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A NEW LOOK AT BRUN'S SIEVE *

by

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1. - Let $\mathcal{A}$ be a sequence of (not necessarily distinct) integers, and let $\mathcal{P}$ be a set of primes. The aim of any 'small' sieve method is to estimate the sifting function

$$S(\mathcal{A};\mathcal{P},z) := |\{a : a\in\mathcal{A}, (a,P(z)) = 1\}|,$$

where $z \geq 2$,

$$P(z) = \prod_{\substack{p\in\mathcal{P}\\ p<z}} p,$$

and $|\{\ \}|$ denotes the cardinality of the set $\{\ \}$. In order to make progress we require some information about the distribution of $\mathcal{A}$ in certain arithmetic progressions ; to this end we postulate the existence of a real number $X > 1$ and a non-negative multiplicative function ω such that

(i) $\omega(p) = 0$ if $p \notin \mathcal{P}$

and (ii) the 'remainders' $R_d := \sum_{\substack{a\in\mathcal{A}\\ a\equiv 0 \bmod d}} 1 - \frac{\omega(d)}{d} X \quad (\mu(d)\neq 0,\ d|P(z))$

are small (perhaps only on average) ; note that $R_1 = |\mathcal{A}| - X$, so that X is seen to be a convenient approximation to the number of elements in $\mathcal{A}$.

Let

$$W(z) = \prod_{p<z} \left(1 - \frac{\omega(p)}{p}\right) ;$$

on probabilistic grounds we expect $XW(z)$ to give the true order of magnitude of $S(\mathcal{A};\mathcal{P},z)$. Indeed, we have easily that

$$S(\mathcal{A};\mathcal{P},z) = \sum_{a\in\mathcal{A}} \sum_{d|(a,P(z))} \mu(d) = XW(z) + \sum_{d|P(z)} \mu(d)\, R_d ,$$

and if we now assume, as we shall do throughout, that

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* This lecture will appear, in expanded form, in a forthcoming book by Halberstam and Richert on Sieve Methods (Makham, Chicago).

(Ω) $\qquad \omega(P) \leq A$ for some constant $A \geq 1$,

and

(P) $\qquad |R_d| \leq \omega(d)$ if $d \mid P(z)$,

then

(1.1) $\qquad S(\mathcal{A};\mathcal{P},z) = XW(z) + O((A+1)^{\pi(z)})$.

It is clear that (1.1) gives an asymptotic formula, or an acceptable estimate, for $S(\mathcal{A};\mathcal{P},z)$ only if z is about of order $\log X$; (1.1) is therefore generally far too weak to be of much use. The first principal result we wish to describe here is the following powerful extension of (1.1) :

THEOREM 1. *Suppose that* (Ω) *and* (R) *hold, and suppose also that*

(Ω_1) $\qquad \frac{\omega(p)}{p} \leq 1 - \frac{1}{A_1}$ *for some constant* $A_1 \geq 1$.

If $U = \log X / \log z$ *and* $X \geq z$, *then*

$$S(\mathcal{A};\mathcal{P},z) = XW(z)\{1+O(e^{-u(\log u - \log\log 3u - \log Ae^2)}) + O(e^{-\sqrt{\log X}})\}.$$

Theorem 1 is of a kind known as a 'fundamental lemma' ; it is important in the study of additive functions (see [1] , lemma 1.4 for a weaker result of this kind) and also in the theory of representations by 'quasi-primes' (numbers free of small prime factors) (see e.g. [2]).

Special interest attaches to Theorem 1 in respect of the *first* error term. Levin [3] pointed out in 1965 that Selberg's sieve method is not capable of yielding a result of this degree of precision ; and he went on to suggest that, by constrast, Theorem 1 could be derived by means of certain elaborate extensions of Brun's sieve (attributed by Levin to Buchstab but not traceable in this form by us). In fact, we can show that Theorem 1 follows from the simpler Rademacher [4]-Tartakovskij [5] formulation of the Brun method ; moreover, by formalizing Brun's ideas along lines suggested by Levin, we are able to give a surprisingly simple account of the Brun method, leading to theorem 2 below. Theorem 1 then follows by easy stages from Theorem 2.

Seen in this new light, Brun's method appears far from exhausted and may well lead to further developments.

THEOREM 2. *Suppose that* (Ω), (Ω_1) *and* (R) *hold. Let* b *be a positive integer and* λ *any positive real number satisfying*

$$\lambda e^{1+\lambda} < 1 . \tag{1.2}$$

Then

$$S(\mathcal{A};\mathcal{P},z) \le XW(z)\{1+2\lambda^{2b+1}\frac{e^{2\lambda}}{1+\lambda^2 e^{2+2\lambda}}\}+O\left(z^{2b+\frac{2.01}{e^{2\lambda/A}-1}}\right)$$

and

$$S(\mathcal{A};\mathcal{P},z) \ge XW(z)\{1-2\lambda^{2b}\frac{e^{2\lambda}}{1+\lambda^2 e^{2+2\lambda}}\}+O\left(z^{2b-1+\frac{2.01}{e^{2\lambda/A}-1}}\right).$$

The conditions (Ω) and (R) could be weakened, without introducing new difficulties ; however, they cover a great many interesting problems, and we have considered it appropriate in this exposition to keep things simple. To illustrate the quality of Theorem 2 we indicate briefly how to prove that

there exist infinitely many integers n *such that both* n *and* $n+2$ *have at most* 7 *prime factors.*

(Of course, more elaborate or deeper methods give much better results - see e.g. [6] or [7]).

We take $\mathcal{A} = \{n(n+2) : n \le x\}$ and $\mathcal{P}$ the set of all primes. We may choose $X = x$, $\omega(2) = 1$ and $\omega(p) = 2$ for $p > 2$, so that (R) is satisfied as well as (Ω) with $A = 2$ and (Ω_1) with $A_1 = 3$. It is easy to deduce from (Ω) and (Ω_1) that $1/W(z) \ll \log^2 z$ and it follows from Theorem 2 (with $b = 1$) that

$$S(\mathcal{A};\mathcal{P},z) \ge \frac{x}{2}\prod_{2<p<z}(1-\frac{2}{p})\{1-\frac{2\lambda^2 e^{2\lambda}}{1-\lambda^2 e^{2+2\lambda}}+O\left(x^{-1}z^{1+\frac{2.01}{e^{\lambda}-1}}\log^2 z\right)\}.$$

Constants λ and u can be found to satisfy

$$1-\frac{2\lambda^2 e^{2\lambda}}{1-\lambda^2 e^{2+2\lambda}} > 0 \quad \text{and} \quad 1+\frac{2.01}{e^{\lambda}-1} < u < 8 ,$$

and we put $z = x^{1/u}$. Then $S(\mathcal{A};\mathcal{P},x^{1/u})$ tends to infinity as $x \to \infty$; in other words, the number of $n \le x$ for which both n and $n+2$ have all their prime factors $\ge x^{1/u}$ tends to infinity with x. Hence each of n, $n+2$ has at most $u < 8$ prime factors, i.e. at most 7, infinitely often.

Another very simple application of (the upper bound in Theorem 2 (again with $b = 1$) gives

THEOREM 3. Suppose that (Ω), (Ω_1) and (R) hold. For any positive number μ,

$$S(\mathcal{A};\mathcal{P},z) = O(XW(z)) \quad \text{if} \quad z \le X^{\mu}$$

and

$$S(\mathcal{A};\mathcal{P},z) = O(XW(X)) \quad \text{if} \quad z \ge X^{1/\mu},$$

where the constants implied by the O-symbols may depend on μ.

Upper estimates of this kind are frequently used in arithmetical investigations, and whenever one meets the phrase " ... by Brun's sieve it follows ... " in the literature, an appeal to Theorem 3 will generally justify the claim.

We shall not prove Theorem 1 in detail. In view of Theorem 3 we may clearly suppose that u is large ; and a careful computation shows that the choices

$$b = [\tfrac{u}{2} - \tfrac{u}{\log u}], \quad \lambda = \frac{A \log u}{u} \quad \text{if} \quad \log z \ge 2 \log^2 u,$$

$$b = [\tfrac{u}{4}], \quad \lambda = \frac{4A}{u} \quad \text{if} \quad \log z < 2 \log^2 u$$

in Theorem 2 lead to Theorem 1.

2. - It remains to prove Theorem 2. In view of (1.1) we may suppose throughout that z is sufficiently large.

Let χ_1 and χ_2 be arithmetical functions, taking the values 0 and 1 only, on the set of positive divisors of $P(z)$, such that for $\nu = 1$ and 2

(2.1) $$\chi_\nu(1) = 1,$$

(2.2) $$\chi_\nu(d) = 1 \text{ implies } \chi_\nu(t) = 1 \text{ for all } t|d$$

(2.3) $$\chi_\nu(t) = 1, \ \mu(t) = (-1)^\nu \text{ imply } \chi_\nu(pt) = 1 \text{ for all}$$

$p < q(t)$, $p|P(z)$, where, if $n > 1$, $q(n)$ is the least prime factor of n, and $q(1) = \infty$. Taking $t = 1$ in (2.3) (and therefore $\nu = 2$) shows that (2.3) incorporates the special condition

(2.4) $$\chi_2(p) = 1 \text{ for all } p|P(z).$$

It is clear that we may think of χ_1, χ_2 as characteristic functions of two sets $\mathcal{B}_1$, $\mathcal{B}_2$ of divisors of $P(z)$, but it turns out to be much more efficient to work directly with χ_1 and χ_2 rather than through the structures of $\mathcal{B}_1$ and $\mathcal{B}_2$. We record the following, virtually obvious, identity.

LEMMA 1. *If* $pt|P(z)$, $p < q(t)$, *then*

$$\chi_\nu(t) - \chi_\nu(pt) = (-1)^{\nu-1}\,\mu(t)\chi_\nu(t)\{1-\chi_\nu(pt)\} \quad (\nu = 1,2).$$

Proof. Both sides vanish if $\chi_\nu(t) = \chi_\nu(pt)$, and by (2.2) this is always the case if $\chi_\nu(pt) = 1$. Hence we may suppose that $\chi_\nu(pt) = 0$, $\chi_\nu(t) = 1$. Then, by (2.3), $\mu(t) = (-1)^{\nu-1}$, and so both sides are equal to 1 .

The relevance of the functions χ_1 and χ_2 is embodied in the next Lemma.

LEMMA 2. *For each* $n|P(z)$, *define*

$$\tau_\nu(n) = \sum_{d|n} \mu(d)\,\chi_\nu(d) \qquad (\nu = 1,2) \ ;$$

then $\tau_1(1) = 1 = \tau_2(1)$, and

$$\tau_2(n) \le \sum_{d|n} \mu(d) \le \tau_1(n) \quad \text{for all} \quad n|P(z) .$$

Proof. If $n = 1$ the result is obvious from (2.1). Suppose then that $n > 1$, in which case we have to show that

$$(-1)^\nu\,\tau_\nu(n) \le 0 \quad \text{if} \quad n > 1 , \quad n|P(z) \quad (\nu = 1,2) . \tag{2.5}$$

But then if $q(n) = p$ and $n = pm$, so that $p < q(t)$ for every $t|m$, we have

$$\begin{aligned}\tau_\nu(n) &= \sum_{t|m} \{\mu(t)\chi_\nu(t)+\mu(pt)\chi_\nu(pt)\} = \sum_{t|m} \mu(t)\{\chi_\nu(t)-\chi_\nu(pt)\} \\ &= (-1)^{\nu-1} \sum_{t|m} \chi_\nu(t)\,\{1-\chi_\nu(pt)\}\end{aligned}$$

by Lemma 1 ; and (2.5) follows at once.

We are now in a position to begin the argument. We have, by lemma 2 ,

$$\begin{aligned}(-1)^\nu S(\mathcal{A};\mathcal{P},z) &= \sum_{a\in\mathcal{A}} (-1)^\nu \sum_{d|(a,P(z))} \mu(d) \ge \sum_{a\in\mathcal{A}} (-1)^\nu \tau_\nu((a,P(z))) \\ &= (-1)^\nu X \sum_{d|P(z)} \frac{\mu(d)\,\chi_\nu(d)}{d}\omega(d) + \\ &\quad + (-1)^\nu \sum_{d|P(z)} \mu(d)\,\chi_\nu(d)\,R_d \\ &\ge (-1)^\nu X L_\nu(z) - \sum_{d|P(z)} \chi_\nu(d)\omega(d) \quad (\nu=1,2)\end{aligned} \tag{2.6}$$

using (R), where

$$L_\nu(z) = \sum_{d|P(z)} \mu(d)\,\chi_\nu(d)\,\frac{\omega(d)}{d} \qquad (\nu = 1,2)\,. \tag{2.7}$$

Our aim must be to construct χ_1, χ_2 consistently with conditions (2.1), (2.2) and (2.3) so that the remainder terms

$$\sum_{d|P(z)} \chi_\nu(d)\,\omega(d) \qquad (\nu = 1,2) \tag{2.8}$$

in (2.6) are small while $L_\nu(z)$ is, for each of $\nu = 1$ and $\nu = 2$, bounded above and below by a constant positive multiple of $W(z)$. We note that $L_\nu(z) = W(z)$ if $\chi_\nu(d) = 1$ for <u>every</u> divisor d of $P(z)$. The connection between $L_\nu(z)$ and $W(z)$ emerge very clearly in the following identity.

LEMMA 3. Let p^+ denote the successor of p in $\mathcal{P}$, and write

$$P(u,v) = \prod_{\substack{u \le p < v \\ p \in \mathcal{P}}} p = P(v) \,/\, P(u)\,.$$

Then, for $\nu = 1$ and 2,

$$L_\nu(z) = W(z)\,\{1+(-1)^{\nu-1} \sum_{p<z} \frac{\omega(p)}{p}\frac{W(p)}{W(z)} \sum_{t|P(p^+,z)} \chi_\nu(t)\{1-\chi_\nu(pt)\}\frac{\omega(t)}{t}\}.$$

<u>Proof.</u> We substitute the (obvious) relation

$$\chi_\nu(d) = 1 - \sum_{p|d} \{\chi_\nu((d,P(p^+,z))) - \chi_\nu((d,P(p,z)))\} \quad (d|P(z))$$

in (2.7), so that

$$L_\nu(z) = \sum_{d|P(z)} \Big(\mu(d) + \sum_{p|d} \mu\Big(\frac{d}{p}\Big)\,\{\chi_\nu((d,P(p^+,z))) - \chi_\nu((d,P(p,z)))\}\Big)\,\frac{\omega(d)}{d}\,;$$

and if we now write $d = \delta p t$ where $\delta|P(p)$, $t|P(p^+,z)$, we obtain

$$L_\nu(z) = W(z) + \sum_{p<z} \frac{\omega(p)}{p} \sum_{\delta|P(p)} \mu(\delta)\,\frac{\omega(\delta)}{\delta} \sum_{t|P(p^+,z)} \mu(t)\,\frac{\chi_\nu(t)-\chi_\nu(pt)}{t}\,\omega(t)\,.$$

An application of Lemma 1 to the innermost sum on the right completes the proof. (Note that division by $W(z)$ is justified in view of condition (Ω_1) which ensures that $\omega(p)/p$ is bounded away from 1).

We now introduce a partition

$$2 = z_r < z_{r-1} < \dots < z_1 < z_0 = z \tag{2.9}$$

of $[2,z]$ and group the primes of $\mathcal{P}$ that are less than z into sub-sets corresponding to the intervals $[z_n,z_{n-1})$ $(n=1,2,\dots,r)$. Using the obvious fact that

$$W(p) \le W(z_n) \quad \text{if} \quad z_n \le p < z_{n-1} ,$$

we derive the

COROLLARY.

$$L_\nu(z) = W(z)\ \{1+\theta \sum_{n=1}^{r} \frac{W(z_n)}{W(z)} \sum_{z_n \le p < z_{n-1}} \frac{\omega(p)}{p} \sum_{t|P(p^+,z)} \frac{\chi_\nu(t)\{1-\chi_\nu(pt)\}}{t}\ \omega(t)\}$$

where $\nu = 1$ or 2 and $|\theta| \le 1$.

There is, as Levin has indicated, considerable latitude in the choice of the functions χ_ν (or, what amounts to the same thing, the sets $\mathcal{B}_\nu$) . However, for our present purpose Brun's choice, as modified by Tartakovskij, suffices. Let b be a positive integer. For $\nu = 1$ or 2 , and each $n = 1,\dots,r$, put

$$\chi_\nu(d) = 1 \quad \text{if} \quad d|P(z_n,z) \quad \text{and} \quad \nu(d) \le 2b-\nu+2n-1 \tag{2.10}$$
$$= 0 \quad \text{if} \quad d|P(z) \quad \text{otherwise.}^*$$

Then χ_1 , χ_2 obviously satisfy (2.1) and (2.2) . To check (2.3) , suppose that $t|P(z_n,z)$ and $\chi_\nu(t) = 1$. Then $\nu(t) \le 2b-\nu+2n-1$; if also $\mu(t) = (-1)^\nu = (-1)^{2b-\nu+2n-2}$, then $\nu(t) = 2b-\nu+2n-1$ is impossible. Hence $\nu(t) < 2b-\nu+2n-1$, and so, if $p < q(t)$ and $p|P(z)$, $\nu(pt) \le 2b-\nu+2n-1$; but $pt|P(z_m,z)$ for some $m \le n$, and it follows that $\chi_\nu(pt) = 1$. Thus χ_1 , χ_2 satisfy (2.3) too.

Let us now interpret the innermost sum (over t) on the right of Lemma 3 , Corollary, in the light of the choice (2.10). Here t makes a contribution to the sum only if $\chi_\nu(t) = 1$, $\chi_\nu(pt) = 0$; since both, t and pt divide $P(z_n,z)$, it follows that $\nu(t) \le 2b-\nu+2n-1$ and $\nu(pt) > 2b-\nu+2n-1$, whence $\nu(t) = 2b-\nu+2n-1$. Writing $pt = d$ we have, therefore, that

$$L_\nu(z) = W(z)\ \{1+\theta \sum_{n=1}^{r} \frac{W(z_n)}{W(z)} \sum_{\substack{d|P(z_n,z) \\ \nu(d)=2b-\nu+2n}} \frac{\omega(d)}{d}\ \} , \quad |\theta| \le 1 \qquad (\nu = 1,2) ,$$

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* In other words, $\mathcal{B}_\nu$ consists of all divisors d of $P(z)$ with the property that if $d|P(z_n,z)$ then $\nu(d) \le 2b-\nu+2n-1$ ($\nu(d)$ denotes the number of prime factors of d) ; thus if $d \in \mathcal{B}_\nu$, d has relatively few large prime factors.

and so, by an elementary inequality for elementary symmetric functions (clearly the sum over d is the $(2b-\nu+2n)$-th elementary symmetric function of the arguments $\omega(p)/p$, $z_n \le p < z$) ,

$$(2.11) \qquad L_\nu(z) = W(z)\{1+\theta \sum_{n=1}^{r} \frac{W(z_n)}{W(z)} \frac{1}{(2b-\nu+2n)!} \Big(\sum_{z_n \le p < z} \frac{\omega(p)}{p} \Big)^{2b-\nu+2n}\}, \quad |\theta| \le 1; \quad \nu = 1,2 .$$

Also (cf. (2.6) and (2.8)) a simple combinatorial argument shows that, with our choice (2.10) ,

$$(2.12) \qquad \sum_{d|P(z)} \chi_\nu(d)\ \omega(d) \le \Big(1+\sum_{p<z} \omega(p)\Big)^{2b-\nu+1} \prod_{n=1}^{r-1} \Big(1+\sum_{p<z_n} \omega(p)\Big)^2$$
$$\le (1+A\pi(z))^{2b-\nu+1} \prod_{n=1}^{r-1} (1+A\pi(z_n))^2$$

by (Ω) ; and to take the estimations in (2.11) and (2.12) further we must now choose a convenient partition (2.9).

Let Λ be a real number satisfying $0 < \Lambda \le 1$, and define $\{z_n\}$ by

$$(2.13) \qquad z_r = 2 \ , \quad \log z_n = e^{-n\Lambda} \log z \quad (n = 1,2,\ldots,r-1) ,$$

where r is chosen so that $\log z_{r-1} = e^{-(r-1)\Lambda} \log z > \log 2$ but $e^{-r\Lambda} \log z \le \log 2$, in other words, so that

$$(2.14) \qquad e^{(r-1)\Lambda} < \frac{\log z}{\log 2} \le e^{r\Lambda} .$$

Remembering that we may assume z to be sufficiently large, a simple calculation combining (2.12) , (2.13) and (2.14) shows that

$$(2.15) \qquad \sum_{d|P(z)} \chi_\nu(d)\ \omega(d) = O\left(z^{2b-\nu+1+\frac{2}{e^\Lambda-1}}\right) ; \quad \nu = 1,2 .$$

It remains to deal with $L_\nu(z)$. We shall see that Λ can be chosen in such a way that

$$(2.16) \qquad W(z_n)/W(z) \le e^{2n\lambda} \qquad (n = 1,2,\ldots,r) ;$$

if we <u>assume</u> (2.16), then

$$\sum_{z_n \le p < q} \frac{\omega(p)}{p} \le \sum_{z_n \le p < z} \log(1-\frac{\omega(p)}{p})^{-1} = \log \frac{W(z_n)}{W(z)} \le 2n\lambda$$
$$(n = 1,\ldots,r) ,$$

so that, by (2.11) ,

$$L_\nu(z) = W(z)\{1+\theta \sum_{n=1}^{r} e^{2n\lambda} \frac{(2n\lambda)^{2b-\nu+2n}}{(2b-\nu+2n)!}\}, \quad |\theta|\le 1; \quad \nu=1,2.$$

Since $(2b-\nu+2n)! \ge (2n)!\,(2n)^{2b-\nu}$, the sum on the right is at most

$$\sum_{n=1}^{r} \lambda^{2b-\nu} \frac{(2n/e)^{2n}}{(2n)!} (\lambda e^{1+\lambda})^{2n};$$

but $(me^{-1})^m/m!$ is a decreasing function of m, so that this sum is, in turn, at most

$$2e^{-2}\lambda^{2b-\nu} \sum_{n=1}^{\infty} (\lambda e^{1+\lambda})^{2n} = 2\lambda^{2b-\nu+2} \frac{e^{2\lambda}}{1-\lambda^2 e^{2+2\lambda}}.$$

It follows form this, (2.6) and (2.I5) that, for $\nu = 1,2$,

$$(2.I7) \qquad (-1)^\nu S(\mathcal{A};\mathcal{P},z) \ge (-1)^\nu XW(z)\{1+\theta 2\lambda^{2b-\nu+2} \frac{e^{2\lambda}}{1-\lambda^2 e^{2+2\lambda}}\} + O(z^{2b-\nu+1+\frac{2}{e^\Lambda-1}}),$$

provided that Λ, $0 < \Lambda \le 1$, is such that (2.I6) is true. (The reader will have noticed that various approximations have been made in the preceding argument which could be improved or avoided, e.g. for small n, if anything were to be gained by it).

We therefore consider (2.I6). By a well-known result we deduce from (Ω) that

$$\sum_{w\le p<z} \frac{\omega(p)\log p}{p} \le A(\log\frac{z}{w}+1) \qquad \text{if} \quad 2\le w\le z,$$

and from this we can deduce, using (Ω_1), that

$$\frac{W(w)}{W(z)} \le (\frac{\log z}{\log w})^A \{1+O(\frac{1}{\log w})\} \qquad \text{if} \quad 2\le w\le z.$$

Hence

$$\frac{W(z_n)}{W(z)} \le e^{n\Lambda A}(1+\frac{Be^{n\Lambda}}{\log z}) = \exp\{n\Lambda A + \log(1+B\frac{e^{n\Lambda}}{\log z})\}, \quad 1\le n\le r$$

for some suitable constant B. By (2.I4)

$$\frac{1}{n}\log(1+B\frac{e^{n\Lambda}}{\log z}) \le \frac{1}{r}\log(1+B\frac{e^{r\Lambda}}{\log z}) \le \frac{\Lambda}{\log(\frac{\log z}{\log 2})}\log(1+B\frac{e}{\log 2}),$$

whence

$$\frac{W(z_n)}{W(z)} \leq \exp\ n\Lambda A(1 + \{\frac{B_1}{\log\log z}\})\} \ , \quad n = 1,\ldots,r \ .$$

Thus (2.I6) follows on choosing

$$\Lambda = \frac{2\lambda}{A}\,(1 + \frac{B_1}{\log\log z})^{-1} \ , \tag{2.I8}$$

and $0 < \Lambda \leq 1$ since $\lambda < \frac{1}{2}$, $A \geq 1$ and z is large. Our argument is now complete, and Theorem 2 follows readily from (2.I7) and (2.I8).

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