u \in A^*(E,F) \) if there exists a constant \( M \) such that for each finite dimensional Banach spaces \( X, Y \) and operators \( v \in L(X,E), w \in L(Y,F) \) and \( z \in A(Y,X) \) there holds
\[
\text{trace } (wuvz) \leq M \|w\| \|v\| \alpha(z).
\]
The least such constant \( M \) is denoted by \( \alpha^*(u) \).

It is easy to check that \( |A^*, \alpha^*| \) is a maximal normed ideal of operators. We call it the dual ideal of \( |A, \alpha| \). Moreover, given normed linear ideal \( |A, \alpha| \) we define the following ideals:

- **right injective envelope of \( |A, \alpha| \)**, denoted \( |A\backslash, \alpha\backslash| \), as follows:
  \( u \in A\backslash(E,F) \) if for some Banach space \( G \) and isometric embedding \( i \) of \( F \) into \( G \) it is \( iu \in A(E,G) \),
  \( a\backslash(u) = \inf a(iu) \), where infimum is taken over all such \( G \) and \( i \).

- **left injective envelope of \( |A, \alpha| \)**, denoted by \( |A/\alpha| \), as follows:
  \( u \in A/\alpha(E,F) \) if for some Banach space \( H \) and normed surjection \( j \) of \( H \) on \( E \) (i. e. \( j \) maps the unite disk in \( H \) on the unite disk in \( E \)) \( u j \in A(H,F) \),
  \( a/\alpha(u) = \inf a(uj) \), \( H,j \)

- **right projective envelope of \( |A, \alpha| \)**, denoted by \( |A/, \alpha/| \), as follows:
  \( u \in A/,\alpha/ (E,F) \) if for each Banach space \( H \) and a normed surjection \( j \) of \( H \) onto \( F \) there exists \( v \in A(E,H^\ast) \) such that \( iu = ttjv \), \( i \) is the cannonical injection of \( F \) in \( F'' \) and \( ttj \) is the second adjoint of \( j \).

- **left projective envelope of \( |A, \alpha| \)**, denoted by \( |A\backslash, \alpha\backslash| \), as follows:
  \( u \in A\backslash(E,F) \) if for each Banach space \( G \) and isometric embedding \( i \) of \( E \) into \( G \) there exists \( v \in A(G,F^\ast) \) such that \( ju = vi \), \( j \) is the cannonical injection of \( F \) in \( F'' \).

One can verify the following

- **I.1.** if \( |A, \alpha| \) is maximal then each of the above defined ideals is maximal also,
- **I.2.** if \( |A, \alpha| \) is maximal then \( |(A^\ast)^\wedge, (\alpha^\ast)^\wedge| \) is equal to \( |A, \alpha| \),
- **I.3.** \( |(A/\alpha)^\wedge, (\alpha/\alpha)^\wedge| \) is equal to \( |A^\ast, \alpha^\ast| \),
- **I.4.** \( |(A\backslash, \alpha\backslash)^\wedge, (\alpha\backslash, \alpha\backslash)^\wedge| \) is equal to \( |A^\ast, \alpha^\ast| \).

**Example I.** Ideal of p-absolutely summing operators, \( |\Pi_p, \pi_p| \)

\( u \in \Pi_p(E,F) \) if for some constant \( M \) for each \( x_1, \ldots, x_n \in E \) there holds
\[
\sum_{i=1}^{n} \|u(x_i)\|^p \leq M \sup_{x' \in E'} \sum_{i=1}^{n} \langle x_i, x' \rangle \|x_i\| \|x'\|,
\]
\( \pi_p(u) \) is the least such constant \( M \).
Example 2. Ideal of $p$-integral operators, $|I_p, ^p_p|

$u \in I_p(E,F)$ if there exists a probability measure space $(\Omega, \mathcal{M}, \mu)$ and operators $v \in L(E, L^\infty(\Omega, \mu))$ and $w \in L(L_p(\Omega, \mu), F')$ such that $w v = i u$, where $j$ is the canonical injection of $L^\infty(\Omega, \mu)$ into $L_p(\Omega, \mu)$ and $i$ the canonical injection of $F$ into $F'$.

$I_p(u)$ is defined as $\inf \|v\| \|w\|$, infimum is taken over all such probability measure spaces $(\Omega, \mathcal{M}, \mu)$ and operators $v$ and $w$.

It was proved by A. Pietsch that

\[ I_p \backslash \gamma_p \backslash \frac{1}{p} \text{ is equal to } |I_p, ^p_p|, \]
\[ I_p \backslash \gamma_p \backslash \frac{1}{p} \text{ is equal to } |I_q, ^q_q| \left( \frac{1}{p} + \frac{1}{q} = 1 \right). \]

$\S$ 2. Ideal of $L_p$ factorizable operators

By $L_p$ space we shall mean any Banach space isometric with the space $L_p(\Omega, \mu)$ for some measure space $(\Omega, \mathcal{M}, \mu)$.

We say that $u \in \Gamma_p(E,F)$ if for some $L_p$ space there exist operators $v \in L(E, L_p)$ and $w \in L(L_p(F'), F')$ such that $i u = w v$, $i$ is the canonical injection of $F$ into $F'$.

$\gamma_p(u)$ is defined as $\inf \|v\| \|w\|$, $v$ and $w$ are as in the definition of $\Gamma_p(E,F)$.

Proposition I. Let $1 \leq p < \infty$, $\gamma_p(\Gamma_p, \gamma_p)$ is a maximal normed ideal of operators.

Proof. We shall make use of the following equality

\[ \inf \left( \frac{1}{p} t^p a^p + \frac{1}{q} t^q b^q \right) \]

which is valid for positive numbers $a$, $b$ and $q$ defined by $\frac{1}{p} + \frac{1}{q} = 1$.

Let for $k = 1, 2$ $u_k \in \Gamma_p(E,F)$ and let $i u_k = v_k$, where $v_k \in L(E, L_p(\Omega_k, \mu_k))$, $w_k \in L(L_p(\Omega_k, u_k), F')$ and $\|v_k\| \|w_k\| = \gamma_p(u_k) + \varepsilon$ (cf. the definition of $|\Gamma_p, \gamma_p|$).

Let $\Omega_0$ be the disjoint sum of $\Omega_1$ and $\Omega_2$ and let $u_1 = \frac{1}{2}(u_1 + u_2)$.

We define $v_0 \in L(E, L_p(\Omega_0, u_0))$ and $w_0 \in L(L_p(\Omega_0, u_0), F')$ as follows $v_0(x)$ is a function on $\Omega_0$ which coincides with $v_1(x)$ on $\Omega_1$ and with $v_2(x)$ on $\Omega_2$, $w_0(x) = v_1(f_1) + w_2(f_2)$, where $f_1 = f|\Omega_1$ and $f_2 = f|\Omega_2$.

Simple computations show that $i(u_1 + u_2) = w_0 v_0$ and

\[ \|v_0\| \leq \frac{1}{2} \|v_1\|^p + \frac{1}{2} \|v_2\|^p \]
\[ \|w_0\| \leq \left( \frac{1}{2} \|v_1\|^q + \frac{1}{2} \|v_2\|^q \right)^{\frac{1}{q}}. \]
Applying 2.1 we obtain
\[ \|v_0\| \|w_0\| \leq p^{-1}\|v_0\|^p + q^{-1}\|w_0\|^q. \]
Hence and by 2.2, 2.3
\[ \|v_0\| \|w_0\| \leq \frac{1}{2}\|v_1\|^{p-1} + \frac{4}{p}\|w_1\|^{q-1} + \frac{1}{2p^{q-1}}\|v_2\|^p + \frac{4}{q}\|w_2\|^q. \]
But we can replace \( v_1 \) by \( t_1 v_1 \) and \( w_1 \) by \( t_1^{-1} v_1 \) and the same with \( v_2 \) and \( w_2 \).
Taking the infimum with respect to \( t_1, t_2 \) the right side of the above inequality is equal to \( \|v_1\| \|w_1\| + \|v_2\| \|w_2\| \).
This proves that \( u_1 + u_2 \in r_p(E,F) \) and \( \gamma_p(u_1 + u_2) \leq \gamma_p(u_1) + \gamma_p(u_2) \).
If \( u \in r_p(E,F) \) then \( tu \) also and \( \gamma_p(tu) = |t|\gamma_p(u) \). Thus \( r_p(E,F) \) is a linear space and \( \gamma_p \) a norm on it.
Properties 1., 2., 3. are obvious.
The maximality of \( r_p, \gamma_p \) may be obtained by the methods from the theory of ultraproducts of Banach spaces, developed by J. Krivine and D. Dacunha-Castelle, cf. [1].

**Proposition 2.** Let \( 1 < p \leq \infty, 1/p + 1/q = 1 \). Then

\( u \in r_p(E,F) \) if and only if there exist Banach spaces \( G \) and operators

\( v \in H_q(E,G), t v \in H_p(F', G') \) such that \( u = wv \),

\( \gamma_p(u) = \inf \gamma_q(v) \pi_p(w) \), infimum is taken over all such \( G,v \) and \( w \).

**Proof.** Suppose \( u \in r_p(E,F) \). By the definition for each \( h \in L(l^1_p,E), g \in L(F,l^1_p) \) and \( i \)-identity operator in \( l^1_p \) there holds

\[ |\text{trace}(guh)| \leq \frac{1}{p}\gamma_p(u) \|g\| \|h\| \gamma_p(i). \]

Since \( \gamma_p(i) = 1 \) this is equivalent to: for each \( x_1, \ldots, x_n \in E, y'_1, \ldots, y'_n \in F' \)

\[ \sum_{i=1}^n <u(x_i), y'_i> \leq \gamma_p(u) \sup_{x' \in K_1} \left( \sum_{i=1}^n <x_i, x'>^{q-1} \sup_{y \in K_2} \left( \sum_{i=1}^n |<y, y'_i>|^p \right)^{1/p} \right), \]

where \( K_1 \) and \( K_2 \) are unite discs in \( E' \) and \( F' \) correspondingly.

Applying 2.1 we get

\[ \sum_{i=1}^n <u(x_i), y'_i> \leq \gamma_p(u) \sup_{x' \in K_1, y \in K_2} \left( \sum_{i=1}^n |<x_i, x'>|^q + \sum_{i=1}^n |<y, y'_i>|^p \right). \]

By the theorem on separations of cones in locally convex spaces it is equivalent to the existence of a probability measure \( \mu \) on \( K \) - the cartesian product of \( K_1 \) and \( K_2 \) such that for each \( x \in E \) and \( y' \in F' \)
Operators factorizable through $L_p$-spaces

$$|\langle u(x), y' \rangle| \leq \gamma_p(\mu)(q^{-1} \int_K |\langle x, x' \rangle|^q d\mu(x') + p^{-1} \int_K |\langle y, y' \rangle|^p d\mu(y')).$$

Replacing $x$ by $tx$ and $y'$ by $t^{-1}y'$ and taking infimum we have by 2.1

$$|\langle u(x), y' \rangle| \leq \gamma_p(\mu) \left( \frac{1}{q} \int_K \langle x, x' \rangle^q d\mu(x') \right)^{\frac{1}{q}} \left( \frac{1}{p} \int_K |\langle y, y' \rangle|^p d\mu(y') \right)^{\frac{1}{p}}.$$

Let $v \in L(E, L_q(K, \mu))$ be defined by $v(x)(x'y') = \langle x, x' \rangle$ on $K$, similarly $w_0 \in L(F', L_p(K, \mu))$ is defined by $w_0(y')(x'y'') = \langle y', y'' \rangle$ on $K$.

Let $G$ denote the closure of $v(E)$ in $L_q(K, \mu)$ and $H$ the closure of $w_0(F')$ in $L_p(K, \mu)$.

By Pietsch theorem 1.5 $v \in \Pi_q(E, G)$, $w_0 \in \Pi_p(F', H)$ and $\pi_q(v), \pi_p(w_0) \leq 1$.

The inequality 2.4 implies the existence of an operator $z \in L(G, H')$ such that $\|z\| \leq \gamma_p(\mu)$ and $t_{w_0}z = iv$, $i$ being the canonical injection of $F$ into $F''$. The image of $G$ by $t_{w_0}z$ is in $F$, so let $v = t_{w_0}z$ be considered as a member of $L(G, F)$. Then

$$\pi_p(t_v) \leq \pi_p(w_0) \|z\| \leq \gamma_p(\mu).$$

Thus $G$, $v$ and $w$ satisfy the required conditions of Proposition 2, moreover

$$\pi_q(v) \pi_p(t_w) \leq \gamma_p(\mu).$$

This proves the necessity.

Now assume $u = vw$, where $v \in \Pi_q(E, G)$ and $t_w \in \Pi_p(F', G')$.

Let $X$ and $Y$ be finite dimensional Banach spaces, $h \in L(X, E)$, $g \in L(F, Y)$ and $z \in \Pi_p(Y, X)$. We have to prove

$$|\text{trace}(zguh)| \leq \pi_q(v) \pi_p(t_w) \gamma_p(z) \|g\| \|h\|.$$

Let $z = z_1 z_2$, $z_1 \in L(L_p, X)$, $z_2 \in L(Y, L_p)$ and $\|z_1\| \|z_2\| \leq \gamma_p(z) + \epsilon$.

Then $vzh_1 \in \Pi_q(L_p, G)$ and $t(z_2g)w \in \Pi_p(L_p, G')$. It was proved by A. Perrson [8], that if $r \in \Pi_p(L_p, G')$ then $r \in \Pi_p(G, L_p)$ and $\gamma_p(r) \leq \Pi_p(t_r)$. Applying this we obtain that

$$z_2gw \in \Pi_p(G, L_p) \text{ and } \gamma_p(z_2gw) \leq \gamma_p(t(z_2gw)).$$

Since $\Pi_p, i_p$ is the dual ideal of $\Pi_q, \pi_q$ we have

$$|\text{trace}(z_2gwzh_1)| \leq i_p(z_2gw)\pi_q(vzh_1) \leq \pi_q(t_w g t z_2) \pi_q(vzh_1).$$

Hence
Because \( \| z_1 \| \| z_2 \| \leq \gamma_p(z) + \varepsilon \) and \( \varepsilon \) is arbitrary small, this ends the proof.

**Corollary 1.** \( u \in \mathcal{L}(E,F) \) is factorizable through \( \mathcal{L}_p \) space (i.e., \( u \in \mathcal{L}_p(E,F) \)) if and only if for each Banach space \( G \) and \( v \in \mathcal{H}_q(F,G) \) it is \( t(vu) \in \mathcal{I}_q(G',E') \).

**Proof.** Let \( u \in \mathcal{L}_p(E,F) \) and \( v \in \mathcal{H}_q(F,G) \). By Proposition 2 if \( t_w \in \mathcal{H}_p(E',G') \) then \( wv \in \mathcal{H}_p(F,E) \). From this we deduce that \( t(vu) \in \mathcal{H}_p(G',E') \) and hence \( t(vu) \in \mathcal{I}_q(G',E') \).

Conversely, if \( u \) satisfies the condition of Corollary then \( u \) belongs to the dual ideal of \( \mathcal{H}_p, \mathcal{H}_q \). In view of the maximality of \( \mathcal{H}_p, \mathcal{H}_q \), by 1.2, \( u \) is its member.

**Corollary 2.** Let \( 1 \neq p \neq \infty \). \( E \) is isomorphic with a complemented subspace of \( \mathcal{L}_p \) if and only if for each Banach space \( G \) and \( v \in \mathcal{H}_q(E,G) \) it is \( t_v \in \mathcal{I}_q(G',E') \).

**Proof.** By Corollary 1 we obtain that the identity operator in \( E \) belongs to \( \mathcal{L}_p(E,E) \). This implies that \( E \) is reflexive and \( E \) isomorphic with a complemented subspace of \( \mathcal{L}_p \).

§ 3. Some related ideals.

By \( \mathcal{S}_p \) space, resp. \( \mathcal{Q}_p \) space, resp. \( \mathcal{SQ}_p \) space, we shall mean any Banach space isometric with a subspace of \( \mathcal{L}_p \), resp. with a quotient of \( \mathcal{L}_p \), resp. with a subspace of a quotient of \( \mathcal{L}_p \).

We say that Banach space is of \( \mathcal{S}_p \) type, resp. \( \mathcal{Q}_p \) type, resp. \( \mathcal{SQ}_p \) type, if it is isomorphic with \( \mathcal{S}_p \) space, resp. \( \mathcal{Q}_p \) space, resp. \( \mathcal{SQ}_p \) space.

One can easily verify the following properties

3.1 \( u \in \mathcal{L}_p(E,F) \) if and only if for some \( \mathcal{S}_p \) space there exist \( v \in \mathcal{L}(E,\mathcal{S}_p) \) and \( w \in \mathcal{L}(\mathcal{S}_p,F) \) such that \( u = vw \). Moreover \( \gamma_p(u) = \inf \| v \| \| w \| \), infimum is taken over all such \( \mathcal{S}_p \) spaces, \( v \) and \( w \).
\[ |\gamma_p, \gamma_p| \text{ is denoted by } |\gamma_p, \sigma_p|, \]

\[ u \in /\gamma_p(E,F) \text{ if and only if for some } Q_p \text{ space there exist } v \in L(E,Q_p) \text{ and } w \in L(Q_p,F^*) \text{ such that } iu = wv. \text{ Moreover} \]

\[ /\gamma_p(u) = \inf \|v\| \|w\|, \text{ infimum is taken over all such } Q_p \text{ spaces, } v \text{ and } w. \]

The ideal \( |/\gamma_p, /\gamma_p| \) is denoted by \( |\gamma_p, \gamma_p| \).

\[ u \in /\gamma_p \setminus (E,F) \text{ if and only if for some } SQ_p \text{ space there exist } v \in L(E, SQ_p) \text{ and } w \in L(SQ_p,F) \text{ such that } u = wv. \text{ Moreover} \]

\[ /\gamma_p(u) = \inf \|v\| \|w\|, \text{ infimum is taken over all such } SQ_p \text{ spaces, } v \text{ and } w. \]

The ideal \( |/\gamma_p \setminus , /\gamma_p| \) is denoted by \( |\gamma_p, \sigma_p| \).

Taking into account the properties 1.3 - 1.6 and Proposition 2 we get

**Proposition 3.** \( u \in \gamma_p^*(E,F) \) if and only if there exist Banach space \( G \) and operators \( v \in \gamma_q(G,F) \) and \( w \in \Pi_q(F,G') \) such that \( iu = wv \), is the canonical injection of \( \gamma_p \) in \( \gamma_p^* \). \( \gamma_p(u) = \inf \gamma_q(v) \Pi_q(tw), \text{ infimum is taken over all such } G, v \text{ and } w. \)

Similar arguments to those used in the proofs of Corollaries 1, 2 give

**Proposition 4.** \( u \in \gamma_p^*(E,F), \text{ i.e. } u \text{ is factorizable through } G \text{ space,} \)

if and only if for each Banach space \( G \) and \( v \in \gamma_q(F,G) \) it is \( \gamma_q(vu) \subseteq \gamma_q(G', E') \)

**Corollary 3.** \( u \in \gamma_p(E,F), \text{ i.e. } u \text{ is factorizable through } G \text{ space,} \)

if and only if for each Banach space \( G \) and \( v \in \gamma_q(F,G) \) it is \( \gamma_q(vu) \subseteq \gamma_q(G', E') \).

The dual results to these are the following

**Proposition 4.** \( u \in \gamma_p^*(E,F) \) if and only if there exist Banach space \( G \)

and operators \( v \in \Pi_q(E,G) \) and \( tw \in \Pi_q(F', G') \) such that \( u = wv, \)

\( \gamma_p(u) = \inf \gamma_q(v) \Pi_q(tw), \text{ infimum is taken over all such } G, v \text{ and } w. \)

**Corollary 5.** \( u \in \gamma_p(E,F), \text{ i.e. } u \text{ is factorizable through } G \text{ space,} \)

if and only if for each Banach space \( G \) and \( v \in \gamma_q(F,G) \) it is \( t(vu) \subseteq \gamma_q(G', E') \)
Corollary 6. Let $1 \leq p \leq \infty$. $E$ is of $Q_p$ type if and only if for each Banach space $G$ and operator $v \in \Pi_p^q(E, G)$ it is $t^p v \in \Pi^q (G', E')$.

Now, combining the above results and again the properties 1.3 - 1.6, we arrive at

Proposition 5. $u \in \mathcal{M}_p^p(E, F)$ if and only if there exist Banach space $G$ and operators $v \in \Pi_q^p(E, G)$ and $w \in \Pi_q^p(F, G)$ such that $iu = tv$ is the canonical injection of $F$ into $F$,

$$
\varphi_p^q(u) = \inf \Pi_q^p(v) \Pi_p^q(tw), \text{ infimum is taken over all such } G, v \text{ and } w.
$$

Corollary 7. $u \in \mathcal{M}_p^p(E, F)$, i.e. $u$ is factorizable through $SQ_p$ space, if and only if for each Banach space $G$ and $v \in \Pi_q^p(F, G)$ it is $t^q(vu) \in \Pi_q^p (G', E')$.

Corollary 8. $E$ is of $SQ_p$ type if and only if for each Banach space $G$ and an operator $v \in \Pi_q^p(E, G)$ it is $t^q v \in \Pi_q^p (G', E')$.


The following result is an answer to Problem 6 of [7].

Theorem 1. Let $1 \leq s \leq p \leq r \leq \infty$ and let $u \in L_p(L_r, L_s)$, then $u$ is factorizable through $L_p$ space.

Proof. By Corollary 2 it is enough to prove that $t^q v \in \Pi_q^p (L_r^p, L_s^r)$ whenever $v \in \Pi_q^p (L_s^r, G)$. If $v \in \Pi_q^p (L_s^r, G)$ then $v \in \Pi_s^p (L_s^r, G)$, because $q \leq s'$, where $s'$ is defined by the equality $1_s + 1_s = 1$. By A. Persson theorem $t^q v \in \Pi_s^p (G', L_s^r)$ and hence $t^q v \in \Pi_q^p (G', L_r^p)$. But for $s, p < r \leq 2$

$I_q^r (F, L_r^p)$ is equal to $I_q^r (F, L_r^p)$ for each Banach space $F$.

This is obtained from the dual equality $\Pi_q^p (L_r^p, F) = \Pi_p^q (L_r^p, F)$ for $s, p < r \leq 2$, which is an easy consequence of Theorem 4 of [5], also cf. [10].

This proves the theorem in the case of $s, p < r \leq 2$. The case $2 \leq s, p \leq r$ is obtained by considering the adjoint operator $t^q v$. The remaining case may be also derived from Corollary 2. Since this case was proved by J. Lindenstrauss and A. Pelczynski we omit it, cf. [7].

If $(\mu, \mathcal{M}, \mu)$ is a measure space and $E$ is Banach space then by $L_p(E, \mu, \mu)$, briefly $L_p(E)$, we denote Banach space of all measurable vector valued in $E$ functions on $\mu$ which are strongly $p$-integrable.
Theorem 2. \( E \) is of \( SQ_p \) type if and only if for each operator \( u \in L_p(L_p, L_p) \) there corresponds an operator \( U \in L(L(E), L_p(E)) \) such that

\[
<U(f), x'> = u(<f, x'>) \quad \text{for each} \quad x' \in E' \quad \text{and} \quad f \in L_p(E).
\]

Proof. Let us observe that Theorem holds for \( E = L_q \) and that if it holds for any Banach space then for its subspaces and quotients also. These two observations prove the necessity, since \( SQ_p \) space is a subspace of a quotient of \( L_p \) space.

Let \( p \neq 1, \infty \). By Corollary 8 it is enough to prove that if \( G \) is Banach space and \( v \in I_q(E, G) \) then \( {}^tv \in \Pi_q(G', E') \). By Theorem 1 of [5] \( E' \) separable \( {}^tv \in \Pi_q(G', E') \) if and only if for each \( w \in L(G, L_q) \) the operator \( wv \) is \( q \)-decomposable, cf. [5]. Let \( iv = v_2jv_1 \), where \( v_1 \in L(E, L_q) \), \( v_2 \in L(L_q G'') \) and \( j \) is the canonical injection of \( L_q \) into \( L_q \), be a factorization of \( q \)-integral operator, cf. § 1. Let \( w \in L(G, L_q) \) and let us denote by \( \hat{w} \) the canonical extension of \( w \) to an element of \( L(G, L_q) \). The operator \( jv_1 \) may be represented in the form \( <', f'> \) for some fixed \( f' \in L_q(E') \), i.e. \( jv_1(x) = <x, f'> \). Now, let \( U \in L(L_p(E), L_p(E)) \) denote the operator corresponding to the operator \( {}^t(\hat{w}v_2) \in L(L_p, L_p) \), according to the assumption of Theorem. Then \( {}^tU \in L(L_q(E'), L_q(E')) \); and it is seen that \( wv = \hat{w}v_2jv_1 \) is represented by \( <', {}^tU(f') > \) and this denotes that \( wv \) is \( q \)-decomposable operator. This ends the proof. For \( p \neq 1, \infty \).

The case of \( p = 1, \infty \) is much more simpler, and we omit it. Let us observe that in this case each Banach space is of \( SQ_p \) type.

The case when \( E' \) is not separable follows from the fact that if each adjoint separable quotient of \( E \) is of \( SQ_p \) type then \( E \) is of \( SQ_p \) type.

Remark 1. All the propositions and corollaries of § 3 remain true if we replace everywhere in their formulations "Banach space \( G \)" by "\( L_q \) space", resp. by "\( L_p \) space". We do not know if it is true with Proposition 2, cf. Problem 1. If we replace "Banach space \( G \)" by "\( L_q \) space" in Corollary 4 then it becomes a characterization of subspaces of \( L_p \), given independently by J. Holub, cf. [4].

Remark 2. In this paper we started with the ideal \( \Gamma_p, \gamma_p \) and then using the transformations of ideals defined in § 1 some related ideals were introduced, cf. § 3. It is possible to give a full list of ideals which may be obtained in this way. There is only finite number of them. In the case of \( p = 1, 2, \infty \) it was done by A. Grothendieck, cf. [3].

Remark 3. Another version of Theorem 2 is the following...
Theorem 2'. $E$ is of SQ$_p$ type if and only if there exists a constant $M$ such that
for each matrix $(a_{i,j})$ defining an operator $u \in L(l_p, l_p)$ and each sequence $(x_i)$ of elements from $E$ there holds
$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{i,j}| x_j^p \leq M \| u \| \sum_{i=1}^{\infty} \| x_i \|^p.$$ 

Remark 4. Theorem 2' is especially interesting in the case of $p = 2$. Because spaces $S_2$, $Q_2$ and SQ$_2$ are Hilbert spaces we obtain a characterization of Banach spaces isomorphic with Hilbert space.

For $p = 2$ Corollary 4 coincides with a theorem proved by J. Cohen [2] and S. Kwapien [6].

Problem 1. Let $1 < p < \infty$. Is it true that Banach space of $S_p$ type as well as of $Q_p$ type is isomorphic with a complemented subspace of $L_p$?

Problem 2. Is the space $L_2(L_p)$ of SQ$_p$ type for $s < r < 2$ or $2 < r < s$?

Problem 3. Let $1 < p < \infty$, and let $u \in \Gamma_p(E,F)$, i.e. $iu = vw$ where $v \in L(E,L_p)$, $w \in L(L_p,F')$ and $i$ is the canonical injection of $F$ into $F'$. Can $u$ be represented in the form $u = w' v'$, where $v' \in L(E,L_p)$ and $w' \in L(L_p,F)$?

REFERENCES


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