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Equivariant brauergroups in algebraic number theory

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1. The Equivariant Brauer Group

This section contains the bare minimum of general theory required in the sequel. We shall avoid going into the categorical generalities which underlie a systematic treatment. (See however our paper in the Proceedings of the Hull conference on K-theory (Springer Notes 108) for the notion of a group graded category $\mathcal{C}$.) Those familiar with this paper will realize that what we are considering here are examples of categories $\text{Rep}(\mathcal{C})$.

We give ourselves a pair $(R,\Gamma)$, where $\Gamma$ is a 2-graded group whose underlying group we shall denote by $\Gamma_+$ with grading map $w: \Gamma_+ \to \pm 1$ (units of $\mathbb{Z}$) and where $R$ is a commutative ring (always with 1) and a $\Gamma_+$-module, $\Gamma_+$ acting by ring automorphisms. We shall be interested specifically in two particular cases, namely (a) direct action when $w: \varepsilon: \Gamma_+ \to 1$ is the null map, i.e., $\Gamma = \Gamma_+$, and (b) involution when $w: \Gamma = \pm 1$ is an isomorphism.

Let $M$, $N$ be $R$-modules. An additive map $f: M \to N$ is said to have grade $\gamma (\gamma \in \Gamma_+)$, if

$$f(r \cdot m) = \gamma(r) f(m), \quad r \in R, m \in M.$$

In the case of direct action an $(R,\Gamma)$-module $(M,g)$ consists of an $R$-module $M$ and a homomorphism $g: \Gamma \to \text{Aut}_R(M)$ so that, for all $\gamma$, $g_{\gamma}$ is of grade $\gamma$. In the case of involution an $(R,\Gamma)$-module $(M,g)$ consists of an $R$-module $M$ and a non-singular Hermitian form $h_g$ on $M$ over $R$, with respect to the involution on $R$ induced by the generator $\gamma$ of $\Gamma$. There is of course a general definition applying to all cases, but we shall not need this here. We shall however give the general definition of an $(R,\Gamma)$-algebra $(A,g)$. This is an $(R,\Gamma_+)$-module, with $A$ as $R$-algebra, and so that the $g_{\gamma}$ act on the ring $A$ by automorphisms when $\gamma$ is even (i.e., $w(\gamma) = 1$) and by anti-automorphisms when $\gamma$ is odd (i.e., $w(\gamma) = -1$). Thus in case (b) $A$ is just an $R$-algebra with involutory anti-automorphism compatible with the involution on $R$.

(*) This is a version of the talk given by Fröhlich at the Bordeaux Colloquium. A detailed account of the underlying theory and its applications will be published elsewhere. No proofs will be given here.
The $(R,\Gamma)$-modules $(M,g)$ for which $M$ is an $R$-progenerator form a category $\mathcal{Gen}(R,\Gamma)$ with product $\otimes_R$ (diagonal action of $\Gamma$) and identity object given by $R$. The morphisms of $\mathcal{Gen}(R,\Gamma)$ are to be just the isomorphisms of grade 1 (of course commuting with the $\Gamma$-action). Similarly the $(R,\Gamma)$-algebras $(A,g)$ with $A$ central separable, and their isomorphisms of grade 1 form a category $\mathcal{A}_2(R,\Gamma)$ with product $\otimes_R$ and identity object. The isomorphism classes in each of these two categories form an Abelian monoid, which we shall denote by $\mathcal{Gen}(R,\Gamma)$, and $\mathcal{A}_2(R,\Gamma)$ respectively. The classes in $\mathcal{Gen}(R,\Gamma)$ with underlying modules of rank one form the maximal subgroup $C(R,\Gamma)$ of $\mathcal{Gen}(R,\Gamma)$, the equivariant class-group or Picard group. Moreover one can define in general a product preserving functor

$$\text{End} : \mathcal{Gen}(R,\Gamma) \to \mathcal{A}_2(R,\Gamma).$$

We only describe it in our two special cases. When the action is direct, then $\text{End}(M,g)$ is just $\text{End}_R(M)$ with $\Gamma$ acting by conjugation, and in the case of involution then it is $\text{End}_R(M)$ with the adjoint involution of $h_g$. We now get a monoid map

$$\text{End} : \mathcal{Gen}(R,\Gamma) \to \mathcal{A}_2(R,\Gamma),$$

whose cokernel is a group, the equivariant Brauer group $B(R,\Gamma)$. To establish the group property one has to generalize the known isomorphism

$$A \otimes_R A^{\text{op}} \cong \text{End}_R(A).$$

Finally forgetting the $\Gamma$-action one gets a map from $B(R,\Gamma)$ into the ordinary Brauer group $B(R)$, and we shall write

$$B_0(R,\Gamma) = \text{Ker}[B(R,\Gamma) \to B(R)].$$

It is this group in which we shall be interested mainly.

The cohomology groups of the graded group $\Gamma$ with coefficients in $U(R)$ (group of units) and in $C(R)$ (ordinary Picard group) are defined via the obvious action of $\Gamma_+$ twisted by the grading $w$. Thus if $(\gamma, u) \mapsto \gamma u$ is the originally given action of $\Gamma_+$ on $R$, then the twisted action of $\Gamma_+$ on $U(R)$ used to define $H^i(\Gamma_+, U(R))$ is $(\gamma, u) \mapsto (\gamma u)^w(\gamma)$. Thus in case (a)

$$H^i(\Gamma_+, U(R)) = H^i(\Gamma_+, U(R)),$$

in case (b) $H^i(\Gamma, U(R)) = H^{i+1}(\Gamma_+, U(R))$ ($i \geq 1$). Similarly for $C(R)$.

From now on assume $\Gamma$ finite.
THEOREM 1. There is an exact sequence

\[ 0 \to H^1(\Gamma, U(R)) \to C(R, \Gamma) \to H^0(\Gamma, C(R)) \to H^2(\Gamma, U(R)) \to B_0(\Gamma, \Gamma) \to H^1(\Gamma, C(R)) \to H^3(\Gamma, U(R)) . \]

Remarks

1) This is the top row of a larger diagram involving $B(R, \Gamma)$ and other versions of the Brauer group.

2) The sequence (1) is derived from an infinite exact sequence

\[ 0 \to H^1(\Gamma, U(R)) \to ... \to H^i(\Gamma, U(R)) \to H^i(\mathcal{C}(R, \Gamma)) \to \]
\[ H^{i-1}(\Gamma, C(R)) \to H^{i+1}(\Gamma, U(R)) \to ... \]

where the $H^i(\mathcal{C}(R, \Gamma))$ are cohomology groups of a certain complex. One gets (1) via suitable isomorphisms for the lowest terms. We shall describe one example of this (cf. (2)). The only property of the $H^i(\mathcal{C}(R, \Gamma))$ we shall need is

THEOREM 2. The groups $H^i(\mathcal{C}(R, \Gamma))$ are annihilated by $\text{card } \Gamma$.

This result is of interest in connection with

THEOREM 3. Every class in $B_0(R, \Gamma)$ is represented by an $(R, \Gamma)$-algebra $(\text{End}_R(M), g)$ with rank $(M) = \text{card } \Gamma$. If $R$ is connected then the class in $B_0(R, \Gamma)$ of any $(R, \Gamma)$-algebra $(\text{End}_R(M), g)$ is annihilated by rank $(M)$.

Examples

(i) - If $w$ is null, $R/R^\Gamma$ Galois with group $\Gamma$ then

\[ C(R^\Gamma) \cong C(R, \Gamma), \quad B(R^\Gamma) \cong B(R, \Gamma) \]
\[ \text{Ker } [B(R^\Gamma) \to B(R)] \cong B_0(R, \Gamma) \]

and our sequence (1) yields one which looks like that of Chase-Harrison-Rosenberg.

(ii) - When $R$ is a field then (1) yields an isomorphism

\[ H^2(\Gamma, U(R)) \cong B_0(R, \Gamma) . \]

It is instructive to interpret this explicitly in the well known cases

(a) $\Gamma$ acts directly as Galois group, (b) $\Gamma$ acts trivially on $R$ with direct action, (c) $\Gamma \cong \pm 1$ with non-trivial involution, (d) $\Gamma \cong \pm 1$ with trivial involution.
2. - **Algebraic integers with involution**

To begin with $R$ can still be an arbitrary commutative ring, $\omega : \Gamma \cong \pm 1$, and $\gamma$ denotes the generator of $\Gamma$.

Consider pairs $(P, f)$, $P$ a rank 1 projective, $f$ an automorphism of $P$ of grade $\gamma$ with $f^2 = 1$. If $Q$ is any rank 1-projective and $\gamma Q$ its image under some bijection $q \mapsto \gamma q$ of grade $\gamma$ then for $P = \gamma Q \otimes_R Q$ we may take $f(\gamma q_1 \otimes q_2) = \gamma q_2 \otimes q_1$. Call this a trivial pair. The isomorphism classes of pairs $(P, f)$ modulo those of trivial pairs form an Abelian group under $\otimes_R$ and this is $H^2(\mathbb{F} (R, \Gamma))$ in our simple case. The general construction is really quite analogous. (There is also a special feature of the quadratic case tying up equivariant class groups and Brauer groups for opposite gradings).

Next we describe the isomorphism

\[
(2) \quad \psi : H^2(\mathbb{F}(R, \Gamma)) \cong B^0(R, \Gamma).
\]

Let a pair $(P, f)$, as above, be given. The associated Brauer class is then that of the pair $(\text{End}_R(M), i_h)$ where (i) $M$ is an $R$-progenerator, (ii) $h : M \times M \to P$ is a non-singular pairing which is $R$-linear in the first argument and so that $h(m_2, m_1) = f h(m_1, m_2)$ (in other words $h$ is a "non-singular Hermitian form from over $(P, f)$") (iii) $i_h$ is the adjoint involution of $h$ in $\text{End}_R(M)$ (this exists!). Note that by Theorems 2 and 3 we could manage with an $M$ of rank 2 and, except for the trivial class, not with $M$ of rank 1. In fact we can choose

\[
(3) \quad M = R \otimes P, \quad h((r_1, p_1), (r_2, p_2)) = r_1 \cdot f p_2 + \gamma r_2 \cdot p_1.
\]

Viewing $\psi$ as an identification the relevant maps of (1) have now an obvious description. Namely $B^0(R, \Gamma) \to H^1(\Gamma, C(R)) = H^0(\Gamma, C(R))$ (Tate cohomology) takes $\text{cl}(P, f)$ into $\text{cl}(P)$. On the other hand let $u \in U(R), \gamma u, u = 1$. Then under $H^1(\Gamma, U(R)) = H^2(\Gamma, U(R)) \to B^0(R, \Gamma)$ the class of $u$ goes into the class of $(R, \Gamma)$, $f_u(r) = u \cdot \gamma r$. The module $M$ in (3) is now free, $\text{End}_R(M)$ is the $2 \times 2$ matrix ring over $R$ and

\[
i_u \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \gamma a_{21} & \gamma a_{12} \\ \gamma a_{22} & \gamma u \end{pmatrix}.
\]

Every full matrix ring over $R$ with involution is Brauer equivalent to one of this type and criteria for equivalence can be derived from (1).
From now let \( R \) be the ring of integers in a finite algebraic number field \( L \).

If first the involution on \( R \) is trivial then (1) reduces to

\[
\begin{align*}
\text{(4)} & \quad C(R, \Gamma) \cong (U(R)/U(R)^2) \times C(R)_2 \\
& \quad B_0(R, \Gamma) \cong \{ \pm 1 \} \times (C(R)/C(R)^2),
\end{align*}
\]

where the subscript 2 denotes the kernel of multiplication by 2. If the involution is non-trivial then (2) yields

\[
\begin{align*}
\text{(5)} & \quad B_0(R, \Gamma) \cong \text{Cok} [\tilde{H}^0(\Gamma^+, L^*) \to \tilde{H}^0(\Gamma^+, U(R))] ,
\end{align*}
\]

where \( L^* = U(L) , I(R) = \text{group of fractional ideals} \). Hence \( B_0(R, \Gamma) \) is an elementary 2-group and

\[
\begin{align*}
\text{(6)} & \quad \text{card } B_0(R, \Gamma) = \sup (2, 2^d) \\
& \quad d = \text{number of ramified prime ideals in } R/R^\Gamma.
\end{align*}
\]

3. - Algebraic integers with direct action of a Galois group

\( L \) is again a finite algebraic number field with subfield \( K \), \( \Gamma = \text{Gal}(L/K) \) with null grading \( \omega = \varepsilon \), \( R = \text{integers in } L \), \( T = \text{integers in } K \). The subscript \( p \) denotes completion at \( p \), with respect to a prime \( p \) in the base field \( K \).

Thus if \( p \) is finite then \( R_p = \prod R_{\mathfrak{p}} \) (all \( \mathfrak{p} \) in \( L \) above \( p \)). One knows that \( B(R_p) = 0 \) whence \( B(R_p, \Gamma) = B_0(R_p, \Gamma) \). Also \( B(R) \to B(L) \) is injective, and we may identify \( B(R) \) with the group of those Brauer classes over \( L \) which split at all finite primes. Moreover, as by (1) \( H^2(\Gamma, U(R)) = B_0(R, \Gamma) \), these groups vanish at all non-ramified prime ideals. Beyond this one has

**Theorem 4.** The sequences

\[
0 \to \text{Ker} [B(T)\to B(R)] \to B_0(R, \Gamma) \to \frac{\prod B_0(R_p, \Gamma)}{p \text{ finite}}
\]

are exact and

\[
B_0(R, \Gamma) \to B_0(L, \Gamma) , B(R, \Gamma) \to B(L, \Gamma)
\]

are injective.

Let \( J_L \) be the idele group of \( L \) and

\[
U_L = \prod_{\text{p finite}} U(R_p) \times \prod_{\text{p finite}} U(L_p).
\]

Then we have
THEOREM 5. In the commutative diagram

\[
\begin{array}{ccc}
0 & \to & B_o(R,\Gamma) \\
& \downarrow & \downarrow \text{inv} \\
& H^2(\Gamma, J_L/L^*) & \to H^2(\Gamma, J_L/L^*)
\end{array}
\]

the first row is exact (and so is of course by classfield theory the second row).

Let for the moment $B_o(L/K)$ denote the subgroup of $B(K)$ of Brauer classes which split in $L$, as well as at all finite, non-ramified $\mathfrak{p}$ and which have at all finite ramified primes cocycles in the group of units. From the last theorem we have an isomorphism

\[\theta : B_o(L/K) \cong B_o(R,\Gamma).\]

We shall describe $\theta$ explicitly.

Let $A$ be a central simple $K$-algebra whose class lies in $B_o(L/K)$. Then $A \otimes_K L \cong \text{End}_L(V)$, $V$ an $L$-vector space. The $\Gamma$-structure, given by the action on $L$, is reflected in a $\Gamma$-structure on $\text{End}_L(V)$ given by conjugation with automorphisms $f_\gamma$ of grade $\gamma$ on $V$, so that $f_\gamma a = f_\gamma f_\gamma^\alpha \text{ (mod } L^*)$. One can then construct an $R$-lattice $M$ spanning $V$ and fractional $R$-ideals $a_\gamma$ so that $f_\gamma M = a_\gamma M$. This yields an $R$-algebra $\text{End}_R(M) \subset \text{End}_L(V)$ stable under the $f_\gamma$. Its class is the required image in $B_o(R,\Gamma)$. Moreover the ideal classes $\text{cl}(a_\gamma)$ define its image under $B_o(R,\Gamma) \to H^2(\Gamma,\mathbb{C}(R))$.

We shall finally compute the order of $B_o(R,\Gamma)$. Let $\mathfrak{p}$ be a finite prime in $L$, $L_\mathfrak{p}$ the completion, $U_\mathfrak{p}$ the group of units of $R_\mathfrak{p}$ and consider the exact valuation sequence

\[\theta : U_\mathfrak{p} \to L_\mathfrak{p}^* \to \mathfrak{p} \to \mathbb{Z} \to 0 .\]

If $e_\mathfrak{p} = e_\mathfrak{p}$ is the ramification index over $K$, then $v_\mathfrak{p}|K = e_\mathfrak{p} v_\mathfrak{p}$. It follows that effectively $H^2(\text{Gal}(L_\mathfrak{p}/K), L_\mathfrak{p}^*) \to H^2(\text{Gal}(L_\mathfrak{p}/K), \mathbb{Z})$ is multiplication by $e_\mathfrak{p}$ and hence that $H^2(\text{Gal}(L_\mathfrak{p}/K), U_\mathfrak{p})$ is cyclic of order $e_\mathfrak{p}$. Going over to the global field and taking into account the infinite primes we conclude that $H^2(\Gamma, U_L)$ is the direct product of cyclic groups of order $e_\mathfrak{p}$, $\mathfrak{p}$ running through all primes of $K$, with the obvious meaning of $e_\mathfrak{p}$ for infinite $\mathfrak{p}$. On the other hand the image of $\text{inv}$ from $H^2(\Gamma, U_L)$ clearly has order the least common multiple of the $e_\mathfrak{p}$. Hence finally

\[\text{card } B_o(R,\Gamma) = \prod_{e_\mathfrak{p}} \frac{1}{\text{lcm}_{\mathfrak{p}}} .\]