ESAIM: M2AN 47 (2013) 169–181
DOI: 10.1051/m2an/2012024

**ESAIM: Mathematical Modelling and Numerical Analysis**
www.esaim-m2an.org

# THE DISCRETE COMPACTNESS PROPERTY FOR ANISOTROPIC EDGE ELEMENTS ON POLYHEDRAL DOMAINS *

Ariel Luis Lombardi[1,2]

**Abstract.** We prove the discrete compactness property of the edge elements of any order on a class of anisotropically refined meshes on polyhedral domains. The meshes, made up of tetrahedra, have been introduced in [Th. Apel and S. Nicaise, *Math. Meth. Appl. Sci.* **21** (1998) 519–549]. They are appropriately graded near singular corners and edges of the polyhedron.

**Mathematics Subject Classification.** 65N30.

Received July 28, 2011. Revised February 29, 2012.
Published online August 31, 2012.

## 1. Introduction

In the theoretical analysis of finite element methods for Maxwell's equations we can distinguish two basic problems. The first one is to compute the eigenvalues (or resonant frequencies) of a bounded cavity. The second one is to compute the electromagnetic field in the cavity due to a known current source (at a nonresonant frequency). Edge finite elements have been used to approximate both problems, and the convergence was studied in several papers. The discrete compactness property is a useful tool for this analysis.

Assuming that $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz polyhedral domain, with boundary $\partial\Omega$ and unit outward normal $\mathbf{n}$, the eigenvalue problem is to find an electric field $E \neq 0$ and an electric eigenvalue $\lambda$ such that

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,E &= \lambda E && \text{in } \Omega,\\ \operatorname{div} E &= 0 && \text{in } \Omega,\\ \mathbf{n}\times E &= 0 && \text{on } \partial\Omega. \end{aligned}$$

On the other hand, given a wave number $k > 0$ of the time harmonic field in $\Omega$ such that $k^2$ is not an electric eigenvalue of $\Omega$, and a divergence free current distribution $J$, the source problem is to find the electric field $E$

*Keywords and phrases.* Discrete compactness property, edge elements, anisotropic finite elements, Maxwell equations.

* *Supported by ANPCyT (grant PICT 2010-1675 and PICTO 2008-00089) and by CONICET (grant PIP 11220090100625).*

[1] Instituto de Ciencias, Universidad Nacional de General Sarmiento, J.M. Gutierrez 1150, Los Polvorines, B1613GSX Provincia de Buenos Airesn, Argentina. aldoc7@dm.uba.ar

[2] Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, 1428 Buenos Aires, Member of CONICET, Argentina

which satisfies

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,E - k^2 E = -J \qquad & \text{in } \Omega, \\ \operatorname{div} E = 0 \qquad & \text{in } \Omega, \\ \mathbf{n} \times E = 0 \qquad & \text{on } \partial\Omega. \end{aligned}$$

The theory of both problems is well studied in the literature, see for example [11] and references therein. Also, the numerical approximation was considered in several articles (see [3,8,13] and their references). Let us mention a few of them.

In [9] the approximation of the eigenvalue problem by edge elements was considered. There, the discrete compactness property was first introduced, and proved, for elements of lowest degree on shape-regular tetrahedral meshes.

In [6] the authors proved, using and inductive approach, the validity of the discrete compactness property for tetrahedral edge elements (actually, for Nédélec elements of first and second class) of any order on regular triangulations of a polyhedron. Using this result, they have obtained the convergence of the corresponding approximations of the Maxwell eigenvalue problem. Precisely, they proved that the edge elements give spurious-free approximations of the eigenvalue problem, in the sense of [5]. We point out that in [6] a quite general setting is considered, including anisotropic and discontinuous materials and mixed boundary conditions.

Also in [2], for the eigenvalue problem (with a different boundary condition), the good approximation properties of edge elements of any order are shown on regular tetrahedral meshes. There, the author consider a mixed method, which is equivalent, in the present situation, to the primal method of [6] (see [3], Sect. 5). The discrete compactness property (in the form considered here) is then indirectly established.

In [14], the approximation of Maxwell eigenvalue and source problems by edge elements of any order on tetrahedral and hexahedral meshes was analyzed following the theory of collectively compact operators. The discrete compactness property of the approximation spaces is used there to verify that indeed that theory can be applied. A weakly quasi-uniform assumption on the meshes is made in this paper, besides the standard shape regularity hypothesis. We also refer to [13], Chapter 7, where improvements and extensions of [14] are presented.

We observe that narrow elements and anisotropically refined meshes are excluded in the analysis of the mentioned papers. However, the validity of the discrete compactness property on meshes adapted to edge or corner singularities was also considered. In [4] the discrete compactness property was obtained for edge elements on suitably refined meshes. This property combined with interpolation results was used to prove optimal algebraic convergence for both the source and eigenvalue problems. The results hold for elements of arbitrary order on hexahedral meshes, but in the case of tetrahedral meshes the analysis is limited to the lowest order case.

The problem was also considered in [16]. There, edge spaces of the lowest order on a polyhedral domain with an edge were considered on meshes which are obtained from an appropriated refined tensor product pentahedral mesh, by dividing each pentahedron into three tetrahedra. Corner singularities were not considered.

The goal of this paper is to prove that the discrete compactness property holds for tetrahedral edge elements of any order, on anisotropically refined meshes on a general Lipschitz polyhedral domain. We consider edge and corner refinements. More precisely, our meshes are proposed in order to be able to adequately approximate a homogeneous Dirichlet problem for the Laplace operator with a right hand side in $L^p$ for some $p > 2$. These meshes were designed in [1]. So, the results contained in that article become fundamental for our approach.

The meshes considered in [4] satisfy our requirements. Therefore, we are extending the results of [4] to the case of tetrahedral meshes and elements of any order.

Our analysis has similarities with the ones developed in [4, 16]. In particular, the key ingredients of our approach are:

- Suitable decompositions of certain vector fields in $H_0(\mathbf{curl}\,, \Omega)$.
- Interpolation error estimates for edge elements of any order on anisotropic meshes satisfying the maximum angle condition [12].

- Control by below of the volume of the elements of the mesh in terms of the mesh-size parameter.
- Accurate error estimates for a continuous piecewise polynomial interpolation of the $H_0^1(\Omega)$-solution of the scalar Laplace equation with right hand side in $L^p$ [1].

The last point deserves a further comment. In our analysis we use known estimates for Lagrange interpolation on anisotropic meshes, and the major assumptions on the meshes are in order to ensure that such results indeed hold true. However, the approach allows to replace Lagrange by other scalar piecewise polynomials interpolations, changing eventually the mesh restrictions. This issue, which is a difference with the theory of [4], may be further analyzed in the future. In [4] the scalar interpolation that commutes with the edge interpolation in the De Rham diagram needed to be used.

In Section 2 we introduce the notation, definitions and preliminary results used in the paper. In Section 3 the assumptions on the meshes are introduced. Further specifications on the parameters introduced in this Section are given later. In Section 4 auxiliary propositions that are used to prove the main result are established. Finally, in Section 5 the discrete compactness property of the finite element spaces introduced in Section 2 is proved.

## 2. Notation and preliminaries

We assume that the domain $\Omega \subset \mathbb{R}^3$ is a Lipschitz polyhedron. We consider the space $H(\mathbf{curl}\,,\Omega)$ of vector fields in $[L^2(\Omega)]^3$ with $\mathbf{curl}$ in $[L^2(\Omega)]^3$. It is a Hilbert space with the scalar product

$$\langle \mathbf{v}, \mathbf{w}\rangle = (\mathbf{v}, \mathbf{w}) + (\mathbf{curl}\,\mathbf{v}, \mathbf{curl}\,\mathbf{w}), \qquad \mathbf{v}, \mathbf{w} \in H(\mathbf{curl}\,,\Omega),$$

and the norm $\|\cdot\|_{H(\mathbf{curl}\,,\Omega)}$ induced by this product. Here, $(\cdot,\cdot)$ indicates the usual $L^2(\Omega)$ scalar product. The set of functions $\mathbf{v} \in H(\mathbf{curl}\,,\Omega)$ such that $\mathbf{v}\times\mathbf{n} = 0$ on $\partial\Omega$ is denoted by $H_0(\mathbf{curl}\,,\Omega)$, where $\mathbf{n}$ is the unitary exterior normal to $\partial\Omega$.

It is known that the embedding of $H_0(\mathbf{curl}\,,\Omega)$ into $[L^2(\Omega)]^3$ is not compact, but one obtains a compact embedding if $H_0(\mathbf{curl}\,,\Omega)$ is replaced by its subspace $X$ defined by

$$X = \{\mathbf{v} \in H_0(\mathbf{curl}\,,\Omega) : \operatorname{div}\mathbf{v} = 0 \text{ on } \Omega\}\,.$$

In order to introduce the discrete spaces, we assume that a family of conforming meshes $\{\mathcal{T}_h : h \in \mathrm{I}\}$ made up of tetrahedra is given on $\Omega$. We assume, as usual, that $h$ denotes the mesh-size parameter and I is a denumerable and bounded set of positive numbers having zero as the only limit point. From now on, $h$ will always denote an element of I (or some subset J of I). Further conditions on the meshes shall be assumed in Section 3.

Let $k \geq 1$ be a natural number (fixed along the paper), and let $K \subset \mathbb{R}^3$ be a tetrahedron. Denote by $P_l(K)$ the set of polynomials on $K$ of degree less than or equal $l$, and by $\widetilde{P}_l(K)$ its subspace of homogeneous polynomials.

The first Nédélec family of edge elements [15] on $K$ of degree $k$, $\mathcal{N}_k(K)$, is the subspace of $[P_k(K)]^3$ given by (see for example [7])

$$\mathcal{N}_k(K) = [P_{k-1}(K)]^3 \oplus \mathcal{J}_k(K)$$

where

$$\mathcal{J}_k(K) = \left\{\mathbf{p} \in [\widetilde{P}_k(K)]^3 : \mathbf{p}(\mathbf{x})\cdot\mathbf{x} \equiv 0\right\}.$$

It is not difficult to check [12] that

$$\mathcal{N}_k(K) = [P_{k-1}(K)]^3 \oplus [\widetilde{P}_{k-1}(K)]^3 \times \mathbf{x}.$$

Now the space of edge elements, $V_h$, is

$$V_h = \{\mathbf{v}_h \in H_0(\mathbf{curl}\,,\Omega) : \mathbf{v}_h|_K \in \mathcal{N}_k(K), \forall K \in \mathcal{T}_h\}\,.$$

The divergence free condition of the functions in $X$ can not be forced into the discrete spaces, but only imitated in the form of "discrete divergence free condition": the discrete counterpart of $X$ is $X_h$ defined as

$$X_h = \{\mathbf{v}_h \in V_h : (\nabla p_h, \mathbf{v}_h) = 0, \forall p_h \in S_h\},$$

where

$$S_h = \left\{p_h \in H^1_0(\Omega) : p_h|_K \in \mathcal{P}_k(K), \forall K \in \mathcal{T}_h\right\}.$$

Then we have $\nabla S_h \subset V_h$ and (see, for example, Cor. 5.1 of [7])

$$V_h = \nabla S_h \oplus X_h.$$

A question that naturally arises is whether, and in which sense, the spaces $X_h$ inherit the compactness property of the space $X$. This question can be treated in terms of the following definition. Here, J denotes an arbitrary denumerable subset of I.

**Definition 2.1.** We say that the family of spaces $\{X_h\}_{h\in \mathrm{I}}$ satisfies the "discrete compactness property" if for each sequence $\{\mathbf{v}_h\}_{h\in \mathrm{J}}$, $\mathrm{J} \subseteq \mathrm{I}$, verifying

$$\mathbf{v}_h \in X_h, \qquad \forall h \in \mathrm{J}, \tag{2.1}$$

$$\|\mathbf{v}_h\|_{H_0(\mathbf{curl}\,,\Omega)} \leq C, \quad \forall h \in \mathrm{J}, \tag{2.2}$$

there exists a function $\mathbf{v} \in X$ and a subsequence $\{\mathbf{v}_{h_n}\}_{n\in\mathbb{N}}$ such that (for $n \to \infty$)

$$\mathbf{v}_{h_n} \to \mathbf{v} \quad \text{in } L^2(\Omega) \tag{2.3}$$

$$\mathbf{v}_{h_n} \rightharpoonup \mathbf{v} \quad \text{weakly in } H_0(\mathbf{curl}\,,\Omega). \tag{2.4}$$

Finally, we need to discuss the regularity of the solution of the scalar Laplace equation

$$-\Delta u = f, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega,$$

with $f \in L^p(\Omega)$, $p > 2$, remembering that $\Omega$ is a Lipschitz polyhedron. This regularity will be used to define the meshes, in Section 3, in order to be able to estimate the $H^1$-seminorm of the linear Lagrange interpolation error for a solution of this problem, for values of $p$ close to 2, in a satisfactory way. We follow the article [1].

Let $S$ be a corner of $\Omega$. Let $C_S$ be the infinite polyhedral cone that coincides with $\Omega$ in a neighborhood of $S$. Define $G_S = C_S \cap \mathcal{S}^2(S)$, where $\mathcal{S}^2(S)$ is the unit sphere centered at $S$. Then, the vertex singular exponent related to $S$ is given by $\lambda_{v,S} = -\frac{1}{2} + \sqrt{\lambda_{S,1} + \frac{1}{4}}$, where $\lambda_{S,k} > 0, k = 1, \ldots$, are the eigenvalues, in increasing order, of the Laplace-Beltrami operator on $G_S$ with Dirichlet boundary conditions. Note that $\lambda_{v,S} > 0$. We say that the vertex $S$ is $p$-singular if $\lambda_{v,S} < 2 - \frac{3}{p}$.

Now, let $A$ be an edge of $\Omega$. The edge singular exponent related to $A$ is $\lambda_{e,A} = \pi/\omega_A$, with $\omega_A$ being the angle between the two faces containing $A$. Note that $\lambda_{e,A} > \frac{1}{2}$. We say that $A$ is $p$-singular if $\lambda_{e,A} < 2 - \frac{2}{p}$.

If a vertex or edge is not $p$-singular, we say that it is $p$-regular.

We note that the singular exponents are independent of $p$, and that if a corner or an edge is $\bar{p}$-singular, then they are also $p$-singular for all $p > \bar{p}$. We make the following definition.

**Definition 2.2.** A corner (resp. an edge) of $\Omega$ is singular if it is $p$-singular for some $p \leq 2$. In the other case, we say that the corner (resp. the edge) is regular.

It follows that we can decompose the set $\mathcal{C}$ of the corners of $\Omega$ into two disjoint subsets $\mathcal{C}_s$ and $\mathcal{C}_r$ containing the singular and regular corners, respectively. A similar decomposition $\mathcal{E} = \mathcal{E}_s \cup \mathcal{E}_r$ is done for the set $\mathcal{E}$ of edges of $\Omega$. Then there exists a number $p_* > 2$ such that

$$S \in \mathcal{C}_r \quad \Rightarrow \quad S \text{ is } p\text{-regular } \forall p < p_*,$$

$$A \in \mathcal{E}_r \quad \Rightarrow \quad A \text{ is } p\text{-regular } \forall p < p_*.$$

This number $p_*$ (that is not unique) will be kept fixed throughout the whole paper.

## 3. Graded meshes

In this section we enumerate the hypotheses on the meshes which we need to assume in order to obtain the main result. For each value of the global mesh parameter $h \in \mathrm{I}$, we consider a conforming subdivision $\mathcal{T}_h$ of the polyhedron $\Omega$ made up of tetrahedra such that for a constant $C$ independent of $h$ it holds

$$h_K \leq Ch \qquad \forall K \in \mathcal{T}_h,$$

where $h_K$ denotes de diameter of the element $K$.

### 3.1. The maximum angle condition

We assume that there exists a constant $\bar{\psi}$ such that for all $h \in \mathrm{I}$ and each element $K \in \mathcal{T}_h$ the following property holds: the maximum angle between faces of $K$ and the maximum angle inside the faces of $K$, are less than or equal $\bar{\psi}$. We denote this property of the family of meshes $\mathcal{T}_h$ by $MAC(\bar{\psi})$.

The maximum angle condition for tetrahedral meshes was first introduced in [10], as a generalization of the Synge's condition for triangles. Under this condition, uniform and anisotropic error estimates have been obtained for different interpolation operators. In particular, we will use the results obtained for Lagrange interpolation in [1] and for Nédélec interpolation in [12].

### 3.2. Number of elements and their sizes

The number $N_{el,h}$ of elements in $\mathcal{T}_h$ is related to the global mesh parameter $h$ by

$$N_{el,h} \leq Ch^{-3}, \qquad \forall h \in \mathrm{I},$$

with $C$ independent of $h$. That is, the number of elements in the mesh $\mathcal{T}_h$ is comparable with the one for a quasiuniform mesh with elements of size $\sim h$.

Furthermore, we need to have a control on the size of the smallest element on $\mathcal{T}_h$. So we assume that for a number $\sigma > 0$ independent of $h$ we have

$$|K| \geq Ch^{\sigma}, \qquad \forall K \in \mathcal{T}_h, h \in \mathrm{I}, \tag{3.5}$$

with $C$ independent of $h$.

### 3.3. Grading

We will use the finite element estimates for the Poisson equation with homogeneous Dirichlet conditions on polyhedral domains obtained in [1]. So we need to introduce the kind of meshes used there. These meshes have a particular refinement near some corners or edges of the domain, and those refinements are associated with a parameter, $\mu_\ell$ for edges and $\nu_\ell$ for corners. Here we describe the meshes in terms of these abstract parameters, and in the next section we specify them.

Let us introduce a decomposition $\bar{\Omega} = \cup_{\ell=1}^{L} \bar{\Lambda}_\ell$ of the polyhedron $\Omega$ into $L$ tetrahedral subdomains $\Lambda_\ell$, such that each subdomain contain at most one singular edge and at most one singular corner. On each subdomain $\Lambda_\ell$ is defined a Cartesian coordinate system $(x_1^\ell, x_2^\ell, x_3^\ell)$ such that one vertex of $\Lambda_\ell$ is located in the origin and one edge of $\Lambda_\ell$ is contained on the $x_3^\ell$-axis. These vertex or edge coincide with the singular ones, if $\Lambda_\ell$ possesses them. Also we introduce for each $\ell$, refinement parameters $\mu_\ell, \nu_\ell \in (0, 1]$, that will be specified in the next section. Here only the condition $\mu_\ell \leq \nu_\ell$ if $\mu_\ell < 1$ is imposed.

We assume that the partitions $\mathcal{T}_h$ fit the subdivisions $\{\Lambda_\ell\}$ for all $h \in \mathrm{I}$. Now, let $K \in \mathcal{T}_h$ fixed. Then there exists a unique $\ell$ such that $K \subseteq \Lambda_\ell$. Then we define:

$$r_K = \inf_{x \in K} [(x_1^\ell)^2 + (x_2^\ell)^2]^{\frac{1}{2}}$$
$$R_K = \inf_{x \in K} [(x_1^\ell)^2 + (x_2^\ell)^2 + (x_3^\ell)^2]^{\frac{1}{2}},$$

that is, the distances from $K$ to the $x_3^\ell$-axis and the origin in the $\ell$ local cartesian system. Consider the following size parameters:

$$\zeta_{K,e} = \begin{cases} h^{\frac{1}{\mu_\ell}} & \text{if } r_K = 0, \\ h r_K^{1-\mu_\ell} & \text{if } r_K > 0, \end{cases} \qquad \zeta_{K,v} = \begin{cases} h^{\frac{1}{\nu_\ell}} & \text{if } 0 \le R_K \le h^{\frac{1}{\nu_\ell}}, \\ h R_K^{1-\nu_\ell} & \text{if } R_K \ge h^{\frac{1}{\nu_\ell}}, \end{cases}$$

and let $h_{K,i}$ be the length of the projection of $K$ on the axis $x_i^\ell$, $i = 1, 2, 3$. Then we assume:

- if $\mu_\ell < 1$ then $h_{K,1} \sim \zeta_{K,e}, h_{K,2} \sim \zeta_{K,e}$, and $h_{K,3} \le \zeta_{K,v}$, and if $r_K = 0$ we demand $h_{K,3} \sim \zeta_{K,v}$;
- if $\mu_\ell = 1$ then $h_{K,i} \le \zeta_{K,v}$, $i = 1, 2, 3$, and $h_{K,i} \sim \zeta_{K,v}$ if $R_K = 0$.

Finally, it is assumed that if $\Lambda_\ell$ possesses neither singular vertex nor singular edge, then its subdivision inherited from $\mathcal{T}_h$ is shape regular uniformly in $h$.

### 3.4. Existence of meshes with all the requirements

A construction of a mesh on a general polyhedron is carefully showed in [1]. There, the authors construct partitions of each $\Lambda_\ell$ by distinguishing between four cases: (i) $\Lambda_\ell$ contains neither singular corner nor singular edge, (ii) $\Lambda_\ell$ contains a singular corner but no singular edge, (iii) $\Lambda_\ell$ contains a singular edge but no singular corner, and (iv) $\Lambda_\ell$ contains both singular corner and singular edge.

We point out that the smallest elements are the ones that have a vertex on a singular corner or an edge on a singular edge. Their sizes are of order $h^{3/\mu_\ell}$ or $h^{3/\nu_\ell}$, so requirement (3.5) holds with $\sigma = 3\max_\ell\{1/\mu_\ell, 1/\nu_\ell\}$.

**Remark 3.1.** Let us show that the conditions on the meshes just introduced include Assumption $2_{k,\beta}$ of Section 5.1 of [4]. To do that, we will assume that $\mu_\ell = \nu_\ell = \tau$ for all $\ell$ (we can take $\tau$ as the minimum between the refinement parameters, thus obtaining, possibly, an over-refined mesh), and in our assumptions we change the sign $\le$ by $\sim$.

Taking into account that in [4] a local coordinate system is considered near each edge, we have to compare the lengths $h_{K,i}$ of our elements, with the lengths $d_i$ of elements there, according to their relative position. In both cases, the third coordinate direction, corresponds to the direction of the edge that is considered. Using the notation $\mathcal{V}_{\mathbf{c}}^0, \mathcal{V}_{\mathbf{e}}^0$ and $\mathcal{V}_{\mathbf{e}}^{\mathbf{c}}$ of [4], we have

1. if $K \in \mathcal{V}_{\mathbf{c}}^0$, then $R_k \sim r_K$, and then $\zeta_{K,v} \sim \zeta_{K,e}$. It follows that $h_{K,i} \sim \zeta_{K,v}$ as in [4] if $\tau = \beta/k$;
2. if $K \in \mathcal{V}_{\mathbf{e}}^0$, then $K$ is far away from the corners, so $R_K \sim 1$. It follows that $h_{K,3} \sim \zeta_{K,v} \sim h$, while $h_{K,i} \sim \zeta_{K,e}$ (if $\tau < 1$). This coincides with [4] if $\tau = \beta/k$;
3. if $K \in \mathcal{V}_{\mathbf{e}}^{\mathbf{c}}$, then we are taking $h_{K,1} \sim \zeta_{K,e}, h_{K,2} \sim \zeta_{K,e}$, and $h_{K,3} \sim \zeta_{K,v}$. This coincides with [4] if $\tau = \beta/k$. Observe that, with $0 < \tau < 1$, if $R_K \le h^{\frac{1}{\tau}}$ (resp. $R_K \sim h^{\frac{1}{\tau}}$) then $hR_K^{1-\tau} \le h^{\frac{1}{\tau}}$ (resp. $hR_K^{1-\tau} \sim h^{\frac{1}{\tau}}$);
4. finally, elements in $\mathcal{V}^0$ are far away the corners and edges, and their lengths are of order $h$ in both cases.

## 4. Auxiliary propositions

In this Section we obtain some results that will be used in what follows to prove the main theorem of the paper. We consider a sequence $\{\mathbf{v}_h\}_{h\in \mathrm{J}}, \mathrm{J} \subseteq \mathrm{I}$, of discrete functions satisfying the conditions (2.1) and (2.2). This sequence is maintained fixed along the rest of the paper.

For each $h \in \mathrm{J}$, define $\mathbf{v}^h$ as the solution of the problem

$$\mathbf{v}^h \in H_0(\mathbf{curl}\,, \Omega), \quad \mathbf{curl}\,\mathbf{v}^h = \mathbf{curl}\,\mathbf{v}_h, \quad \operatorname{div}\mathbf{v}^h = 0.$$

The existence of $\mathbf{v}^h$ is a consequence of Theorem 3.6 in [7]. Now, we know from Lemma 12 in [4] that for each $p \ge 2$ we can split $\mathbf{v}^h$ as

$$\mathbf{v}^h = \mathbf{w}^h + \nabla q^h \tag{4.6}$$

with
$$\mathbf{w}^h \in W_0^{1,p}(\Omega)^3, \quad q^h \in H_0^1(\Omega), \quad \Delta q^h \in L^p(\Omega), \tag{4.7}$$
and
$$\|\mathbf{w}^h\|_{W^{1,p}(\Omega)} + \|\Delta q^h\|_{L^p(\Omega)} \le \|\mathbf{v}_h\|_X + \|\mathbf{curl}\,\mathbf{v}_h\|_{L^p(\Omega)}. \tag{4.8}$$
As was pointed in [4] (Proof of Cor. 3), the functions $q^h$ verify
$$q^h \in H^{\frac{3}{2}+s}(\Omega) \quad \text{for some } s > 0,$$
and so they are in $C(\bar{\Omega})$. Let
$$I_h : C(\bar{\Omega}) \to \{p \in H^1(\Omega) : p|_K \in P_1(K), \forall K \in \mathcal{T}_h\}$$
be the piecewise linear Lagrange interpolation operator. It follows that $I_h(q^h)$ is well defined, for all $h$. We also need the Nédélec interpolation operator
$$\Pi_h : W^{1,p}(\Omega) \to \{\mathbf{v}_h \in H(\mathbf{curl}\,, \Omega) : \mathbf{v}_h|_K \in \mathcal{N}_k(K), \forall K \in \mathcal{T}_h\}, \qquad p > 2.$$
For a function $\mathbf{w}$ it is defined by
$$(\Pi_h \mathbf{w})|_K = \Pi_K(\mathbf{w}|_K), \qquad \forall K \in \mathcal{T}_h$$
where $\Pi_K \mathbf{w}$ is the unique function in $\mathcal{N}_k(K)$ such that:
$$\int_e \Pi_K \mathbf{v} \cdot \mathbf{t}\, q = \int_e \mathbf{v} \cdot \mathbf{t}\, q, \qquad \forall q \in P_{k-1}(e), \forall e \text{ edge of } K, \tag{4.9}$$
$$\int_f \Pi_K \mathbf{v} \times \mathbf{n} \cdot \mathbf{q} = \int_f \mathbf{v} \times \mathbf{n} \cdot \mathbf{q}, \qquad \forall \mathbf{q} \in [P_{k-2}(f)]^2, \forall f \text{ face of } K, \tag{4.10}$$
$$\int_K \Pi_K \mathbf{v} \cdot \mathbf{q} = \int_K \mathbf{v} \cdot \mathbf{q}, \qquad \forall \mathbf{q} \in [P_{k-3}(K)]^3 \tag{4.11}$$
($\mathbf{t}$ denotes a unitary tangent field on the edge $e$, $\mathbf{n}$ denotes the exterior normal field to $K$ on the face $f$). It is well known [7,15] that the degrees of freedom (4.9)–(4.11) define $\Pi_K \mathbf{v}$ uniquely.

We observe that $\{\mathbf{v}^h\} \subset X$, and $X \subset [L^2(\Omega)]^3$ with compact inclusion. So, we can hope to obtain convergence properties for $\{\mathbf{v}_h\}$ from properties of $\{\mathbf{v}^h\}$. The next Proposition is a step in this direction. To prove it, we use the commutative property
$$\mathbf{curl}\,(\Pi_K \mathbf{w}) = \mathcal{RT}_K(\mathbf{curl}\,\mathbf{w}) \tag{4.12}$$
valid for fields $\mathbf{w}$ such that $\mathbf{w}|_K$ and its $\mathbf{curl}$ are in $W^{1,p}(K)$ (see, for example, Sects. 2 and 3 of [4]). Here, $\mathcal{RT}_K$ denotes the Raviart-Thomas interpolation of degree $k$ on $K$, we refer to [17] for its definition and properties. We will use that
$$\mathcal{RT}_K \mathbf{w} = \mathbf{w}, \qquad \forall \mathbf{w} \in [P_{k-1}(K)]^3. \tag{4.13}$$

**Proposition 4.1.** *Suppose that $\mathbf{w}^h$ and $q^h$ satisfy (4.6)–(4.8) for some $p > 2$. With $\Pi_h$ and $I_h$ the Nédélec and Lagrange interpolation operators introduced before, we have*
$$\|\mathbf{v}^h - \mathbf{v}_h\|_{H_0(\mathbf{curl}\,,\Omega)} \le \|\mathbf{w}^h - \Pi_h \mathbf{w}^h\|_{L^2(\Omega)} + \|\nabla(q^h - I_h q^h)\|_{L^2(\Omega)}. \tag{4.14}$$

*Proof.* Since $\mathbf{curl}\,(\mathbf{v}^h - \mathbf{v}_h) = 0$ we have $\|\mathbf{v}^h - \mathbf{v}_h\|_{H_0(\mathbf{curl}\,,\Omega)} = \|\mathbf{v}^h - \mathbf{v}_h\|_{L^2(\Omega)}$. Then
$$\begin{aligned}\|\mathbf{v}^h - \mathbf{v}_h\|^2_{L^2(\Omega)} &= (\mathbf{v}^h - \mathbf{v}_h, \mathbf{v}^h - \mathbf{v}_h)\\ &= (\mathbf{v}^h - \mathbf{v}_h, \mathbf{w}^h + \nabla q^h - \mathbf{v}_h)\\ &= (\mathbf{v}^h - \mathbf{v}_h, \mathbf{w}^h - \Pi_h \mathbf{w}^h + \nabla q^h - \nabla I_h q^h)\\ &\quad + (\mathbf{v}^h - \mathbf{v}_h, \Pi_h \mathbf{w}^h + \nabla I_h q^h - \mathbf{v}_h).\end{aligned}$$

Since $\mathbf{v}_h \in X_h$ and $I_h q^h \in S_h$, it follows that

$$(\mathbf{v}_h, \nabla I_h q^h) = 0.$$

On the other hand, by using the commutative property (4.12) relating Nédélec and Raviart-Thomas operators we have for all $K \in \mathcal{T}_h$

$$\begin{aligned}\mathbf{curl}\,[\Pi_K(\mathbf{w}^h|_K)] &= \mathcal{RT}_K[\mathbf{curl}\,(\mathbf{w}^h|_K)]\\ &= \mathcal{RT}_K[\mathbf{curl}\,(\mathbf{v}^h|_K)]\\ &= \mathcal{RT}_K[\mathbf{curl}\,(\mathbf{v}_h|_K)]\\ &= \mathbf{curl}\,(\mathbf{v}_h|_K).\end{aligned}$$

We used property (4.13) in the last equality. Hence

$$\mathbf{curl}\,(\Pi_h\mathbf{w}^h - \mathbf{v}_h) = 0,$$

and therefore $\Pi_h\mathbf{w}^h - \mathbf{v}_h = \nabla q_h$, for some $q_h \in S_h$ (see Rem. 5.7 of [7]), and now, since $\mathbf{v}_h \in X_h$ we obtain

$$(\mathbf{v}_h, \Pi_h\mathbf{w}^h - \mathbf{v}_h) = 0.$$

But we have then obtained that

$$\Pi_h\mathbf{w}^h + \nabla I_h q^h - \mathbf{v}_h = \nabla(q_h - I_h q^h), \qquad q_h, q^h \in H_0^1(\Omega),$$

so, using that $\operatorname{div}\mathbf{v}^h = 0$, we have

$$(\mathbf{v}^h, \Pi_h\mathbf{w}^h + \nabla I_h q^h - \mathbf{v}_h) = 0.$$

Therefore

$$(\mathbf{v}^h - \mathbf{v}_h, \Pi_h\mathbf{w}^h + \nabla I_h q^h - \mathbf{v}_h) = 0,$$

and we can conclude that

$$\|\mathbf{v}^h - \mathbf{v}_h\|^2_{L^2(\Omega)} = (\mathbf{v}^h - \mathbf{v}_h, \mathbf{w}^h - \Pi_h\mathbf{w}^h + \nabla q^h - \nabla I_h q^h)$$

from which (4.14) follows. □

Next, we estimate the edge interpolation error $\mathbf{w}^h - \Pi_h\mathbf{w}^h$. We will use the following result.

**Proposition 4.2.** *Let $k \geq 1$. Let $K$ be a tetrahedron satisfying MAC($\bar{\psi}$). There exist three edges of $K$, $\ell_i, i = 1, 2, 3$, and a constant $C$, such that if $p > 2$, then for all $\mathbf{u} \in [W^{1,p}(K)]^3$ with $\nabla\mathbf{curl}\,\mathbf{u} \in [L^p(K)]^3$, we have*

$$\|\mathbf{u} - \Pi_k\mathbf{u}\|_{L^p(K)} \leq C\left\{\sum_{i=1}^3 h_i\left\|\frac{\partial\mathbf{u}}{\partial\xi_i}\right\|_{L^p(K)} + h\|\mathbf{curl}\,\mathbf{u}\|_{L^p(K)} + h\sum_{i=1}^3 h_i\left\|\frac{\partial\mathbf{curl}\,\mathbf{u}}{\partial\xi_i}\right\|_{L^p(K)}\right\}, \tag{4.15}$$

*where $h_i$ denotes the lengths of $\ell_i$, $\xi_i = \ell_i/\|\ell_i\|$, $i = 1, 2, 3$, and $h$ is the diameter of $K$. The constant $C$ depends only on $\bar{\psi}$, $k$ and $p$, and it is independent of the function $\mathbf{u}$. Furthermore, $C$ can be chosen such that, in addition, if $M \in \mathbb{R}^{3\times3}$ is the matrix made up of $\xi_i$ as columns, then $\|M\|, \|M^{-1}\| \leq C$.*

*Proof.* This Proposition, but changing the last term in the right hand side of (4.15) by $h^2\|\nabla\mathbf{curl}\,\mathbf{u}\|_{L^p(K)}$ is contained in Theorem 6.1 of [12]. One can check that (4.15) can be obtained by the same techniques used in that paper. □

**Proposition 4.3.** *Suppose that $\mathbf{w}^h$ (and $q^h$) satisfies (4.6)–(4.8) for some $p > 2$. If the mesh satisfies $MAC(\bar{\psi})$ then*

$$\|\mathbf{w}^h - \Pi_h \mathbf{w}^h\|_{L^2(\Omega)} \le Ch \left( \|\mathbf{v}_h\|_{H_0(\mathbf{curl},\Omega)} + \|\mathbf{curl}\,\mathbf{v}_h\|_{L^p(\Omega)} \right), \tag{4.16}$$

*with the constant $C$ depending on $\bar{\psi}$.*

*Proof.* We have

$$\begin{aligned}
\|\mathbf{w}^h - \Pi_h \mathbf{w}^h\|^2_{L^2(\Omega)} &= \sum_{K\in\mathcal{T}_h} \|\mathbf{w}^h - \Pi_h \mathbf{w}^h\|^2_{L^2(K)} \\
&= \sum_{K\in\mathcal{T}_h} \int_K |\mathbf{w}^h - \Pi_h \mathbf{w}^h|^2 \\
&\le \sum_{K\in\mathcal{T}_h} \left( \int_K |\mathbf{w}^h - \Pi_h \mathbf{w}^h|^p \right)^{\frac{2}{p}} |K|^{1-\frac{2}{p}} \\
&= \sum_{K\in\mathcal{T}_h} \|\mathbf{w}^h - \Pi_h \mathbf{w}^h\|^2_{L^p(\Omega)} |K|^{1-\frac{2}{p}}.
\end{aligned}$$

Now, fix an element $K \in \mathcal{T}_h$. From Section 3, we know that $K$ satisfies $MAC(\bar{\psi})$ for some $\bar{\psi}$ independent of $K$ and $h$. From Proposition 4.2 we then have ($h_K \le Ch$)

$$\|\mathbf{w}^h - \Pi_k \mathbf{w}^h\|_{L^p(K)} \le C \left\{ \sum_{i=1}^3 h_i^K \left\| \frac{\partial \mathbf{w}^h}{\partial \xi_i^K} \right\|_{L^p(K)} + h\|\mathbf{curl}\,\mathbf{w}^h\|_{L^p(K)} + h \sum_{i=1}^3 h_i^K \left\| \frac{\partial \mathbf{curl}\,\mathbf{w}^h}{\partial \xi_i^K} \right\|_{L^p(K)} \right\},$$

where $h_i^K = h_i$ and $\xi_i^K = \xi_i$ are the ones given by the Proposition. Now, by Hölder inequality and adding on all the elements $K \in \mathcal{T}_h$ we have

$$\begin{aligned}
\|\mathbf{w}^h - \Pi_h \mathbf{w}^h\|^2_{L^2(\Omega)} \le{}& C \sum_{K\in\mathcal{T}_h} \left( \sum_{i=1}^3 h_i^K \left\| \frac{\partial \mathbf{w}^h}{\partial \xi_i^K} \right\|_{L^p(K)} \right)^2 |K|^{1-\frac{2}{p}} + C \sum_{K\in\mathcal{T}_h} \left( h\|\mathbf{curl}\,\mathbf{w}^h\|_{L^p(K)} \right)^2 |K|^{1-\frac{2}{p}} \\
&+ C \sum_{K\in\mathcal{T}_h} \left( \sum_{i=1}^3 h_i^K \left\| \frac{\partial \mathbf{curl}\,\mathbf{w}^h}{\partial \xi_i^K} \right\|_{L^p(K)} \right)^2 |K|^{1-\frac{2}{p}} \\
=:{}& C(I + II + III).
\end{aligned}$$

Use now the discrete Hölder inequality to obtain

$$\begin{aligned}
I &= \sum_{K\in\mathcal{T}_h} \left( \sum_{i=1}^3 h_i^K \left\| \frac{\partial \mathbf{w}^h}{\partial \xi_i^K} \right\|_{L^p(K)} \right)^2 |K|^{1-\frac{2}{p}} \\
&\le \left[ \sum_{K\in\mathcal{T}_h} \left( \sum_{i=1}^3 h_i^K \left\| \frac{\partial \mathbf{w}^h}{\partial \xi_i^K} \right\|_{L^p(K)} \right)^p \right]^{\frac{2}{p}} \left( \sum_{K\in\mathcal{T}_h} |K| \right)^{1-\frac{2}{p}} \\
&\le h^2 \left| \mathbf{w}^h \right|^2_{W^{1,p}(\Omega)} |\Omega|^{1-\frac{2}{p}}.
\end{aligned}$$

We have used that due to the properties of matrix $M_K = M$ of Proposition 4.2, we have

$$|\mathbf{w}^h|_{W^{1,p}(K)} \sim \sum_{i=1}^3 \left\| \frac{\partial \mathbf{w}^h}{\partial \xi_i^K} \right\|_{L^p(K)},$$

for all $K \in \mathcal{T}_h$.

Since $\mathbf{curl}\,\mathbf{w}^h|_K = \mathbf{curl}\,\mathbf{v}_h|_K \in [P_{k-1}(K)]^3$ for all $K \in \mathcal{T}_h$, we can use an inverse estimate obtaining

$$\|\mathbf{curl}\,\mathbf{w}^h\|^2_{L^p(K)} \le |K|^{\frac{2}{p}-1}\|\mathbf{curl}\,\mathbf{w}^h\|_{L^2(K)}. \tag{4.17}$$

Therefore

$$II \le h^2\|\mathbf{curl}\,\mathbf{w}\|^2_{L^2(\Omega)}.$$

Using now the inverse inequality (valid since $\mathbf{curl}\,\mathbf{w}^h|_K \in [P_{k-1}(K)]^3$)

$$h_i^K \left\|\frac{\partial \mathbf{curl}\,\mathbf{w}^h}{\partial \xi_i^K}\right\|_{L^p(K)} \le C\|\mathbf{curl}\,\mathbf{w}^h\|_{L^p(K)}$$

and again (4.17), we obtain

$$III \le h^2\|\mathbf{curl}\,\mathbf{w}^h\|^2_{L^2(\Omega)}.$$

Hence, we have arrived at

$$\|\mathbf{w}^h - \Pi_h\mathbf{w}^h\|_{L^2(\Omega)} \le Ch\left(\left|\mathbf{w}^h\right|_{W^{1,p}(\Omega)} + \|\mathbf{curl}\,\mathbf{w}^h\|_{L^2(\Omega)}\right).$$

Then, by noting that $\mathbf{curl}\,\mathbf{w}^h = \mathbf{curl}\,\mathbf{v}_h$ and using (4.8) we have

$$\begin{aligned}\left|\mathbf{w}^h\right|_{W^{1,p}(\Omega)} + \|\mathbf{curl}\,\mathbf{w}^h\|_{L^2(\Omega)} &= C\left(\|\mathbf{v}_h\|_{H_0(\mathbf{curl},\Omega)} + \|\mathbf{curl}\,\mathbf{v}_h\|_{L^p(\Omega)}\right) + \|\mathbf{curl}\,\mathbf{v}_h\|_{L^2(\Omega)}\\ &\le C\left(\|\mathbf{v}_h\|_{H_0(\mathbf{curl},\Omega)} + \|\mathbf{curl}\,\mathbf{v}_h\|_{L^p(\Omega)}\right).\end{aligned}$$

Therefore, inequality (4.16) follows. □

It remains to deal with the Lagrange interpolation error $q^h - I_h q^h$. We can use directly the results of [1], which we include below for the sake of completeness. We remark that we are assuming the conditions on the mesh (and on the refinement parameters $\mu_\ell$ and $\nu_\ell$) established in Section 3. To obtain the following Proposition, restrictions on the refinement parameters need to be added. First we introduce some notation.

Recall the decomposition $\bar{\Omega} = \cup_\ell^L \bar{\Lambda}_\ell$ of Section 3.3. For each $\ell = 1, \dots, L$, we put $\lambda_v^\ell = \lambda_{S,v}$ if $\lambda_\ell$ contains one singular vertex $S$ of $\Omega$, otherwise we put $\lambda_v^\ell = +\infty$. Similarly, we put $\lambda_e^\ell = \lambda_{e,A}$ if $\Lambda_\ell$ contains one singular edge $A$ of $\Omega$, otherwise we take $\lambda_e^\ell = +\infty$.

In addition to the number $p_* > 2$ introduced in Section 2, we define a number $p_+$ satisfying $2 < p_+ \le 6$ and

$$\lambda_v^\ell \ge 1 - \frac{2}{p_+}, \qquad \lambda_e^\ell \ge 1 - \frac{1}{p_+},$$

for all $\ell = 1, \dots, L$. We observe that being $\lambda_v^\ell > 0$ and $\lambda_e^\ell > \frac{1}{2}$ for all $\ell$, such a number $p_+$ can always be chosen.

We are ready to state the next result.

**Proposition 4.4.** *Let $p$ be such that $2 < p < \min\{p_*, p_+\}$. Suppose that for all $\ell = 1, \dots, L$, $\mu_\ell$ and $\nu_\ell$ satisfy the following conditions:*

$$\mu_\ell < \lambda_e^\ell \frac{p}{2p-2}, \tag{4.18}$$

$$\nu_\ell < \left(\lambda_v^\ell + \frac{1}{2}\right)\frac{2p}{5p-6}, \tag{4.19}$$

$$\frac{1}{\nu_\ell}\left(\frac{5}{2} - \frac{3}{p}\right) + \frac{1}{\mu_\ell}\left(\lambda_v^\ell - 2 + \frac{3}{p}\right) > 1. \tag{4.20}$$

*Suppose that $q^h$ (and $\mathbf{w}^h$) satisfies (4.6)–(4.8) for some $p > 2$. Then we have*

$$\|\nabla(q^h - I_h q^h)\|_{0,\Omega} \le Ch\left(\|\mathbf{v}_h\|_X + \|\mathbf{curl}\,\mathbf{v}_h\|_{L^p(\Omega)}\right). \tag{4.21}$$

*Proof.* Since for all $h \in \mathrm{J}$

$$\|\Delta q^h\|_{L^p(\Omega)} \leq C\left(\|\mathbf{v}_h\|_X + \|\mathbf{curl}\,\mathbf{v}_h\|_{L^p(\Omega)}\right),$$

the proposition follows from Theorem 5.1 of [1]. □

Note that for $p > 2$ and $\mu_\ell = \nu_\ell$, (4.19) implies (4.20).

We close this section with a definition that will be used to prove the compactness result.

**Definition 4.5.** We say that the refinement parameters verify property $(U)$ if they satisfy conditions (4.18)–(4.20) uniformly for all $2 < p < p_0$ for some $p_0 > 2$.

We point out that it is always possible to take $\mu_\ell$ and $\nu_\ell$ satisfying property $(U)$. Indeed, we can proceed as follows:

- If $\lambda_e^\ell \leq 1$, fix $p_0 > 2$ and take

$$\mu_\ell = \nu_\ell < \min\left\{\lambda_e^\ell \frac{p_0}{2p_0 - 2}, \left(\lambda_v^\ell + \frac{1}{2}\right)\frac{2p_0}{5p_0 - 6}\right\}.$$

  Then conditions (4.18) and (4.19) hold for all $2 < p < p_0$, and also condition (4.20) holds, since, in this case, it follows from (4.19).
- If $\lambda_e^\ell > 1$, then fix $p_0 > 2$ such that $\lambda_e^\ell \frac{p_0}{2p_0-2} > 1$. Then take $\mu_\ell = 1$ and $\nu_\ell$ verifying

$$\nu_\ell < \left(\lambda_v^\ell + \frac{1}{2}\right)\frac{2p_0}{5p_0 - 6}$$

  and small enough to have

$$\frac{1}{\nu_\ell}\left(\frac{5}{2} - \frac{3}{p}\right) + \lambda_v^\ell - 2 + \frac{3}{p} > 1 \qquad \forall\, 2 < p < p_0.$$

## 5. The discrete compactness property

In this section we prove that the family of discrete spaces $\{X_h\}_{h\in\mathrm{I}}$ associated with the graded meshes introduced in Section 3 verifies the Discrete Compactness Property. We will make use of the next Proposition that is an easy consequence of the results of Section 4. We continue using the notation introduced there, in particular, $\{\mathbf{v}_h\}_{h\in J}$ is a sequence of discrete functions satisfying the conditions (2.1) and (2.2) (J is a denumerable subset of I), and $\mathbf{v}^h, h \in \mathrm{J}$, are defined in the beginning of Section 4.

**Proposition 5.1.** *Suppose that the family of meshes $\mathcal{T}_h$ verifies the requirements of Section 3 with refinements parameters $\mu_\ell$ and $\nu_\ell$ satisfying property $(U)$. Then, the sequence $\{\mathbf{v}^h\}_{h\in\mathrm{J}}$ verifies*

$$\|\mathbf{v}^h - \mathbf{v}_h\|_{H(\mathbf{curl}\,,\Omega)} \to 0, \qquad \textit{for } h \to 0 \ (h \in \mathrm{J}).$$

*Proof.* Fixed $p > 2$, we consider, for each $h$, the decomposition (4.6) verifying (4.7) and (4.8). Since $\mathbf{curl}\,\mathbf{v}^h = \mathbf{curl}\,\mathbf{v}_h$, we have $\|\mathbf{v}^h - \mathbf{v}_h\|_{H(\mathbf{curl}\,,\Omega)} = \|\mathbf{v}^h - \mathbf{v}_h\|_{L^2(\Omega)}$. It follows from Propositions 4.1, 4.3 and 4.4, that

$$\|\mathbf{v}^h - \mathbf{v}_h\|_{L^2(\Omega)} \leq Ch\left(\|\mathbf{v}_h\|_X + \|\mathbf{curl}\,\mathbf{v}_h\|_{L^p(\Omega)}\right).$$

We emphasize that the constant $C$ in the previous equation depends on $p$.

Let $\sigma = 3\max_\ell\{1/\mu_\ell, 1/\nu_\ell\}$. Then, using an inverse inequality, we obtain

$$\|\mathbf{curl}\,\mathbf{v}_h\|_{L^p(\Omega)} \leq Ch^{-\sigma\left(\frac{1}{p}-\frac{1}{2}\right)}\|\mathbf{curl}\,\mathbf{v}_h\|_{L^2(\Omega)}.$$

Therefore,

$$\|\mathbf{v}^h - \mathbf{v}_h\|_{L^2(\Omega)} \le Ch\|\mathbf{v}_h\|_X + Ch^{1-\sigma\left(\frac{1}{p}-\frac{1}{2}\right)}\|\mathbf{curl}\,\mathbf{v}_h\|_{L^2(\Omega)}.$$

Now, taking into account the $(U)$ property of the refinement parameters, choose $p > 2$ close enough to 2, such that

$$\alpha_0 = 1 - \sigma\left(\frac{1}{p} - \frac{1}{2}\right) > 0,$$

while conditions (4.18)–(4.20) remain valid for all $\ell$. So, using this value of $p$ in the previous computations, it follows that

$$\|\mathbf{v}^h - \mathbf{v}_h\|_{L^2(\Omega)} \le Ch^{\alpha_0}\|\mathbf{v}_h\|_X,$$

from which the proof is concluded, since $\{\mathbf{v}_h\}$ is bounded in $X$. □

Now we are ready to prove the main result of the paper.

**Theorem 5.2.** *If the meshes $\mathcal{T}_h$ satisfy the assumptions of Section 3, then the family of spaces $\{X_h\}$ introduced in Section 2 verifies the discrete compactness property.*

*Proof.* Since $\{\mathbf{v}_h\}$ is bounded in $H_0(\mathbf{curl}\,,\Omega)$ and $\operatorname{div}\mathbf{v}^h = 0$ for all $h \in \mathrm{J}$, it follows from Proposition 5.1 that $\{\mathbf{v}^h\}$ is a bounded sequence in $X$. Taking into account that $\Omega$ is a bounded Lipschitz polyhedron, it follows from the Weber's compactness result (Thms. 2.1 and 2.2 of [18]) that $\{\mathbf{v}^h\}$ has an $L^2(\Omega)$-convergent subsequence $\{\mathbf{v}^{h_n}\}_{n\in\mathbb{N}}$ to a function $\mathbf{v} \in L^2(\Omega)$. We have $\operatorname{div}\mathbf{v} = 0$. Since $H_0(\mathbf{curl}\,,\Omega)$ (resp. $X$) is reflexive and $\{\mathbf{v}^{h_n}\}$ is bounded in $H_0(\mathbf{curl}\,,\Omega)$ (resp. $X$), taking a subsequence if necessary, we check that $\mathbf{v} \in H_0(\mathbf{curl}\,,\Omega)$ (resp. $\mathbf{v} \in X$) and $\mathbf{v}^{h_n} \rightharpoonup \mathbf{v}$ weakly in $H_0(\mathbf{curl}\,,\Omega)$.

Now, using again Proposition 5.1 on one hand, and the reflexivity of $H_0(\mathbf{curl}\,,\Omega)$ on the other hand, we have that $\{\mathbf{v}_{h_n}\}$ converges strongly to $\mathbf{v}$ in $L^2(\Omega)$, and, taking a subsequence if necessary, weakly in $H_0(\mathbf{curl}\,,\Omega)$. □

## 6. Conclusions

We have obtained the validity of the discrete compactness property for edge elements of any order on tetrahedral triangulations of a general Lipschitz polyhedron. The meshes allowed include the standard graded ones proposed in the literature to deal with edge and corner singularities.

The restrictions on the family of meshes are essentially imposed to ensure the validity of some estimates for the Lagrange interpolation of the solution of a homogeneous Dirichlet problem for the Laplace operator with right hand side in $L^p$ with $p > 2$. However, the analysis allows the use of interpolations other than the Lagrange one. Further research with other interpolations is needed.

*Acknowledgements.* I thank Irene Drelichman for her careful reading of the manuscript.

## References

[1] T. Apel and S. Nicaise, The finite element method with anisotropic mesh grading for elliptic problems in domains with corners and edges, *Math. Meth. Appl. Sci.* **21** (1998) 519–549.

[2] D. Boffi, Fortin operator and discrete compactness for edge elements. *Numer. Math.* **87** (2000) 229–246.

[3] D. Boffi, Finite element approximation of eigenvalue problems. *Acta Numer.* **19** (2010) 1–120.

[4] A. Buffa, M. Costabel and M. Dauge, Algebraic convergence for anisotropic edge elements in polyhedral domains. *Numer. Math.* **101** (2005) 29–65.

[5] S. Caorsi, P. Fernandes and M. Raffetto, On the convergence of Galerkin finite element approximations of electromagnetic eigenproblems, *SIAM J. Numer. Anal.* **38** (2000) 580–607.

[6] S. Caorsi, P. Fernandes and M. Raffetto, Spurious-free approximations of electromagnetic eigenproblems by means of Nedelec-type elements. *Math. Model. Numer. Anal.* **35** (2001) 331–354.

[7] V. Girault and P.A. Raviart, Finite Element Methods for Navier-Stokes Equations, in *Theory and Applications*. Springer-Verlag, Berlin (1986).

[8] R. Hiptmair, Finite elements in computational electromagnetism. *Acta Numer.* **11** (2002) 237–339.

[9] F. Kikuchi, On a discrete compactness property for the Nédélec finite elements. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **36** (1989) 479–490.

[10] M. Krízek, On the maximum angle condition for linear tetrahedral elements. *SIAM J. Numer. Anal.* **29** (1992) 513–520.

[11] R. Leis, *Initial Boundary Value Problems in Mathematical Physics.* John Wiley, New York (1986).

[12] A.L. Lombardi, Interpolation error estimates for edge elements on anisotropic meshes. *IMA J. Numer. Anal.* **31** (2011) 1683–1712.

[13] P. Monk, *Finite Element Methods for Maxwell's Equations.* Oxford University Press, New York (2003).

[14] P. Monk and L. Demkowicz, Discrete compactness and the approximation of Maxwell's equations in $\mathbb{R}^3$. *Math. Comp.* **70** (2001) 507–523.

[15] J.C. Nédélec, Mixed finite elements in $\mathbb{R}^3$. *Numer. Math.* **35** (1980) 315–341.

[16] S. Nicaise, Edge elements on anisotropic meshes and approximation of the Maxwell equations. *SIAM J. Numer. Anal.* **39** (2001) 784–816.

[17] P. A. Raviart and J.-M. Thomas, A mixed finite element method for second order elliptic problems, in Mathematical Aspects of the Finite Element Method, edited by I. Galligani and E. Magenes. *Lect. Notes Math.* **606** (1977).

[18] Ch. Weber, A local compactness theorem for Maxwell's equations. *Math. Meth. Appl. Sci.* **2** (1980) 12–25.