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Equivariant brauergroups in algebraic number theory

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1. - The Equivariant Brauer Group

This section contains the bare minimum of general theory required in the sequel. We shall avoid going into the categorical generalities which underlie a systematic treatment. (See however our paper in the Proceedings of the Hull conference on K-theory (Springer Notes 108) for the notion of a group graded category \( \mathcal{G} \). Those familiar with this paper will realize that what we are considering here are examples of categories \( \text{Rep}(\mathcal{G}) \).

We give ourselves a pair \((R, \Gamma)\), where \( \Gamma \) is a 2-graded group whose underlying group we shall denote by \( \Gamma^+ \) with grading map \( \omega : \Gamma^+ \to \pm 1 \) (units of \( \mathbb{Z} \)) and where \( R \) is a commutative ring (always with 1) and a \( \Gamma^+ \)-module, \( \Gamma^+ \) acting by ring automorphisms. We shall be interested specifically in two particular cases, namely (a) direct action when \( \omega = e : \Gamma^+ \to 1 \) is the null map, i.e., \( \Gamma = \Gamma^+ \), and (b) involution when \( \omega : \Gamma = \pm 1 \) is an isomorphism.

Let \( M, N \) be \( R \)-modules. An additive map \( f : M \to N \) is said to have grade \( \gamma (\gamma \in \Gamma^+) \), if

\[
f(r m) = \gamma r f(m) , \quad r \in R , \quad m \in M .
\]

In the case of direct action an \((R, \Gamma)\)-module \((M, g)\) consists of an \( R \)-module \( M \) and a homomorphism \( g : \Gamma \to \text{Aut}_R(M) \) so that, for all \( \gamma \), \( g_\gamma \) is of grade \( \gamma \). In the case of involution an \((R, \Gamma)\)-module \((M, g)\) consists of an \( R \)-module \( M \) and a non-singular Hermitian form \( h_g \) on \( M \) over \( R \), with respect to the involution on \( R \) induced by the generator \( \gamma \) of \( \Gamma \). There is of course a general definition applying to all cases, but we shall not need this here. We shall however give the general definition of an \((R, \Gamma)\)-algebra \((A, g)\). This is an \((R, \Gamma^+)\)-module, with \( A \) as \( R \)-algebra, and so that the \( g_\gamma \) act on the ring \( A \) by automorphisms when \( \gamma \) is even (i.e., \( \omega (\gamma) = 1 \)) and by anti-automorphisms when \( \gamma \) is odd (i.e., \( \omega (\gamma) = -1 \)). Thus in case (b) \( A \) is just an \( R \)-algebra with involutory anti-automorphism compatible with the involution on \( R \).

(*) This is a version of the talk given by Fröhlich at the Bordeaux Colloquium. A detailed account of the underlying theory and its applications will be published elsewhere. No proofs will be given here.
The \((R,\Gamma)\)-modules \((M, g)\) for which \(M\) is an \(R\)-progenerator form a category \(\text{Gen}(R, \Gamma)\) with product \(\otimes_R\) (diagonal action of \(\Gamma\)) and identity object given by \(R\). The morphisms of \(\text{Gen}(R, \Gamma)\) are to be just the isomorphisms of grade 1 (of course commuting with the \(\Gamma\)-action). Similarly the \((R,\Gamma)\)-algebras \((A, g)\) with \(A\) central separable, and their isomorphisms of grade 1 form a category \(\text{A}_Z(R, \Gamma)\) with product \(\otimes_R\) and identity object. The isomorphism classes in each of these two categories form an Abelian monoid, which we shall denote by \(\text{Gen}(R, \Gamma)\), and \(\text{A}_Z(R, \Gamma)\) respectively. The classes in \(\text{Gen}(R, \Gamma)\) with underlying modules of rank one form the maximal subgroup \(C(R, \Gamma)\) of \(\text{Gen}(R, \Gamma)\), the equivariant class-group or Picard group. Moreover one can define in general a product preserving functor
\[
\text{End} : \text{Gen}(R, \Gamma) \to \text{A}_Z(R, \Gamma)
\]
We only describe it in our two special cases. When the action is direct, then \(\text{End}(M, g)\) is just \(\text{End}_R(M)\) with \(\Gamma\) acting by conjugation, and in the case of involution then it is \(\text{End}_R(M)\) with the adjoint involution of \(h_g\). We now get a monoid map
\[
\text{End} : \text{Gen}(R, \Gamma) \to \text{A}_Z(R, \Gamma)
\]
whose cokernel is a group, the equivariant Brauer group \(B(R, \Gamma)\). To establish the group property one has to generalize the known isomorphism
\[
A \otimes_R A^{\text{op}} \cong \text{End}_R(A)
\]
Finally forgetting the \(\Gamma\)-action one gets a map from \(B(R, \Gamma)\) into the ordinary Brauer group \(B(R)\), and we shall write
\[
B_0(R, \Gamma) = \text{Ker}[B(R, \Gamma) \to B(R)].
\]
It is this group in which we shall be interested mainly.

The cohomology groups of the graded group \(\Gamma\) with coefficients in \(U(R)\) (group of units) and in \(C(R)\) (ordinary Picard group) are defined via the obvious action of \(\Gamma\) twisted by the grading \(\omega\). Thus if \((\gamma, u) \mapsto \gamma u\) is the originally given action of \(\Gamma\) on \(R\), then the twisted action of \(\Gamma\) on \(U(R)\) used to define \(H^i(\Gamma, U(R))\) is \((\gamma, u) \mapsto (u)\omega(\gamma)\). Thus in case (a)
\[
H^i(\Gamma, U(R)) = H^i(\Gamma_+, U(R)), \quad \text{in case (b) } H^i(\Gamma, U(R)) = H^{i+1}(\Gamma_+, U(R)) (i \geq 1).
\]
Similarly for \(C(R)\).

From now on assume \(\Gamma\) finite.
THEOREM 1. There is an exact sequence

\[ 0 \to H^1(\Gamma, U(R)) \to C(R, \Gamma) \to H^0(\Gamma, C(R)) \to H^2(\Gamma, U(R)) \to \]
\[ B_0(R, \Gamma) \to H^1(\Gamma, C(R)) \to H^3(\Gamma, U(R)) \to \]

Remarks
1) This is the top row of a larger diagram involving \( B(R, \Gamma) \) and other versions of the Brauer group.

2) The sequence (1) is derived from an infinite exact sequence

\[ 0 \to H^1(\Gamma, U(R)) \to \ldots \to H^i(\Gamma, U(R)) \to H^i(\mathcal{C}(R, \Gamma)) \to \]
\[ \to H^{i-1}(\Gamma, C(R)) \to H^{i+1}(\Gamma, U(R)) \to \ldots \]

where the \( H^i(\mathcal{C}(R, \Gamma)) \) are cohomology groups of a certain complex. One gets (1) via suitable isomorphisms for the lowest terms. We shall describe one example of this (cf. (2)). The only property of the \( H^i(\mathcal{C}(R, \Gamma)) \) we shall need is

THEOREM 2. The groups \( H^i(\mathcal{C}(R, \Gamma)) \) are annihilated by \( \text{card} \ \Gamma \).

This result is of interest in connection with

THEOREM 3. Every class in \( B_0(R, \Gamma) \) is represented by an \( (R, \Gamma) \)-algebra \( ( \text{End}_R(M), g ) \) with rank \( (M) = \text{card} \ \Gamma \). If \( R \) is connected then the class in \( B_0(R, \Gamma) \) of any \( (R, \Gamma) \)-algebra \( ( \text{End}_R(M), g ) \) is annihilated by \( \text{rank} \ (M) \).

Examples
(i) - If \( \omega \) is null, \( R/R^\Gamma \) Galois with group \( \Gamma \) then
\[ C(R^\Gamma) \cong C(R, \Gamma) \], \( B(R^\Gamma) \cong B(R, \Gamma) \)
\[ \text{Ker} [ B(R^\Gamma) \to B(R) ] \cong B_0(R, \Gamma) \]
and our sequence (1) yields one which looks like that of Chase-Harrison-Rosenberg.

(ii) - When \( R \) is a field then (1) yields an isomorphism
\[ H^2(\Gamma, U(R)) \cong B_0(R, \Gamma) \].

It is instructive to interpret this explicitly in the well known cases

(a) \( \Gamma \) acts directly as Galois group , (b) \( \Gamma \) acts trivially on \( R \) with direct action , (c) \( \Gamma \cong \pm 1 \) with non-trivial involution, (d) \( \Gamma \cong \pm 1 \) with trivial involution.
2. - Algebraic integers with involution

To begin with $R$ can still be an arbitrary commutative ring, $\omega : \Gamma \cong \pm 1$, and $\gamma$ denotes the generator of $\Gamma$.

Consider pairs $(P, f)$, $P$ a rank 1 projective, $f$ an automorphism of $P$ of grade $\gamma$ with $f^2 = 1$. If $Q$ is any rank 1-projective and $\gamma Q$ its image under some bijection $q \mapsto \gamma q$ of grade $\gamma$ then for $P = \gamma Q \otimes_R Q$ we may take $f(\gamma q_1 \otimes q_2) = \gamma q_2 \otimes q_1$. Call this a trivial pair. The isomorphism classes of pairs $(P, f)$ modulo those of trivial pairs form an Abelian group under $\otimes_R$ and this is $H^2(\mathbb{Z}(R, \Gamma))$ in our simple case. The general construction is really quite analogous. (There is also a special feature of the quadratic case tying up equivariant classgroups and Brauer groups for opposite gradings).

Next we describe the isomorphism

$$\psi : H^2(\mathbb{Z}(R, \Gamma)) \cong B_0(R, \Gamma).$$

Let a pair $(P, f)$, as above, be given. The associated Brauer class is then that of the pair $(\text{End}_R(M), i_h)$ where (i) $M$ is an $R$-progenerator, (ii) $h : M \times M \to P$ is a non-singular pairing which is $R$-linear in the first argument and so that $h(m_2, m_1) = fh(m_1, m_2)$ (in other words $h$ is a "non-singular Hermitian form from over $(P, f)$") (iii) $i_h$ is the adjoint involution of $h$ in $\text{End}_R(M)$ (this exists!). Note that by Theorems 2 and 3 we could manage with an $M$ of rank 2 and, except for the trivial class, not with $M$ of rank 1. In fact we can choose

$$M = R \otimes P, \quad h((r_1, p_1), (r_2, p_2)) = r_1 f p_2 + \gamma r_2 p_1.$$

Viewing $\psi$ as an identification the relevant maps of (1) have now an obvious description. Namely $B_0(R, \Gamma) \to H^1(\Gamma, C(R)) = H^0(\Gamma, C(R))$ (Tate cohomology) takes $\text{cl}(P, f)$ into $\text{cl}(P)$. On the other hand let $u \in U(R)$, $\gamma u = 1$. Then under $H^1(\Gamma, U(R)) = H^2(\Gamma, U(R)) \to B_0(R, \Gamma)$ the class of $u$ goes into the class of $(R, f_u)$, $f_u(r) = u \gamma r$. The module $M$ in (3) is now free, $\text{End}_R(M)$ is the $2 \times 2$ matrix ring over $R$ and

$$i_u \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix} = \begin{pmatrix} \gamma a_2 & \gamma a_1 \\ \gamma a_1 & a_2 \end{pmatrix}. $$

Every full matrix ring over $R$ with involution is Brauer equivalent to one of this type and criteria for equivalence can be derived from (1).
From now let $R$ be the ring of integers in a finite algebraic number field $L$. If first the involution on $R$ is trivial then (1) reduces to

$$
\begin{align*}
C(R,\Gamma) &= (U(R)/U(R)^2) \times C(R) \\
B_0(R,\Gamma) &= \{ \pm 1 \} \times (C(R)/C(R)^2)
\end{align*}
$$

where the subscript 2 denotes the kernel of multiplication by 2. If the involution is non-trivial then (2) yields

$$
B_0(R,\Gamma) = \text{Cok} \left[ \tilde{H}^0(\Gamma, L^*) \to \tilde{H}^0(\Gamma, I(R)) \right],
$$

where $L^* = U(L)$, $I(R) =$ group of fractional ideals. Hence $B_0(R,\Gamma)$ is an elementary 2-group and

$$
\begin{align*}
\text{card } B_0(R,\Gamma) &= \sup (2, 2^d) \\
d &= \text{number of ramified prime ideals in } R/R^\Gamma.
\end{align*}
$$

3. - Algebraic integers with direct action of a Galois group

$L$ is again a finite algebraic number field with subfield $K$, $\Gamma = \text{Gal}(L/K)$, with null grading $w = \varepsilon$, $R =$ integers in $L$, $T =$ integers in $K$. The subscript $p$ denotes completion at $p$, with respect to a prime $p$ in the base field $K$. Thus if $p$ is finite then $R_p = \prod R_{\mathfrak{p}}$ (all $\mathfrak{p}$ in $L$ above $p$). One knows that $B(R_p) = 0$ whence $B(R_p, \Gamma) = B_0(R_p, \Gamma)$. Also $B(R) \to B(L)$ is injective, and we may identify $B(R)$ with the group of those Brauer classes over $L$ which split at all finite primes. Moreover, as by (1) $H^2(\Gamma, U(R))^* = B_0(R_p, \Gamma)$, these groups vanish at all non-ramified prime ideals. Beyond this one has

**Theorem 4.** The sequences

$$
0 \to \text{Ker} [B(T) \to B(R)] \to B_0(R,\Gamma) \to \prod B_0(R_p,\Gamma)_{p \text{ finite}}
$$

are exact and

$$
B_0(R,\Gamma) \to B_0(L,\Gamma), \quad B(R,\Gamma) \to B(L,\Gamma)
$$

are injective.

Let $J_L$ be the idele group of $L$ and

$$
U_L = \prod_{p \text{ finite}} U(R_p) \times \prod_{p \text{ finite}} U(L_p).
$$

Then we have
THEOREM 5. In the commutative diagram

\[
\begin{array}{ccc}
0 & \to & B_0(O,\Gamma) \\
& \downarrow & \downarrow \text{inv} \\
0 & \to & H^2(\Gamma, U_L) \\
& \to & H^2(\Gamma, J_L/L^*)
\end{array}
\]

the first row is exact (and so is of course by classfield theory the second row).

Let for the moment \( B_0(L/K) \) denote the subgroup of \( B(K) \) of Brauer classes which split in \( L \), as well as at all finite, non-ramified \( p \) and which have at all finite ramified primes cocycles in the group of units. From the last theorem we have an isomorphism

\[
\theta : B_0(L/K) \cong B_0(O,\Gamma) .
\]

We shall describe \( \theta \) explicitly.

Let \( A \) be a central simple \( K \)-algebra whose class lies in \( B_0(L/K) \). Then \( A \otimes_K L \cong \text{End}_L(V) \), \( V \) an \( L \)-vector space. The \( \Gamma \)-structure, given by the action on \( L \), is reflected in a \( \Gamma \)-structure on \( \text{End}_L(V) \) given by conjugation with automorphisms \( f_\gamma \) of grade \( \gamma \) on \( V \), so that \( f_\gamma \alpha \beta f_\gamma^{-1} \equiv \alpha \beta \pmod{L^*} \). One can then construct an \( R \)-lattice \( M \) spanning \( V \) and fractional \( R \)-ideals \( \alpha_\gamma \) so that \( f_\gamma M = \alpha_\gamma M \). This yields an \( R \)-algebra \( \text{End}_R(M) \subset \text{End}_L(V) \) stable under the \( f_\gamma \). Its class is the required image in \( B_0(O,\Gamma) \). Moreover the ideal classes \( \text{cl}(\alpha_\gamma) \) define its image under \( B_0(O,\Gamma) \to H^2(\Gamma, C(R)) \).

We shall finally compute the order of \( B_0(O,\Gamma) \). Let \( \mathfrak{p} \) be a finite prime in \( L \), \( L_\mathfrak{p} \) the completion, \( U_\mathfrak{p} \) the group of units of \( R_\mathfrak{p} \) and consider the exact valuation sequence

\[
\theta : U_\mathfrak{p} \to L_\mathfrak{p}^* \to \mathfrak{p} \to \mathbb{Z} \to 0 .
\]

If \( e_\mathfrak{p} = e_p \) is the ramification index over \( K_p | \mathfrak{p} \) then \( \gamma_\mathfrak{p} | K_p = e_p \gamma_\mathfrak{p} \). It follows that effectively \( H^2(\text{Gal}(L_\mathfrak{p}/K_p), L_\mathfrak{p}^*) \to H^2(\text{Gal}(L_\mathfrak{p}/K_p), \mathbb{Z}) \) is multiplication by \( e_\mathfrak{p} \) and hence that \( H^2(\text{Gal}(L_\mathfrak{p}/K_p), U_\mathfrak{p}) \) is cyclic of order \( e_\mathfrak{p} \). Going over to the global field and taking into account the infinite primes we conclude that \( H^2(\Gamma, U_L) \) is the direct product of cyclic groups of order \( e_p \), \( p \) running through all primes of \( K \), with the obvious meaning of \( e_p \) for infinite \( p \). On the other hand the image of \( \text{inv} \) from \( H^2(\Gamma, U_L) \) clearly has order the least common multiple of the \( e_p \). Hence finally

\[
\text{card } B_0(O,\Gamma) = \prod_{\mathfrak{p}} \frac{1}{e_\mathfrak{p}} .
\]