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Well-distributed sequences

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Colloque Th. Nombres [1969, Bordeaux]
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WELL-DISTRIBUTED SEQUENCES

by

J. CIGLER

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The purpose of this talk is to give a survey of some methods and results concerning well-distributed sequences. Almost all facts will be known, but some proofs and the chosen point of view may perhaps be interesting.

The prototype of all those results is H. Weyl's observation that for every irrational number θ the sequence $\{n\theta\}$ is uniformly distributed mod. 1 . H. Weyl's proof - in his time a spectacular event - is now routine matter, due to the fact that his ingenious ideas have meanwhile become standard facts of contemporary analysis. An optimal framework for his ideas - neither too abstract nor too concrete - seems to be harmonic analysis on compact Abelian groups. A sequence x_n from a compact Abelian group with countable base is called uniformly distributed in G if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} f(x_n) = \lambda(f)$$

for all $f \in C(G)$, where λ denotes (normalized) Haar measure on G . It is called well-distributed in G if even

$$\lim_{N\to\infty} \sup_{k} \left| \frac{1}{N} \sum_{n\leq N} f(x_{n+k}) - \lambda(f)\right| = 0 ,$$

holds.

Choosing $G = T = R/Z$, the one-dimensional torus group, which may be visualized as the interval $[0,1]$ (the end points being identified) and $x_n = n\theta$, we get the special case mentioned in the beginning. I shall give a rather complicated proof of this simple fact which has the advantage that it can be generalized to yield deeper results. Let me begin with a well-known ergodic theorem : assume X is a compact Hansdorff space with countable base, $S : X\to X$ a continuous map and μ a probability measure on X such that $\mu(f\circ S) = \mu(f)$ for all $f \in C(X)$. Assume further that μ is the only such probability measure. Then

$$\lim_{N\to\infty} \sup_{x\in X} \left|\frac{1}{N} \sum_{n\leq N} f(S^n x) - \mu(f)\right| = 0$$

holds for all $f \in C(X)$.

To prove this, we observe that if this relation would not hold there would exist $\varepsilon_0 > 0$, $f_0 \in C(X)$ and infinitely many $N' \in Z^+$ such that

$$(*) \qquad \left|\frac{1}{N'}\sum_{n\le N'} f_o(S^n x_{N'}) - \mu(f_o)\right| \ge \varepsilon_o$$

for suitable $x_{N'} \in X$. Define now Radon measures $\mu_{N'}$ by $\mu_{N'}(f) = \frac{1}{N'}\sum_{n\le N'} f(S^n x_{N'})$ for all $f \in C(X)$.

By the separability of $C(X)$ there exists a subsequence $\{\mu_{N'_k}\}$ of the sequence $\{\mu_{N'}\}$ such that $\mu_{N'_k}(f)$ converges for every $f \in C(X)$. Denote this limit by $\nu(f)$. Then obviously ν is a probability measure on X such that $\nu(f\circ S) = \nu(f)$. But by assumption this implies $\nu = \mu$, which contradicts $(*)$.

H. Weyl's result is a special case of this ergodic theorem. It suffices to choose $X = T$, $\mu = \lambda$, $Sx = x+\theta$, and to observe that every S-invariant measure is invariant with respect to all translations and coincides therefore with Haar measure λ.

The same method can be applied to the sequence

$$\{(p(n), p(n+1), \dots, p(n+k-1))\}$$

in T^k, where $p(x) = a_o x^k + \dots + a_k$ is a polynomial such that $a_o = \theta$ is irrational.

In this case define $S : T^k \to T^k$ by

$$S(x_o, x_1, \dots, x_{k-1}) = \left(x_1, x_2, \dots, \sum_{j=1}^{k} \binom{k}{j} (-1)^{j-1} x_{k-j} + k!\theta\right)$$

and $\mu = \lambda$, Haar measure on T^k.

Using the mean ergodic theorem it can easily be shown (cf. [5]) that λ is the unique S-invariant probability measure on T^k. Now we have

$$S^n(p(0), p(1), \dots, p(k-1)) = (p(n), p(n+1), \dots, p(n+k-1))$$

We get therefore as special case H. Weyl's result, that the sequence $\{p(n)\}$ is well-distributed mod. 1.

It is interesting to observe that the ideas underlying these proofs can be extended to yield a useful characterization of well-distributed sequences. Let again G be a compact Abelian group with countable base. Let $\Omega = \prod_{n=1}^{\infty} G_n$, $G_n = G$, be the product space. The elements $\omega \in \Omega$ may be identified with infinite sequences $\omega = \{x_n\}$. Define now a continuous map $S : \Omega \to \Omega$ by $S\{x_n\} = \{x_{n+1}\}$. For $\omega_o = \{x_n^o\} \in \Omega$ let X_{ω_o} be the closure of the set $\{\omega_o, S\omega_o, S^2\omega_o, \dots\}$ in Ω. Then X_{ω_o} too is a compact Hausdorff space with countable base and S is a continuous map on X_{ω_o}. We shall suppose that ω_o is dense in G. Define now π: $X_{\omega_o} \to G$ by $\pi\{x_n\} = x_1^o$. Then obviously π is continuous. Consider now the functions in $C(X_{\omega_o})$ of the form $f'(\omega) = f(\pi(\omega))$, $f \in C(G)$. These functions form a subalgebra A of $C(X_{\omega_o})$ isomorphic to $C(G)$ (via the map $f \to f\circ\pi$).

Let $\mathcal{J}_S(X_{\omega_o})$ be the set of all S-invariant probability measures on X_{ω_o} and let $M(G)$ be the set of all probability measures on G. Define $\hat{\pi} : \mathcal{J}_S(X_{\omega_o}) \to M(G)$ by $(\hat{\pi}\mu)(f) = \mu(f \circ \pi)$ for all $f \in C(G)$, i.e. let $\hat{\pi}\mu$ be the restriction of μ to A. Then we have the.

THEOREM. <u>The following conditions are equivalent</u> :

1) $\hat{\pi}(\mathcal{J}_S(X_{\omega_o})) = \{\lambda\}$

2) <u>Every</u> $\omega \in X_{\omega_o}$ <u>is uniformly distributed in</u> G.

3) <u>For all</u> $f \in C(G)$

$$\lim_{N\to\infty} \sup_{\omega \in X_{\omega_o}} \left|\frac{1}{N} \sum_{n\le N} f(\pi(S^n \omega)) - \lambda(f)\right| = 0 .$$

4) ω_o <u>is well-distributed in</u> G.

<u>The proof may be found in</u> [3] .

<u>Some applications</u> :

1) Let $a > 1$ be an algebraic integer of degree $k \geq 1$. Then the set of all $x \in T$, such that $\{a^n x\}$ is well-distributed, is a null-set.

Let us restate this result in a more illuminating way :

a) For almost all x the sequence $\{(a^n x, a^{n+1}x, \ldots, a^{n+k-1}x)\}$ is uniformly distributed in T^k.

b) If $(0,0,\ldots,0)$ is a cluster point of the sequence $\{(a^n x, \ldots, a^{n+k-1}x)\}$, then the sequence $\{a^n x\}$ cannot be well-distributed.

a) and b) together yield our result. An unsolved problem is whether there exist well-distributed sequences of this form at all. In this connection it would be interesting to know whether the uniform distribution of the sequence $\{a^n x\}$ implies that $(0,0,\ldots,0)$ is a cluster point of the sequence $\{(a^n x, \ldots, a^{n+k-1}x)\}$ or not (for $k \geq 2$).

Because a) is a well-known result (cf. e.g. [1]) let us only prove b) : Let $\omega_o = (x, ax, a^2x, \ldots)$. Define $\varphi : X_{\omega_o} \to T^k$ by $\varphi(\{x_n\}) = (x_1, x_2, \ldots, x_{k-1})$. Let $a^k = h_1 a^{k-1} + \ldots + h_k$, $h_i \in Z$, and define $S : T^k \to T^k$ by

$$S\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k-1} \\ x_k \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \ldots 0 \\ 0 & 0 & 1 \ldots 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & \ldots\ldots 1 \\ h_k & h_{k-1} & \ldots\ldots h_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k-1} \\ x_k \end{pmatrix}$$

Then $S^n(x, ax, \ldots, a^{k-1}x) = (a^n x, \ldots, a^{n+k-1}x)$. Therefore every $\omega \in X_{\omega_o}$ is uniquely determined by its k first terms and φ defines a bijection from X_{ω_o} to some compact subset of T^k. If ω_o would be well-distributed in T, then by 3) the same would be true of $\omega \equiv \{0\}$, a contradiction.

2) A sequence $\omega \in \Pi G_n$ is called completely uniformly distributed in G if $\{S^n\omega\}$ is uniformly distributed in Ω. (If $G = Z/_{kZ}$ and $\omega = \{x_n\}$, then ω is completely uniformly distributed if and only if

$$x = \sum_1^\infty \frac{x_n}{k^n}$$

is normal to base k).

The individual ergodic theorem gives at once that almost all $\omega \in \Omega$ are completely uniformly distributed in G. Now we shall show that a completely uniformly distributed sequence cannot be well-distributed. Therefore almost no sequence is well-distributed.

Assume ω_o is completely uniformly distributed. This implies first of all that $X_{\omega_o} = \Omega$ because $\{S^n \omega_o\}$ is then dense in Ω. Let μ be any probability measure on G, then the product measure $\Pi \mu_n$, $\mu_n \equiv \mu$, is an S-invariant probability measure on Ω. Thus $\hat{\pi}(\mathcal{J}_S(X_{\omega_o})) = M(G) \neq \{\lambda\}$ if $G \neq \{0\}$. Thus condition 1) of the theorem does not hold and therefore ω_o is not well-distributed.

3) Let $Q_n \geq 1$ be a bounded sequence of integers and $\omega_o = \{Q_1 Q_2 \ldots Q_n x\}$ a sequence in T. Then ω_o is not well-distributed.

Assume that ω_o would be well-distributed, then a fortiori it would be dense in T. Let n_k be an increasing sequence such that $Q_1 Q_2 \ldots Q_{n_k} x \to 0 \pmod 1$. Then also for every fixed h we have $Q_1 \ldots Q_{n_k} \ldots Q_{n_k+h}\, x \to 0$, which implies $(0,0,0,\ldots) \in X_{\omega_o}$, a contradiction.

4) Let $G = T$ and let $\omega_o = \{f(n)\}$ be a uniformly distributed sequence such that $\lim\limits_{n\to\infty} (f(n+h) - f(n)) = 0$ for every $h = 1,2,3,\ldots$. Then ω_o is not well-distributed.

Assume ω_o would be well-distributed ; there would exist a subsequence such that $\lim f(n_k) = 0$. But then also $\lim f(n_k+h) = 0$, thus $(0,0,0,\dots) \in X_{\omega_o}$, a contradiction.

This example can somewhat be improved : Call a sequence $\{f(n)\}$ of real numbers tempered if f is an $(\ell+1)$ -times continuously differentiable real valued function on $[1,\infty]$ such that $f^{(\ell+1)}$ decreases to 0 and

$$\lim_{t\to\infty} f^{(\ell)}(t) = \lim_{t\to\infty} t\, f^{(\ell+1)}(t) = \infty .$$

(Here ℓ denotes some non negative integer). It is well known, that every tempered sequence is uniformly distributed mod. 1 . More precisely one can show that the sequence $\{(f(n),f^1(n) ,\dots, f^{(\ell)}(n))\}$ is uniformly distributed in $T^{\ell+1}$. From this we may conclude in the same way as before, that no tempered sequence is well-distributed mod 1.

Further information and references to the literature on this subject can be found in the following papers :

[1] Der individuelle Ergodensatz in der Theorie der Gleichverteilung mod 1 , J. Reine angew. Math. 205, (1960), 91-100.

[2] Einige Fragen der Theorie der Gleichverteilung, Colloquium "Gelijkverdeling" , 55-70, Math. Centrum 1969.

[3] A characterization of well-distributed sequences, Compositio Math. 17, (1967), 263-267.

[4] Some remarks on the distribution mod 1 of tempered sequences, Nieuw Archief voor Wiskunde 16 (1968), 194-196.

[5] On a theorem of H. Weyl, Compositio Math. 21 (1969), 151-154.

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