W. NARKIEWICZ

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SOME UNSOLVED PROBLEMS

by

W. NARKIEWICZ

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In this paper, which is a slightly extended version of my talk at the Bordeaux Number Theory Colloquium, I shall present some unsolved problems in number theory. The first group of those concerns the theory of factorization in algebraic number fields, whereas the second deals with some questions from other branches of the number theory.

1. Let K be an algebraic number field. An integer in K is called irreducible, if it cannot be decomposed into two factors, neither of them a unit in K. Clearly every integer generating a prime ideal is irreducible, but the converse holds only if the class-number $h(K)$ of the field in question equals unity. Denote by $A(x)$ the number of non-associated irreducible integers of K with their norms not exceeding x in absolute value.

P. Rémond ([14]) proved the following asymptotics for $A(x)$:

$$A(x) \sim c(K)x(\log\log x)^{t-1}(\log x)^{-1}$$

where $t \geq 1$ is a constant, depending only on the class-group $H(K)$ of the field K. It is easy to see that t is the maximal number of prime ideals occuring in the decomposition of an irreducible integer in K, and so t equals the minimal number s with the propery that every system of s elements from $H(K)$ has a subsystem whose product equals 1 . H. Davenport proposed the following

PROBLEM 1. Evaluate $t = t(H(K))$.

It has been conjectured by J.E. Olson ([13]) that if the group H is a product of cyclic groups $C_{n_1},\dots,C_{n_k}$ with $n_1|n_2|\dots|n_k$, then $t(H) = 1 + \sum_{i=1}^{k} (n_i-1)$. This conjecture is trivial for H cyclic, and for abelian p-groups was established by J.E. Olson ([13]). Professor Wirsing kindly informed me that the same result was obtained independently by S. Schanuel (unpublished).

Although this problem has also a meaning for non-abelian groups, it seems that this case is entirely untouched.

2. Now let $B(x)$ be the number of non-associated integers in K with their norms not exceeding x in absolute value, which have unique factorization in K. E. Fogels considered in [3] the field $K = Q((-5)^{1/2})$ with $h = 2$ and proved that in this case $B(x) = o(x)$. I proved later in [4] that a similar result holds for

all fields with $h \neq 1$ (in the case $h = 1$ obviously $B(x) \sim cx$), more precisely, one hase

$$B(x) = O(x(\log\log x)^{a}(\log x)^{-b})$$

with some positive b, and nonnegative a. It can be shown, although it was never published, that the best possible value of b equals $1 - h^{-1}$.

PROBLEM 2. Prove that

$$B(x) \sim d(K)x(\log\log x)^{t}(\log x)^{h^{-1}-1}$$

where $d(K)$ is a positive constant, and the exponent t is the same as in problem 1.

Similarly we may consider the number $C(x)$ of positive rational integers not exceeding x, having unique factorization in K. For the field $K = Q((-5)^{1/2})$ E. Fogels [3] obtained

$$C(x) = O(x(\log\log x)^{4/5}(\log x)^{-1/5})$$

and a similar result (with exponents depending on the field) I proved in [4] for all normal fields (with $h \neq 1$). It can be shown that the best possible value of the exponent of log x equals $(h-1)/hn$ with $n = [K:Q]$.

PROBLEM 3. Prove that for all fields with $h \neq 1$ (not necessarily normal) one has $C(x) = o(x)$, and find the precise asymptotics.

The precise asymptotics is known only in the case of K quadratic. (See [8]).

In [8] it was shown that the natural numbers with unique factorization in a given quadratic field are evenly distributed among the relatively prime residue classes mod N, provided that N is relatively prime to the discriminant of the field. The restriction on N was later ([9]) removed.

PROBLEM 4. Prove the same result for arbitrary normal extensions of Q.

3. L. Carlitz proved in [1] that the class-number of K does not exceed 2 if and only if all factorizations of any given integer in K are of the same length (i.e. have the same number of factors). Evaluations for the number of nonassociated integers in a field with $h \neq 1,2$ which have all factorizations of the same length, and also of natural numbers with the same property, whose norms do not exceed x in absolute value, were given in [4] and [5]. In [8] it was proved that if D has all its factorizations of the same length, and K is a quadratic field with $h \neq 1,2$, then the number of positive rational integers lying in a given residue class $m \pmod n$ with $(m,n) = D$, having all factorizations of the same

length and not exceeding x equals asymptotically

$$c(n,D,K)x(\log\log x)^{a}(\log x)^{-b}$$

with $b = (h-1-c(D))/2h$, where $0 \leq c(D) \leq g$, and g is the number of even invariants of the class-group. For $D = 1$ we have $c(D) = g$ and I conjectured that $c(D) = g$ always. This was shown to be false by A. Schinzel and later I showed [10] that $c(D) = g$ holds for all D and a fixed field K if and only if either $h(K)$ is odd or $H(K) = C_2 \times C_2 \times \ldots \times C_2$.

PROBLEM 5. Determine the precise value of $c(D)$ for all quadratic fields.

Carlitz's proof uses the fact that every ideal class contains prime ideals, which may be not true in other Dedekind domains. In fact, from one result of L. Claborn ([2]) one can obtain an example of a Dedekind domain in which every element has factorizations of the same length, but the class-group has the form $C_2 \times C_2$. It is easy to see that every such example must have its class-group of the form $C_2 \times \ldots \times C_2$ (provided it is finite), but the converse does not obviously hold. Thus we have.

PROBLEM 6. Characterize Dedekind domains in which every element has all its factorizations of the same length.

4. The following problem is due to P. Turán :

PROBLEM 7. Let K be any algebraic number field with $h \neq 1$, and let $f(n)$ be the number of factorizations of n in K . Is it possible to find a nondecreasing normal order for $f(n)$, i.e. a non-decreasing function $F(n)$ such that fo every positive ε and almost all n the inequality $|f(n) - F(n)| < \varepsilon F(n)$ holds ?

I proved in [12] that if K is quadratic with $h = 2$ then such a function $F(n)$ does not exist. The general case is still open. A similar problem may be posed for the function $g(n)$ counting the number of factorizations of different length of n . In the same paper I showed that for quadratic fields with $h = 2$ and for quartic fields with $H = C_2 \times C_2$ such a function $F(n)$ exists and equals $c \log\log n$ with $c = 1/9$ in the first case and $c = 1/8$ in the second.

5. Now some problems which are not connected with factorizations :

PROBLEM 8. (J. Browkin, unpublished). Let K be an algebraic number field such that one can find a sequence of integers in K , say $a_1, a_2, \ldots$ such that for every ideal I in the ring of integers of K all residue classes (mod I) are represented by the sequence

$$a_1, \dots, a_{N(I)} .$$

Prove that K is the field of rational numbers.

It is easy to see that such a sequence cannot exist in an imaginary quadratic field with a discriminant large enough in absolute value. An elementary solution of this problem for all quadratic fields was recently obtained by B. Wańtuła in his M.A. thesis at Wroclaw University (unpublished). The general case seems to be quite difficult.

One can ask also for a sequence $a_1, a_2, \dots$ which has the property stated in problem 8 for all prime ideals of the field. In this case nothing seems to be known.

6. The next two problems concern transformations by polynomials.

PROBLEM 9. Let A be a closed curve in the complex plane, and let $P(z)$ be a polynomial with the property $P(A) = A$. Prove that either A is a circle, or $P(z) = \alpha z$.

Now let K be any field. We shall say that K has the property (P) if for every infinite subset A of K and for every polynomial $P(t)$ over K the equality $P(A) = A$ implies the linearity of $P(t)$. I proved in [6] that every finitely generated extension of the rationals has property (P).

PROBLEM 10. (i) Let K be an algebraic number field of infinite degree generated by a subset with all its elements of bounded degree. Prove that K has the property (P).

(ii) Let K a function field in finite number of variables over a finite field. Prove that K has the property (P).

7. We conclude with two elementary problems.

PROBLEM 11. For which N the values of the function $\sigma(n)$, giving the sum of divisors of n, are equally distributed among the residue classes (mod N), relatively prime to N?

A similar question concerning the number of divisors and the Euler's function was answered in [11] where also a more general necessary and sufficient condition was obtained, which is however very hard to apply in the case of $\sigma(n)$.

In [7] I proved the following result concerning the unitary convolution of arithmetical functions :

If $\sum_{n=1}^{\infty} |f(n)|$ converges, and g denotes the inverse of f under the unitary convolution (if it exists), then the series $\sum_{n=1}^{\infty} |g(n)|$ is also convergent.

We recall that the unitary convolution of two functions $a(n)$ and $b(n)$ is defined by

$$\sum_{\substack{d|n \\ (d,n/d)=1}} a(d)\, b(n/d)$$

and the unit element of the ring so obtained equals the function $\varepsilon(n)$ which is $= 0$ for $n \neq 1$ and $\varepsilon(1) = 1$.

The proof of the mentioned result is non-elementary, as it uses the theory of normed rings.

PROBLEM 12. Find an elementary, direct proof of the result quoted.

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Wrocław 14
Sudecka 112/3
(Pologne)