ESAIM: MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

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ESAIM: Modélisation mathématique et analyse numérique, tome 34, nº 2 (2000), p. 477-499

<http://www.numdam.org/item?id=M2AN_2000__34_2_477_0>

Mathematical Modelling and Numerical Analysis M2AN, Vol. 34, N° 2, 2000, pp. 477–499
Modélisation Mathématique et Analyse Numérique

NUMERICAL APPROXIMATIONS OF THE RELATIVE REARRANGEMENT: THE PIECEWISE LINEAR CASE. APPLICATION TO SOME NONLOCAL PROBLEMS *, **

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Abstract. We first prove an abstract result for a class of nonlocal problems using fixed point method. We apply this result to equations revelant from plasma physic problems. These equations contain terms like monotone or relative rearrangement of functions. So, we start the approximation study by using finite element to discretize this nonstandard quantities. We end the paper by giving a numerical resolution of a model containing those terms.

Mathematics Subject Classification. 76X05, 35R30, 35R35, 28A75, 46E30, 65L60, 65M60, 65N30.

Received: May 27, 1999. Revised: October 26 1999.

1. INTRODUCTION

In the mathematical models appearing in plasma physics either for the Tokamak models or the Stellerators models, the modelling equations may depend not only on the solution u but also on the distribution function associated with that function, that is the volume of a level set $m_u(t) = \text{meas}\,\{x \in \Omega, u(x) > t\}$, $t \in \mathbb{R}$, (here, Ω is the mathematical domain). What is more, it might even depend on the generalized inverse of that function m_u, called the decreasing monotone rearrangement of u and denoted by u_* and its derivatives u'_* or u''_*. For instance, in the Tokamak models, Grad [25] and Shafranov [51], conjectured that the current flux u (associated with the magnetic fields and the pressure) can satisfy an equation of the form

$$-\Delta u(x) - \lambda u''_*(m_u(u(x)) = f(x), x \in \Omega$$

(*see* also Temam [55, 57]). More recently, in the case of a confined plasma in a Stellerator, Díaz-Rakotoson (*see* [15, 16] for the modelization, [19] for the mathematical justification and [5] for the numerical solution)

Keywords and phrases. Monotone decreasing and relative rearrangements, nonlocal problems, numerical approximations.

* *Dedicated to Roger Temam for his 60th birthday*

** *This research has been partially supported by the University of Santiago.*

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established that the current field u satisfies the following equation:

$$(\mathcal{PS})\begin{cases} -\Delta u = a(x)\left[F_v^2 - 2\int_0^{u_+(x)} p'(t)b_{*u}(m_u(t))\mathrm{d}t\right]_+^{1/2} + p'(u)\Big[b(x) - b_{*u}\Big(m_u(u(x))\Big)\Big] \text{ in } \Omega \\ u = \gamma \text{ on } \partial\Omega \end{cases}$$

where

$$b_{*u} = \lim_{t\searrow 0}\frac{(u+tb)_* - u_*}{t}$$

is called, according to Mossino-Temam [34], the relative rearrangement of b with respect to u (see below for more details).

In this article, we wish to present a method for the numerical approximation of the monotone and relative rearrangements by using finite elements $\mathbb{P}_1$. Since very few results are known concerning the regularity of the derivatives of monotone and relative rearrangement functions, we shall only look at the convergence of the scheme according to what we know on the first derivative of the monotone rearrangement. (Almost nothing is known on the second derivatives of these quantities).

As an application in P.D.E, we choose a model which involves the first derivative of the monotone relative rearrangement and whose nonlinearities are of the same type as those in Grad-Shafranov in the Stellerator models, say:

$$(\mathcal{P})\begin{cases} -u''(x) - \lambda u_*'\Big(m_u(u(x))\Big) - a(x)\Big[F_0^2 - \displaystyle\int_{m_u(0)}^{m_u(u_+(x))} p'(u_*)u_*'b_{*u}\mathrm{d}t\Big]_+^{1/2} = f(x) \\ u(0) = u(1) = 0 \end{cases}$$

The variational problem associated with that problem reads:

$$(\mathcal{P}_v)\begin{cases} \text{Find } u \in H_0^1(0,1) \text{ such that} \\ \displaystyle\int_0^1 u'(x)v'(x)\mathrm{d}x - \lambda\int_0^1 u_*'(s)v_{*u}(s)\mathrm{d}s \quad - \int_0^1 F(u)(x)v(x)\mathrm{d}x \\ \qquad\qquad = \displaystyle\int_0^1 f(x)v(x)\mathrm{d}x \quad \text{for all } v \in H_0^1(0,1) \end{cases}$$

where $F(u)(x) = a(x)\Big[F_0^2 - \displaystyle\int_{m_u(0)}^{m_u(u_+(x))} p'(u_*)u_*'b_{*u}\mathrm{d}t\Big]_+^{1/2}$, a and b are in $\mathrm{L}^\infty(0,1)$, $f \in \mathrm{L}^2(0,1)$, $F_0 > 0$ is a constant and p a C^1-function with p' bounded for the sake of simplicity.

Along this paper, we use the fact that $u_* \in H^1(0,1)$ if $u \in H^1(0,1)$ and then $\dfrac{\mathrm{d}^+u_*}{\mathrm{d}s}\Big(m_u(u(x))\Big) = u_*'\Big(m_u(u(x))\Big)$ *a.e.*

In order to give a theoretical and numerical resolution of that problem, we shall consider a family of subspaces V_h of $H_0^1(0,1)$ with finite dimension. Then, we shall prove the existence of a function $u_h \in V_h$ satisfying the following approximate problem

$$(\mathcal{P}_v^h)\begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ \displaystyle\int_0^1 u_h'(x)v_h'(x)\mathrm{d}x - \lambda\int_0^1 u_{h*}'(s)(v_h)_{*u_h}(s)\mathrm{d}s \quad - \int_0^1 F(u_h)(x)v_h(x)\,\mathrm{d}x \\ \qquad\qquad = \displaystyle\int_0^1 f(x)v_h(x)\mathrm{d}x \quad \text{for all } v_h \in V_h. \end{cases}$$

In order to solve problem $(\mathcal{P}_v)$ and $(\mathcal{P}_v^h)$, we introduce an abstract result which will allow us to consider other nonlinearities than in $(\mathcal{P})$.

This abstract result reads as follow:
Let $(V, \|\cdot\|)$ be a Hilbert space continuously and compactly imbedded in a Banach separable space $(H, |\cdot|)$. Assume that there exist a family of finite elements $V_h \subset V$ and a family of linear operators Π_h: $V \longrightarrow V_h$ such that $\lim_{h \to 0} \|v - \Pi_h v\| = 0$ for all $v \in V$. Consider B: $V \times V \longrightarrow \mathbb{R}$ a bilinear coercive and continuous map and G: $V \longrightarrow H'$(dual of H) a nonlinear continuous map from V-strong into $H' - * - \sigma(H', H)$ (weak-star topology), with G having the following growth:

$$|G(v)|_* \le \lambda_0 \|v\| + \lambda_1 \qquad 0 < \lambda_0 < \Big(\inf_{\|v\|=1} B[v,v]\Big) \cdot \Big(\inf_{|v|=1} \|v\|\Big)$$

Then:

i) There exists $u_h \in V_h$ such that

$$B(u_h, v_h) =< G(u_h), v_h >, \quad \text{for all } v_h \in V_h$$

ii) There exists $u \in V$ such that

$$u_h \longrightarrow u \text{ in } V \text{ strong} \qquad B[u,v] =< G(u), v > \qquad \text{for all } v \in V.$$

It happens that for the kind of operators that we meet in the literature, the map G is not continuous on the whole space V but only on a subset $\mathcal{V}$ of V, containing all V_h; in which case we may assume that $H = H'$ and then the continuity of G can be restricted to $\mathcal{V} \subset V \longrightarrow H$-weak, (for instance if $A \in \mathcal{L}(V, V')$ associated with the bilinear form B then $\mathcal{V} = D(A)$ (domain of A) $\bigcup_h V_h$ induced by the norm of V). The above conclusion remains true provided that we show that $u \in \mathcal{V}$.

We shall apply statement i) and ii) with $V = H_0^1(0,1)$ $H = \mathrm{L}^2(0,1)$

$$B(u,v) = \int_0^1 u'v', \qquad < G(u), v >= \lambda \int_0^1 u'_* v_{*u} + \int_0^1 F(u)(x) v(x) \mathrm{d}x + \int_0^1 f(x) v(x) \mathrm{d}x.$$

The "main" difficulty will be to prove the continuity of G from V-strong into $\mathrm{L}^2(0,1)$-weak. So we shall introduce some appropriate new lemmas (see Lem. 3.1 to Lem. 3.4).
As a consequence of this analysis, we derive a stability result for u_h of the form:

$$|u'_h|_{\mathrm{L}^2(0,1)} \le \frac{\sqrt{2}|f|_{\mathrm{L}^2} + |a|_\infty c}{1 - |\lambda|} \text{ for } |\lambda| < 1.$$

We also obtain for statement ii) the convergence of the scheme.
Some qualitative properties for the solutions of the continuous and discrete problems are given: when f is symmetric (that is $f(x) = f(1-x)$), we will show the existence of symmetric solutions. When $\lambda f < 0, \lambda a < 0$, then every solution u of $(\mathcal{P}_v)$ is such that the set $\{x\colon u'(x) = 0\}$ is of measure zero. We shall also provide the rigidity matrix associated with the discretized problem.

For convenience for the reader, we start by recalling some useful notions on the monotone and relative rearrangements.

2. Monotone and relative rearrangements of a function: Definitions and properties

Since the numerical schemes that we shall present below are in one dimension, we restrict the introduction of this section to functions defined on $[0,1]$. Let u be a real valued Lebesgue measurable function defined on

$]0,1[$. For $t \in \mathbb{R}$, we denote by $\{u > t\}$ the set $\{x \in]0,1[, u(x) > t\}$, and by $\{u = t\}$ the set $\{x \in]0,1[, u(x) = t\}$. The Lebesgue measure of any measurable set E is denoted $|E|$ or meas(E); in particular, the measure of the above level sets are denoted by, $|u > t|, |u = t|$.

Definition 2.1. *A measurable function u on $]0,1[$ has a plateau at a value t if $|u = t| > 0$.*

We set $P(u) = \{x \in]0,1[: |u = u(x)| > 0\}$.

The distribution function associated with u is the real valued function $t \in \mathbb{R} \to m_u(t) = |u > t|$.

Definition 2.2. *For a measurable function u, the generalized inverse of its distribution function is called the monotone decreasing rearrangement, that is the function u_* with finite value on on $]0,1[$ given by: if $s \in [0,1[$, then $u_*(s) = \operatorname{Inf}\{t \in \mathbb{R}: |u > t| \leq s\}$ and $u_*(1) = \operatorname{ess\,inf}\{u(x), x \in [0,1]\}$.*

Properties of the monotone rearrangement

i) The monotone decreasing rearrangement u_* of u is equimeasurable to u, that is for all $t \in \mathbb{R}$ $|u > t| = |u_* > t|$. This implies in particular that the integral of $F(u)$ over the level set $\{u > t\}$ is equal to the integral of $F(u_*)$ over the level set $\{u_* > t\}$, whenever F is a real valued Borel function with $F(u)$ integrable on $[0,1]$.

ii) If u belongs to the Sobolev space $W^{1,p}(0,1), 1 \leq p \leq \infty$, then u_* belongs to the same space and we have the inequality:

$$|u'_*|_{L^p(0,1)} \leq |u'|_{L^p(0,1)},$$

where we denote by $|.|_{L^p(0,1)}$ the norm in the Lebesgue space $L^p(0,1)$.

For more details on these properties, see G. Talenti [52], Rakotoson-Temam [49], Mossino [36], Hardy Littlewood and Polya [27].

2.1. Definition and properties of the relative rearrangement

Let $u \in L^1(0,1)$ and $b \in L^p(0,1), 1 \leq p \leq \infty$. For a fixed s in $[0,1]$, we denote by B_s the restriction of b to the level set $\{u = u_*(s)\}$. Define on $[0,1]$ the function w by

$$w(s) = \int_{\{u > u_*(s)\}} b(x)\,\mathrm{d}x + \int_0^{s - |u > u_*(s)|} (B_s)_*(t)\,\mathrm{d}t.$$

Then, w is in the Sobolev space $W^{1,p}(0,1)$ and the quotient $\dfrac{(u+tb)_* - u_*}{t}$ converges to w' as $t \searrow 0$, in $L^p(0,1)$-weak if $1 < p < \infty$, in $L^\infty(0,1)$-weak-$*$ if $p = \infty$ and for the topology $\sigma(L^1(0,1), L^\infty(0,1))$ for $p = 1$. The function w' is called the relative rearrangement of b with respect to u and is denoted by b_{*u}.

Properties of the relative rearrangement

Let $u \in L^1(0,1)$ and $b \in L^p(0,1), 1 \leq p \leq \infty$. Then,

i) The map $b \in L^p(0,1) \to b_{*u} \in L^p(0,1)$ is a contraction. In particular, we have the main inequality:

$$|b_{*u}|_{L^p(0,1)} \leq |b|_{L^p(0,1)}.$$

ii) If Φ is a nondecreasing function on $\mathbb{R}$, then $\Phi(u)_{*u} = \Phi(u_*)$ provided that $\Phi(u) \in L^1(0,1)$.

iii) If u_h is a family of functions such that u_h converges strongly to a function u in $H^1(0,1) = W^{1,2}(0,1)$ and if $|\{x: u'(x) = 0\}| = |\{x: u'_h(x) = 0\}| = 0$, then b_{*u_h} converges strongly to b_{*u} in $L^p(0,1)$, provided that $1 \leq p < \infty$.

One can also define the monotone and relative rearrangements associated with weighted functions. The definitions and properties given above can be carried naturally by making use of weighted spaces when necessary. In particular, if a is a weight function then the distribution function m_u^a of a Lebesgue measurable function u

with respect to the weight a is $m_u^a(t) = \int_{\{u>t\}} a(x)\,\mathrm{d}x$. Its generalized inverse, that is the monotone decreasing rearrangement of u with respect to a, is denoted u_*^a and satisfies for $s \in [0, \int_0^1 a(x)\mathrm{d}x[$

$$u_*^a(s) = \operatorname{Inf}\{t \in \mathbb{R}\colon m_u^a(t) \leq s\}\cdot$$

For more details on weighted relative rearrangement, we refer the reader to Rakotoson-Simon [48]. The link between relative rearrangement and weighted rearrangement is given in the following lemma:

Lemma 2.1. *Let $u \in W^{1,1}(0,1)$ be such that $|\{x\colon u'(x) = 0\}| = 0$ and let $b \in L^\infty(0,1)$ satisfy* $\operatorname{ess\,inf} b > 0$. *Then:*

$$b_{*u}(m_u(u(x))) = \frac{u_*'(m_u(u(x)))}{(u_*^b)'(m_u^b(u(x)))}\cdot$$

This lemma is proven in Díaz-Rakotoson [19].

We shall also use the following mean value formula; for a complete statement, we refer the interested reader to Mossino-Temam [34], Mossino [36], Rakotoson-Simon [47].

Lemma 2.2. *Let u, b be two functions in $L^2(0,1)$. Then, there exists a linear continuous operator from $L^2(0,1)$ into $L^2(0,1)$ denoted by $M_{u,b}$ such that for all $g \in L^2(0,1)$, one has:*

$$\int_0^1 g(s) b_{*u}(s)\,\mathrm{d}s = \int_0^1 M_{u,b}(g)(x) b(x)\,\mathrm{d}x.$$

Furthermore, one has for $x \in]0,1[\backslash P(u)$, $M_{u,b}(g)(x) = g(m_u(u(x)))$: if g vanishes on $P(u_*)$ then $M_{b,u}(g)(x) = 0$, for $x \in P(u)$.

3. Some theoretical results for some variational nonlocal problems

We begin this section by introducing an abstract result which will cover the resolution of $(\mathcal{P}_v)$ and of the discrete problem $(\mathcal{P}_v^h)$.

Through out this paper, we shall consider a Hilbert space $(V, \|\cdot\|)$ and a separable Banach space $(H, |\cdot|)$ satisfying:

H1) V is a continuously and compactly imbedded in H; thus, $\inf_{|v|=1} \|v\| > 0$.

H2) There exist a family of finite elements $V_h \subset V$ and a family of linear operators Π_h such that $\lim_{h\to 0} \|v - \Pi_h v\| = 0$, for all $v \in V$.

We also consider a nonlinear map G from V into the dual space H' of H satisfying:

H3) G is continuous from V-strong into H'-weak-star (*i.e.* for the topology $* - \sigma(H', H)$).

We then have the:

Theorem 3.1. *Assume that H1) to H3) hold and let $B\colon V \times V \to \mathbb{R}$ be a bilinear form which is coercive (in the sense that $\alpha = \inf_{\|v\|=1} B(v,v) > 0$), continuous (i.e. $\sup_{\|u\|=\|v\|=1} B(u,v) = M < +\infty$).*

Assume that G has the following growth:

H4) *There exists $0 < \lambda_0 < \alpha \inf_{|v|=1} \|v\|$ and $0 < \lambda_1 \in \mathbb{R}$ such that: $|G(v)|_* \leq \lambda_0 \|v\| + \lambda_1$ for all $v \in V$. Then,*

i) *there exists $u_h \in V_h$ such that*

$$B(u_h, v_h) = < G(u_h), v_h >, \qquad \forall v_h \in V_h$$

ii) *there exist $u \in V$ and a (subsequence) $u_h \in V$ such that u_h converges strongly to u in V, where u solves: $B(u,v) = < G(u), v >$ for all $v \in V$.*

Proof of Theorem 3.1

Let $m = \dim V_h$ and $\{\varphi_1, \cdots, \varphi_m\}$ be a basis of V_h. Define the following scalar product on V_h for $v = \sum_{j=1}^{m} v_j\varphi_j$ and $w = \sum_{j=1}^{m} w_j\varphi_j$: $[v,w] = \sum_{j=1}^{m} v_j w_j$. We introduce the map T_m: $V_h \to V_h$ by setting $T_m v = \sum_{j=1}^{m}[B(v,\varphi_j) - < G(v), \varphi_j >]\varphi_j$. To prove statement i) of Theorem 3.1, we see that for all $v \in V_h$

$$\begin{aligned}[T_m v, v] = B(v,v) - < G(v), v > \ &\geq \alpha\|v\|^2 - \lambda_0\|v\| \cdot |v| - \lambda_1|v| \\ &\geq \Big(\alpha \inf_{|z|=1} \|z\| - \lambda_0\Big)\|v\||v| - \lambda_1|v|.\end{aligned}$$

Thus, $[T_m v, v] \to +\infty$ as $[v,v] \to +\infty$. Furthermore, T_m is continuous, the continuity of B and G yield that. We conclude with Brouwer's fixed point theorem to obtain: the existence of $u_h \in V_h$ such that $T_m u_h = 0, B(u_h, v_h) =< G(u_h), v_h >$ for all $v_h \in V_h$ and $\|u_h\| \leq \text{Constant} = \dfrac{\lambda_1}{\alpha \inf\limits_{|z|=1} \|z\| - \lambda_0}$.

We consider $u \in V$ and a subsequence still denoted by u_h such that $u_h \rightharpoonup u$ weakly in V and H-strong and $G(u_h) \rightharpoonup \ell_u$ in H'-weak-star.

Let $v \in V$. Then, one has:

$$B(u_h, \Pi_h v) =< G(u_h), \Pi_h v > \tag{1}$$

$$|B(u_h, \Pi_h v) - B(u_h, v)| \leq (\text{Constant}) \cdot \|v - \Pi_h v\| \tag{2}$$

$$| < G(u_h), \Pi_h v > - < G(u_h), v > | \leq C_0\|v - \Pi_h v\|. \tag{3}$$

From relation (1) to (3), we deduce:

$$B(u,v) = \lim_{h\to 0} B(u_h, \Pi_h v) = \lim_{h\to 0} < G(u_h), v >=< \ell_u, v > . \tag{4}$$

Let us show that $\lim\limits_{h\to 0} \|u_h - u\| = 0$. It suffices to show that $\lim\limits_{h\to 0} B(u_h, u_h) = B(u,u)$. One has:

$$\lim_{h\to 0} < G(u_h), u_h >=< \ell_u, u > \ \text{ since } |u_h - u| \xrightarrow[h\to 0]{} 0,$$

that is $\lim\limits_{h\to 0} B(u_h, u_h) = B(u,u)$. By the continuity of G, we deduce that $\ell_u = G(u)$. □

Remark 3.1. Suppose that the map G is only continuous on a subset $\mathcal{V}$ of V containing all the V_h. In that case, statement i) as well as the strong convergence remain true. Furthermore, if we can show that $u \in \mathcal{V}$, then the conclusion of the second statement ii) is also true. Let us give an example of such a situation:

We replace assumption H3) by the following one:

H5) Let A be the linear continuous operator from V to V' defined by $< Av, w >= B(v,w)$ for all v, w and let us denote by $D(A)$ its domain. We assume that $H = H'$, G maps V into H and its restriction to $D(A) \cup \Big(\bigcup\limits_{h>0} V_h\Big) = \mathcal{V}$ is continuous from $(\mathcal{V}, \|\cdot\|)$-strong into H-weak.

Then:

Theorem 3.2. *Assume that H1), H2), H4), and H5) hold. If B is the same bilinear form as in Theorem 3.1, then we have the same conclusions as in Theorem 3.1.*

Proof. The proof of statement i) is the same as in Theorem 3.1, while for the statement ii), the function u is found as a strong limit of a sequence u_h, and belongs to $D(A)$ since $Au = \ell_u \in H$. Thus, $G(u_h) \to G(u)$ in H-weak, which implies that $\ell_u = G(u)$. □

Other situations will be given in the applications below when $\mathcal{V}$ is not a vector space. In order to verify the hypotheses satisfied by G when it involves the relative rearrangement b_{*u}, we introduce the following weak-convergence:

Lemma 3.1. *Let $v \in \mathrm{L}^1(0,1)$, $\theta = \chi_{\mathrm{P}(v_*)}$ the characteristic function of $P(v_*)$. If v_n tends to v in $\mathrm{L}^1(0,1)$-strong then the sequence $(1-\theta)b_{*v_n}$ converges weakly to $(1-\theta)b_{*v}$ in L^p-weak whenever $1 < p < +\infty$ and $b \in \mathrm{L}^p(0,1)$.*

Proof. Let $\varphi \in \mathcal{C}[0,1]$. Without loss of generality, we may assume that θ is continuous and equal to 1 on $P(v_*)$. Then, by the mean value theorem, we have

$$\int_0^1 \varphi(1-\theta)b_{*v_n}\mathrm{d}s = \int_0^1 M_{v_n,b}(\varphi(1-\theta))b(x)\mathrm{d}x. \tag{5}$$

For convenience, we introduce the following notations, for $x \in]0,1[$:

$$\begin{array}{ll} \beta_n(x) = |v_n > v_n(x)| & \gamma_n(x) = \beta_n(x) + |v_n = v_n(x)| \\ \beta(x) = |v > v(x)| & \gamma(x) = \beta(x) + |v = v(x)|. \end{array}$$

From the definition of the mean value operator, one can deduce that:

$$\inf_{\sigma\in[\beta_n(x),\gamma_n(x)]} \varphi(1-\theta)(\sigma) \le M_{v_n,b}(\varphi(1-\theta))(x) \le \sup_{\sigma\in[\beta_n(x),\gamma_n(x)]} \varphi(1-\theta)(\sigma). \tag{6}$$

From relation (6), we then deduce for all x:

$$\begin{aligned} &\inf_{\sigma\in[\beta(x),\gamma(x)]} \varphi(1-\theta)(\sigma) \le \liminf_n M_{v_n,b}(\varphi(1-\theta))(x) \le \\ &\limsup_n M_{v_n,b}(\varphi(1-\theta))(x) \le \sup_{\sigma\in[\beta(x),\gamma(x)]} \varphi(1-\theta)(\sigma). \end{aligned}$$

If $x \in P(v)$, then $[\beta(x),\gamma(x)] \subset P(v_*)$ and $(1-\theta)(\sigma) = 0$ for $\sigma \in [\beta(x),\gamma(x)]$, which implies that $\lim_{n\to\infty} M_{v_n,b}(\varphi(1-\theta))(x) = 0$. If $x \notin P(v)$, then $\beta(x) = \gamma(x) \notin P(v_*)$ and then $\lim_{n\to\infty} M_{v_n,b}(\varphi(1-\theta))(x) = \varphi(\beta)(x)$. We have shown that for all $x \in]0,1[$,

$$\lim_{n\to\infty} M_{v_n,b}(\varphi(1-\theta))(x) = (\varphi(1-\theta))(\beta(x)) = M_{v,b}(\varphi(1-\theta))(x).$$

We conclude with Lebesgue's and the mean value theorems to find:

$$\lim_{n\to\infty}\int_0^1 \varphi(1-\theta)b_{*v_n}\mathrm{d}s = \int_0^1 M_{v,b}(\varphi(1-\theta))b(x)dx = \int_0^1 \varphi(1-\theta)b_{*v}\mathrm{d}s \tag{7}$$

We end by a classical density argument. □

Remark 3.2. This lemma is true in any dimension and also for $p=+\infty$.

Lemma 3.2. *Let $v \in \mathrm{L}^1(0,1)$, $\theta=\chi_{P(v_*)}$ be the characteristic function of $P(v_*)$ and v_n be a sequence of $L^1(0,1)$ converging to v, almost everywhere and in $L^1(0,1)$. We set, for $x\in[0,1]$*

$$I(v_n)(x)=\Big[m_{v_n}\Big(v_{n+}(x)\Big)\Big),\ m_{v_n}(0)\Big]\quad\Big(resp\ I(v)(x)\Big)$$

and we denote by $\chi_{I(v_n)(x)}\Big(resp\ \chi_{I(v)(x)}\Big)$ the characteristic function of $I(v_n)(x)\Big(resp\ I(v)(x)\Big)$. Then, for all $\sigma\in[0,1], \sigma\neq|v>v_+(x)|,\ \sigma\neq|v>0|$, one has

$$\lim_{n\to+\infty}(1-\theta)(\sigma)\chi_{I(v_n)(x)}(\sigma)=(1-\theta)(\sigma)\chi_{I(v)(x)}(\sigma).$$

Proof. Let $\sigma\in[0,1], \sigma\neq|v>0|$ and $\sigma\neq|v>v_+(x)|$.
Note that we always have

$$|v>0|\le \liminf|v_n>0|\le\limsup|v_n>0|\le|v\ge 0|$$

and,

$$|v>v_+(x)|\le\liminf|v_n>v_{n+}(x)|\le\limsup|v_n>v_{n+}(x)|\le|v\ge v_+(x)|$$

So, if $\sigma<|v>0|$ or $\sigma>|v\ge v_+(x)|$, then for large n, $\chi_{I(v_n)(x)}(\sigma)=\chi_{I(v)(x)}(\sigma)=0$.
If $\sigma\in]|v\ge 0|,|v>v_+(x)|[$, then the same conclusion holds, that is $\chi_{I(v_n)(x)}(\sigma)=\chi_{I(v)(x)}(\sigma)=1$ for large n.
If $\sigma\in]|v>0|,|v\ge 0|]$ and $|v=0|>0$, or $\sigma\in]|v>v_+(x)|,|v\ge v_+(x)|]$ and $|v=v_+(x)|>0$, then $(1-\theta)(\sigma)=0$. □

Lemma 3.3. *Under the same assumptions as in Lemma 3.2, if $v_h\in\mathrm{H}_0^1(0,1)$ converges strongly to v, then for all $x\in[0,1]$*

$$\lim_{h\to 0}\int_0^1(1-\theta)\chi_{I(v_h)(x)}(\sigma)v'_{h_*}(\sigma)p'(v_{h_*})(\sigma)b_{*v_h}(\sigma)=\int_0^1(1-\theta)\chi_{I(v)(x)}(\sigma)v'_*(\sigma)p'(v_*)(\sigma)b_{*v}(\sigma)\mathrm{d}\sigma$$

whenever $b\in\mathrm{L}^2(0,1)$; $p'\in C(\mathbb{R})$ and $|p'(t)|\le c_2$, for all $t\in\mathbb{R}$, $p(0)=0$.

Proof. From Coron's result (see [13]), we have $v'_{h_*}\to v'_*$ in $\mathrm{L}^2(0,1)$ and $p'(v_{h_*})\to p'(v_*)$ in $\mathrm{L}^2(0,1)$.
Then, from the above result, we have $(1-\theta)\chi_{I(v_h)(x)}(\cdot)v'_{h_*}p'(v_{h_*})\to(1-\theta)\chi_{I(v)(x)}(\cdot)v'_*p'(v_*)$ in $\mathrm{L}^2(0,1)$-strong and $(1-\theta)b_{*v_h}\rightharpoonup(1-\theta)b_{*v}$ weakly in $\mathrm{L}^2(0,1)$.
Since $(1-\theta)^2=(1-\theta)$ (remember that θ is a characteristic function), we then deduce the result from the two last convergences. □
As a consequence of this lemma, we have the,

Corollary 3.1 (of Lemma 3.3). *Under the same assumptions as in Lemma 3.3, the map $v\in H^1(0,1)\to F(v)\in\mathrm{L}^2(0,1)$ is continuous for the strong topology. Here, $F(v)(x)=a(x)\Big[F_0^2-\displaystyle\int_{m_v(0)}^{m_v(v_+(x))}p'(v_*)v'_*b_{*v}\mathrm{d}t\Big]_+^{1/2}$.*

Proof. Let v_n be a sequence converging to a function v in $H^1(0,1)$. Let $\theta(.)$ be the characteristic function of $P(v_*)$. Since $v'_*(\sigma)=0$ whenever $\sigma\in P(v_*)$, one then has using Coron's continuity result that

$$\lim_{n\to\infty}\int_0^1|v'_{n*}(s)\theta(s)|^2\,\mathrm{d}s=0.$$

Using the fact that

$$|b_{*v_n}|_{L^2} \leq |b|_{L^2},$$

we derive that:

$$\lim_{n\to\infty} \int_0^1 \theta(s) v'_{n*}(s) p'(v_{n*}(s)\chi_{I(v_n)(x)}(s) b_{*v_n} \,\mathrm{d}s = 0. \tag{8}$$

Thus, if we write

$$\begin{aligned} \int_{m_{v_n}(0)}^{m_{v_n}(v_{n+}(x))} p'(v_{n*}) v'_{n*} b_{*v_n} \mathrm{d}t = & \int_0^1 (1-\theta(s)) v'_{n*}(s) p'(v_{n*}(s)\chi_{I(v_n)(x)}(s) b_{*v_n}(s)\,\mathrm{d}s \\ & + \int_0^1 \theta(s) v'_{n*}(s) p'(v_{n*}(s)\chi_{I(v_n)(x)}(s) b_{*v_n}(s)\,\mathrm{d}s, \end{aligned} \tag{9}$$

we can apply Lemma 3.3 and the above convergence to find that $F(v_n)(x) \to F(v)(x)\ \forall x$. From the main estimates on the monotone rearrangement and relative rearrangement (*see* Sect. 1), we derive that:

$$|F(v_n)(x)| \leq |a|_\infty \Big[F_0 + |b|^{1/2}_{L^2(0,1)} |p(v_n)|^{1/2}_{H^1(0,1)} \Big] \leq constant \tag{10}$$

We conclude with Lebesgue's theorem. □

Lemma 3.4. *i) For any $v \in H^1_0(0,1)$, the map $b \in L^2(0,1) \to \int_0^1 v'_*(s) b_{*v}\mathrm{d}s$ is linear and continuous.*

ii) For a fixed $b \in L^2(0,1)$, the map $v \in W^{1,2}(0,1) \to \int_0^1 v'_*(s) b_{*v}\mathrm{d}s$ is continuous for the strong topology of $W^{1,2}(0,1)$.

Proof. From the mean value theorem (see Lem. 2.2), one has $M_{v,b}(v'_*)(x) = v'_*\Big(m_v(v(x))\Big)$. Thus

$$\int_0^1 v'_*(s) b_{*v}\mathrm{d}s = \int_0^1 v'_*\Big(m_v(v(x))\Big) b(x)\mathrm{d}x$$

which shows the linearity of the map. The continuity is a consequence of Schwartz's inequality and the main inequalities for v'_* and b_{*v} (*see* Sect. 1, properties of the monotone and relative rearrangements). For the statement *ii)*, the proof is similar to that performed in Corollary 3.1 of Lemma 3.3. □

Theorem 3.3. *Let V_h be a family of finite elements in $H^1_0(0,1)$ such that there exists a family of linear operators Π_h from $H^1_0(0,1)$ in V_h satisfying $\lim_{h\to 0} |v - \Pi_h v|_{H^1_0(0,1)} = 0$. Then, there exist a solution u_h of $(\mathcal{P}^h_v)$ and a solution $u \in H^1_0(0,1) \cap H^2(0,1)$ of $(\mathcal{P}_v)$, provided that $|\lambda| < 1$.*

Proof. We define a function $G : H^1_0(0,1) \to L^2(0,1)$ by setting

$$(G(v),\varphi) = \lambda \int_0^1 v'_*(s)\varphi_{*v}(s)\mathrm{d}s + \int_0^1 F(v)(x)\varphi(x)\mathrm{d}x + \int_0^1 f(x)\varphi(x)\mathrm{d}x$$

for all $v \in H^1_0(0,1)$ for all, $\varphi \in L^2(0,1)$. We have

$$|G(v)|_{L^2(0,1)} \leq |\lambda||v'|_{L^2(0,1)} + |a|_\infty \Big[F_0 + |b|^{1/2}_{L^2(0,1)} |p(v)|^{1/2}_{H^1_0(0,1)} \Big] + |f|_{L^2(0,1)}. \tag{11}$$

By Young's inequality, one deduces that: $\forall\varepsilon > 0$, $\exists c_\varepsilon$ such that:

$$|G(v)|_{L^2(0,1)} \leq (|\lambda| + \varepsilon)|v|_{H_0^1(0,1)} + |a|_\infty c_\varepsilon. \tag{12}$$

Furthermore, the map G is continuous from $H_0^1(0,1)$-strong into $L^2(0,1)$-weak. (This is a consequence of Corollary 3.1 of Lem. 3.3 and Lem. 3.4). Setting $B(v,\varphi) = \displaystyle\int_0^1 v'\varphi'$, $V = H_0^1(0,1)$, $H = L^2(0,1)$, we then have, choosing $0 < \epsilon < 1 - |\lambda|$:

$$|\lambda| + \varepsilon < 1 \leq \inf_{|v|_V=1} B(v,v) \cdot \inf_{|v|_H=1} |v|_V.$$

We can apply Theorem 3.1 to deduce that there exists $u_h \in V_h$ such that $B(u_h, v_h) = (G(u_h), v_h)$ $\forall v_h \in V_h$ and $u \in V$ such that $B(u,v) = (G(u), v)$ $\forall v \in V$. Since $-u" = G(u) \in L^2(0,1)$, we deduce that $u \in H^2(0,1)$. □
From Theorem 3.1, we also deduce a stability result for the discrete problem, that can be written as:

$$|u_h'|_{\mathrm{L}^2(0,1)} \leq \frac{\sqrt{2}|f|_{\mathrm{L}^2} + |a|_\infty c}{1 - |\lambda|} \text{ for } |\lambda| < 1,$$

where c depends only on $a, b,$ F_0.
As an application of Theorem 3.2, one has the following existence result.

Theorem 3.4. *Let $f \in L^2(0,1), b \in L^\infty(0,1)$ $g \in C(\mathbb{R})$, $g \geq 0$, $b \geq 0$ and $f < 0$. Then, there exists $u \in H^2(0,1) \cap H_0^1(0,1)$ (non trivial) solution of*

$$\int_0^1 u'\varphi' + \int_0^1 g(u_*)b_{*u}\varphi_{*u} = \int_0^1 f\varphi$$

for all $\varphi \in H_0^1(0,1)$.

Proof. We set $\varphi_j(x) = \sqrt{2}\sin(j\pi x)$, $H = L^2(0,1)$, $V = H_0^1(0,1)$, and $V_m = \operatorname{span}\{\varphi_1, \varphi_2, \cdots, \varphi_m\}$

$$B(v,\varphi) = \int_0^1 v'\varphi' =< Av, \varphi >,$$

with $D(A) = H_0^1(0,1) \bigcap H^2(0,1)$, $\mathcal{V} = \{v \in D(A),$ measure $\{x : v'(x) = 0\} = 0\} \bigcup \{0\}$. For $v \in H_0^1(0,1)$, we define $G(v)$ as a solution of $\Big(G(v), \varphi\Big) = -\displaystyle\int_0^1 g(v_*)b_{*v}(s)\varphi_{*v}(s)\mathrm{d}s + \int_0^1 f\varphi$ for all $\varphi \in L^2(0,1)$. Therefore, $G(v) \in L^2(0,1)$ and one has

$$|G(v)|_{L^2(0,1)} \leq |g(v)|_\infty |b|_\infty + |f|_{L^2(0,1)}$$

By Theorem 1 of [46] and Lemma 3.1, we infer G is continuous from $\Big(\mathcal{V}, |\cdot|_{H_0^1(0,1)}\Big)$ into $L^2(0,1)$-weak. Thus H4) is satisfied. Assumptions H1), to H2) are easily checked. Since $b \geq 0$, $g \geq 0$, then there exists $h \geq 0$ such that $-u"(x) + h(x) = f(x)$, and if $|u' = 0| > 0$, then $h(x) = f(x)$ *a.e.* on the set $\{u' = 0\}$, which contradicts the fact that $f < 0$. Therefore, $u \in \mathcal{V}$. We may then apply Remark 1 of Theorem 3.1. Arguing by contradiction, we see that $u \neq 0$. □

4. Some qualitative aspects of a solution of $(\mathcal{P}_v)$ and $(\mathcal{P}_v^h)$

We start this section by studying the existence of symmetric solutions (with respect to $\frac{1}{2}$) whenever f is symmetric.

For a measurable function v on $[0,1]$, we set $v^s(x) = v(1-x),\ x \in [0,1]$.
We shall say that a function $v \in L^1(0,1)$ is symmetric if $v(x) = v(1-x)$ *a.e.* We set

$$L_s^2(0,1) = \{v \in L^2(0,1) \text{ such that } v \text{ is symmetric}\}$$

$$H_{0,s}^1(0,1) = H_0^1(0,1) \cap L_s^2(0,1).$$

Proposition 4.1. *Let u be a solution of $(\mathcal{P})$. Then, the function u^s is also a solution whenever $f \in L_s^2(0,1)$, a and b are also symmetric.*

Proof. First, we observe that u and u^s are equimeasurable (*i.e.* $m_u = m_{u^s}$). Thus, $u_*^s = u_*$ and for all $v \in L^2(0,1)$, we have from the mean value theorem and a change of variables,

$$\begin{aligned}\int_0^1 u_*'(t)v_{*u}^s(t)dt &= \int_0^1 (u_*^s)'\Big(|u^s > u(x)|\Big)v^s(x)\mathrm{d}x \\ &= \int_0^1 (u_*^s)'\Big(|u^s > u^s(x)|\Big)v(x)\mathrm{d}x \\ &= \int_0^1 (u_*^s)'(t)v_{*u^s}(t)\mathrm{d}t.\end{aligned} \tag{13}$$

If b is symmetric, then $b_{*u^s}^s = b_{*u^s} = b_{*u}$. By a simple change of variables, we then have:

$$\int_0^1 F(u)(x)v^s(x)\mathrm{d}x = \int_0^1 F(u^s)(x)v(x)\mathrm{d}x.$$

For any $v \in H_0^1(0,1)$, we have $v^s \in H_0^1(0,1)$ and

$$\int_0^1 u'(x)(v^s)'(x)\mathrm{d}x - \lambda\int_0^1 u_*'(t)v_{*u}^s(t)\mathrm{d}t - \int_0^1 F(u)(x)v^s(x)\mathrm{d}x = \int_0^1 f(x)v^s(x)\mathrm{d}x, \tag{14}$$

that is

$$-\int_0^1 u'(x)v'(1-x)\mathrm{d}x - \lambda\int_0^1 (u_*^s)'(t)v_{*u^s}(t)\mathrm{d}t - \int_0^1 F(u^s)(x)v(x)\mathrm{d}x = \int_0^1 f^s(x)v(x)\mathrm{d}x. \tag{15}$$

So if $f^s = f$, then this last equation reads

$$\int_0^1 (u^s)'(x)v'(x)\mathrm{d}x - \lambda\int_0^1 (u_*^s)'(t)v_{*u^s}(t)\mathrm{d}t - \int_0^1 F(u^s)(x)v(x)\mathrm{d}x = \int_0^1 f(x)v(x)\mathrm{d}x, \tag{16}$$

that is u^s is a solution of $(\mathcal{P})$. $\square$

Theorem 4.1 (Existence of a symmetric solution). *If $f \in L_s^2(0,1)$ and a and b are also symmetric, then there exists a symmetric solution $u \in H_0^1(0,1) \cap H^2(0,1)$.*

Proof. The set $H_{0,s}^1(0,1)$ of symmetric functions is a closed subset of $H_0^1(0,1)$. Thus, this space is a Hilbert separable space endowed with the usual scalar product of $H_0^1(0,1)$. So, let $\{\varphi_1,\cdots,\varphi_m,\cdots\}$ be a hilbertian basis of $H_{0,s}^1(0,1)$. We define V_m^s to be the vector space spanned by $\{\varphi_1,\cdots,\varphi_m\}$. Reconsidering the same operator T_m as in Theorem 2.1 defined by:

$$T_m v(x) = \sum_{j=1}^m a_j\varphi_j(x)$$

with

$$a_j = \int_0^1 v'(x)\varphi_j'(x)\,dx - \lambda \int_0^1 v_*'(x)\varphi_{j*v}(x)\,dx - \int_0^1 F(v)(x)\varphi_j(x)dx - \int_0^1 f(x)\varphi_j(x)\,dx.$$

We have, since φ_j is symmetric, $T_m v \in V_m^s$. So, the same argument as in the preceding paragraph (see Theor. 3.1 using Brouwer's fixed point theorem) shows the existence of $u_m^s \in V_m^s$ such that $T_m u_m^s = 0$. Thus, there exist a function $u \in H_{0,s}^1(0,1)$ and a subsequence still denoted by u_m^s such that $u_m^s \to u$ in $H_{0,s}^1(0,1)$ weak and uniformly in $C[0,1]$. The function u solves for all $v \in H_{0,s}^1(0,1)$

$$\int_0^1 u'(x)v'(x)dx - \lambda \int_0^1 u_*'(t)v_{*u}(t)dt - \int_0^1 F(u)(x)v(x)dx = \int_0^1 f(x)v(x)dx. \tag{17}$$

Let $v \in H_0^1(0,1)$. Then, $w = \dfrac{v+v^s}{2} \in H_{0,s}^1(0,1)$. So, one has from relation (17):

$$\begin{aligned} &\tfrac{1}{2}\Big[\int_0^1 u'(x)v'(x)dx - \lambda \int_0^1 u_*'(t)v_{*u}(t)dt - \int_0^1 F(u)(x)v(x)dx - \int_0^1 fv\Big] \\ &+\tfrac{1}{2}\Big[\int_0^1 u'(t)(v^s)'(t)dt - \lambda \int_0^1 u_*'v_{*u}^s - \int_0^1 F(u^s)(x)v(x)dx - \int_0^1 fv^s\Big] = 0. \end{aligned} \tag{18}$$

By a change of variables, one has:

$$\begin{aligned} &\int_0^1 u'(x)(v^s)'(x)\,dx = \int_0^1 (u^s)'(x)v'(x)\,dx = \int_0^1 u'(x)v'(x)\,dx \\ &\int_0^1 f(x)v^s(x)\,dx = \int_0^1 f^s(x)v(x)\,dx = \int_0^1 f(x)v(x)\,dx \\ &\int_0^1 F(u^s)(x)v^s(x)dx = \int_0^1 F(u)(x)v^s(x)dx = \int_0^1 F(u^s)(x)v(x)dx = \int_0^1 F(u)(x)v(x)dx. \end{aligned} \tag{19}$$

So by relations (13, 18, 19), one finally obtains:

$$\int_0^1 u'(x)v'(x)\,dx - \lambda \int_0^1 u_*'(x)v_{*u}(x)\,dx - \int_0^1 F(u)(x)v(x)dx = \int_0^1 f(x)v(x)\,dx \text{ for all } v \in H_0^1(0,1).$$

□

Theorem 4.2 (Maximum principle). *Let u be a solution of ($\mathcal{P}$). If $\lambda \le 0, a \le 0$ (resp $\lambda \ge 0, a \ge 0$) and $f \ge 0$ (resp $f \le 0$), then $u \ge 0$ (resp $u \le 0$).*

Proof. We set $M(t) = -t_- = \min(t,0)$. Then, $M(u) \in H_0^1(0,1)$. We obtain (using Sect. 1 and the fact that $a \le 0$):

$$\int_{\{u\le 0\}} (u')^2(x)dx - \lambda \int_0^1 u_*' M(u_*) = \int_0^1 fM(u) + \int_0^1 F(u)(x)M(u)(x)dx \le 0, \tag{20}$$

that is

$$\int_{\{u\le 0\}} (u')^2(x)dx + \lambda \int_{u_*(1)}^{u_*(0)} M(t)\,dt \le 0 \tag{21}$$

Since $\lambda \le 0$, one has:

$$\int_{\{u\le 0\}} (u')^2(x)dx = 0 = \int_{\inf u}^{\sup u} t_- dt \ : \ \inf u \ge 0.$$

The proof of the second case (*i.e.* $\lambda \geq 0, a \geq 0,\ f \leq 0$) is the same as above, replacing $-t_-$ by t_+. Indeed, in this case, we have the following identity:

$$\int_{\{u\geq 0\}} (u')^2(x)\mathrm{d}x + \frac{\lambda}{2}u_{+*}^2(0) = \int_0^1 f(x)u_+(x)\,\mathrm{d}x + \int_0^1 F(u)(x)u_+(x)\mathrm{d}x. \tag{22}$$

□

Theorem 4.3. *Let u be a solution of ($\mathcal{P}$). Then*

i) If $f \neq 0, \lambda a \leq 0$ in $[0,1]$, u_- has no plateau, that is $\mathrm{meas}\Big(P(u_-)\Big) = 0$.

ii) If $\lambda f < 0, \lambda a < 0$ in $[0,1]$, then the set $\{x\colon u'(x) = 0\}$ is of measure zero.

Proof. If $\mathrm{meas}(P(u_-)) \neq 0$, then there exists a number $t \leq 0$ such that $|u = t| > 0$. Since $u \in H^2(0,1)$, $u''(x) = 0 = \dfrac{\mathrm{d}^+u_*}{\mathrm{d}s}\Big(m_u(u(x))\Big)$ *a.e.* $x \in \{u = t\}$. Using the equation satisfied by u, one has $f(x) = 0$ *a.e.* on $\{u = t\}$. This contradicts the assumption on f.
The proof of the second statement follows a similar idea, *i.e.* if the measure of the set $\{x\colon u'(x) = 0\}$ is positive, then $u''(x) = 0$ on that set and then equation ($\mathcal{P}$) leads to:

$$0 \leq -\lambda^2\frac{\mathrm{d}^+u_*}{\mathrm{d}s}\Big(|u > u(x)\Big) - \lambda a(x)F_0 = \lambda f(x) < 0.$$

□

5. Numerical approximation of the monotone and relative rearrangements of piecewise linear functions. The monodimensional case

We begin this section by some results concerning the properties of the monotone and relative rearrangements of piecewise linear continuous functions.

Let $\Delta = \{0 = x_0 < x_1 < \dots < x_{N+1} = 1\}$ be a mesh on $[0,1]$. and below $\{\varphi_j\}_{j=0}^{N+1}$ denotes the basis of the piecewise linear functions space relative to Δ, defined by $\varphi_j(x_i) = \delta_{ij}$.

The ordered values

$$\Delta_u = \mathrm{sort}\,\{u(x_0), u(x_1), \dots u(x_{N+1})\} = \{t_0 = \min u_j < t_1 < \dots < t_M = \max u_j\}$$

$$\Delta_m = \{|\Omega| = m_0 = m_u(t_0) \geq m_1 = m_u(t_1) \geq \dots \geq m_M = m_u(t_M) = 0\}$$

give us the meshes of $I_u = [\min u_j, \max u_j]$ and $\Omega_* = (0, |\Omega|)$, (duplicated values must be suppressed and these meshes may have $M+1 < N+2$ points).

Proposition 5.1. *Let $u(x) = \sum_{j=0}^{N+1} \alpha_j\varphi_j$ be a piecewise linear continuous function related to the mesh Δ. Then, the distribution function m_u and the decreasing rearrangement are also the piecewise linear functions related to meshes Δ_u and Δ_m. Furtheremore,*

$$m_u(t) = \begin{cases} \dfrac{m_{i+1}t_i - m_it_{i+1}}{t_i - t_{i+1}} + \dfrac{m_{i+1} - m_i}{t_{i+1} - t_i}t & \text{if } t \in [t_i, t_{i+1}] \qquad i = 0, \dots, M-1, \\ 0 & \text{if } t > t_M \\ |\Omega| & \text{if } t < t_0 \end{cases} \tag{23}$$

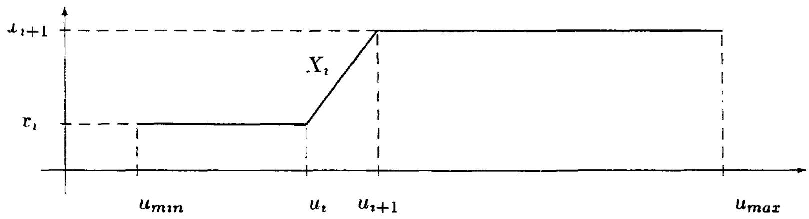

FIGURE 1. Cases $a_i{}^1 > 0$.

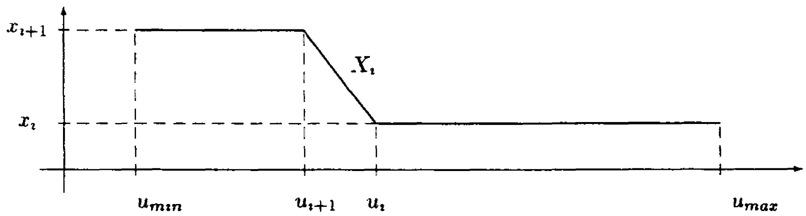

FIGURE 2. Cases $a_i{}^1 < 0$.

and

$$u_*(s) = \frac{t_{i+1} - t_i}{m_{i+1} - m_i}\, s - \frac{m_i t_{i+1} - m_{i+1} t_i}{m_{i+1} - m_i} \quad \textit{if} \;\; s \in [m_{i+1}, m_i]. \tag{24}$$

Proof. Let $a_0^i = \dfrac{u_i x_{i+1} - u_{i+1} x_i}{x_{i+1} - x_i}$ and $a_1^i = \dfrac{u_{i+1} - u_i}{x_{i+1} - x_i}$ be defined on each interval. Let

$$X_i(t) = \begin{cases} x_i & \text{if } t > u_i, a_1^i < 0 \\ x_i & \text{if } t < u_i, a_1^i > 0 \\ \dfrac{t - a_0^i}{a_1^i} & \text{if } t \in (\min\{u_i, u_{i+1}\}, \max\{u_i, u_{i+1}\}) = I_{u_i} \\ x_{i+1} & \text{if } t > u_{i+1}, a_1^i > 0 \\ x_{i+1} & \text{if } t < u_{i+1}, a_1^i < 0 \end{cases}$$

be the extended inverse of $u|_{[x_i, x_{i+1}]}$ showed in Figures 1–2 if $a_1^i \neq 0$.

The measure can be exactly computed by:

$$m_u(t^+) = \sum_{\substack{j \in \mathcal{N}(t) \\ a_1^j > 0}} [x_{j+1} - X_j(t)] + \sum_{\substack{j \in \mathcal{N}(t) \\ a_1^j < 0}} [X_j(t) - x_j] + \sum_{\substack{j \in \mathcal{N}(t) \\ a_1^j = 0}} [x_{j+1} - x_j] \tag{25}$$

where $\mathcal{N}(t) = \{i \colon t < \max\{u_i, u_{i+1}\}\}$. Since the functions X_i are piecewise linear continuous, it is the same for m_u if u has not plateau. In another case, $|u = t_i| > 0$ is the jump at the value t_i. □

Remark 5.1. Derivatives of u_* (respectively m_u) are not defined at the mesh points but they are constant on each interval (m_{i+1}, m_i) (repectively (t_i, t_{i+1})). Inside each interval we have:

$$m_u'(t) = \sum_{\substack{j\in\mathcal{N}(t)\\ a_1^j>0}} -X_j'(t) + \sum_{\substack{j\in\mathcal{N}(t)\\ a_1^j<0}} X_j'(t) = \sum_{j\in\mathcal{N}_*(t)} -\mathrm{sig}(a_1^j)\,\frac{x_{j+1}-x_j}{u_{j+1}-u_j} \tag{26}$$

where $\mathcal{N}_*(t) = \{i.\ \min\{u_i, u_{i+1}\} < t < \max\{u_i, u_{i+1}\}\}$, if $t \neq u_j$ for all $j = 0, 1, \dots N+1$.

Proposition 5.2. *If b is a piecewise linear positive function, $m_u^b(t) = \displaystyle\int_{\{x\ u(x)>t\}} b(x)\,\mathrm{d}x$ is a piecewise quadratic function related to mesh Δ_u.*

Proof. As in the previous proposition, we find the b-weighted measure

$$\begin{aligned} m_u^b(t^+) &= \sum_{\substack{j\in\mathcal{N}(t)\\ a_1^j>0}} \int_{X_j(t)}^{x_{j+1}} b(x)\,\mathrm{d}x + \sum_{\substack{j\in\mathcal{N}(t)\\ a_1^j<0}} \int_{x_j}^{X_j(t)} b(x)\,\mathrm{d}x + \sum_{\substack{j\in\mathcal{N}(t)\\ a_1^j=0}} \int_{x_j}^{x_{j+1}} b(x)\,\mathrm{d}x \\ &= \sum_{\substack{j\in\mathcal{N}(t)\\ a_1^j=0}} [g(x_{j+1}) - g(X_j(t))] + \sum_{\substack{j\in\mathcal{N}(t)\\ a_1^j<0}} [g(X_j(t)) - g(x_j)] + \sum_{\substack{j\in\mathcal{N}(t)\\ a_1^j=0}} g(x_{j+1}) - g(x_j) \end{aligned}$$

where g is a primitive of b, which is a quadratic function. □

Theorem 5.1. *Let b and u be piecewise linear functions, where u has not plateau. Then the relative rearrangement is a piecewise linear function.*

Proof. Following Propositions 5.1 and 5.2, it suffices, in order to compute the b-measures, to solve the linear system:

$$\left.\begin{aligned} \alpha_i t_i^2 + \beta_i t_i + \gamma_i &= m_u^b(t_i) \\ \alpha_i t_{i+\frac{1}{2}}^2 + \beta_i t_{i+\frac{1}{2}} + \gamma_i &= m_u^b(t_{i+\frac{1}{2}}) \\ \alpha_i t_{i+1}^2 + \beta_i t_{i+1} + \gamma_i &= m_u^b(t_{i+1}) \end{aligned}\right\} \tag{27}$$

at extremities and at the middle point, $t_{i+\frac{1}{2}}$, in the interval $[t_i, t_{i+1}]$. Consequently, the relative rearrangement is a piecewise linear function inside each interval (m_{i+1}, m_i):

$$b_{*u}(s) = \frac{m_u^{b'}(u_*(s))}{m_u'(u_*(s))} = \frac{2\alpha_i u_*(s) + \beta_i}{\dfrac{m_{i+1}-m_i}{t_{i+1}-t_i}}, \tag{28}$$

where the coefficients α_i, β_i are given on each interval (t_i, t_{i+1}) by

$$\alpha_i = \frac{2}{\Delta t_i^2}(m_{i+1}^b - 2m_{i+\frac{1}{2}}^b + m_i^b) \tag{29}$$

$$\beta_i = \frac{m_{i+1}^b - m_i^b}{\Delta t_i} - \frac{2}{\Delta t_i^2}(m_{i+1}^b - 2m_{i+\frac{1}{2}}^b + m_i^b)(t_{i+1} + t_i) \tag{30}$$

whith $\Delta t_i = t_{i+1} - t_i$. □

Remark 5.2. If b is continuous and u is monotone, $\mathcal{N}_*(t)$ (*see* Rem. 4.1) must a unique element, and the relative rearrangement is continuous. But, in another case, there are two or more elements in the set $\mathcal{N}_*(t)$, and the continuity can not be assured as we will show in the example below.

Example. Let $\Omega = (0, \frac{5}{2})$, $\Omega_* = (0, |\Omega|) = (0, \frac{5}{2})$, $b(x) = 3x$ and

$$u(x) = \begin{cases} x & 0 \le x \le 1 \\ \frac{1}{2}x + \frac{1}{2} & 1 < x \le 2 \\ \dfrac{7}{2} - x & 2 < x \le \frac{5}{2} \end{cases}$$

In this case

$$b_{*u}(s) = \begin{cases} 6 - s & 0 \le s < \frac{3}{2} \\ 3(\frac{5}{2} - s) & \frac{3}{2} \le s \le \frac{5}{2} \end{cases}$$

which is not continuous at $\frac{3}{2}$.

6. Numerical solution of the variational nonlocal problem

We shall begin by solving the variational problem $(\mathcal{P}_v)$ in the case $a(x) = 0$. If f is a symmetric function and we take a suitable mesh, the discrete problem turns into a linear system with tridiagonal matrix but, in more general cases, the nonlocal terms will be treated by a fixed point algorithm.

In order to find a numerical solution we consider the usual $\mathbb{P}_1$ finite element approach.

Let $\Delta = \{0 = x_0 < x_1 < \cdots < x_{N+1} = 1\}$ be a mesh in the interval $[0, 1]$. We consider the space $V_h = \Big\{v \in C^0[0,1]\colon v(0) = v(1) = 0,\ v|_{[x_i, x_{i+1}]} \in \mathbb{P}_1, \text{ for all } i = 0, 1, \ldots N\Big\}$. Thus, the discrete variational problem reads:

$$(\mathcal{P}^h_{v0}) \begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ \displaystyle\int_0^1 u'_h(x) v'_h(x)\,\mathrm{d}x - \lambda \int_0^1 u'_{h*}(\sigma)(v_h)_{*u_h}(\sigma)\,\mathrm{d}\sigma - \int_0^1 f(x) v_h(x)\,\mathrm{d}x = 0 \text{ for all } v_h \in V_h. \end{cases}$$

We are looking for a solution of $(\mathcal{P}^h_{v0})$ without plateau. In that case, by using previous results, we obtain

$$\begin{aligned} \int_0^1 u'_{h*}(\sigma)(v_h)_{*u_h}(\sigma)\,\mathrm{d}\sigma &= \sum_{i=0}^{M-1} \int_{m_{i+1}}^{m_i} u'_{h*}(\sigma)(2\alpha_i u_{h*}(\sigma) + \beta_i)\left(\frac{t_{i+1} - t_i}{m_{i+1} - m_i}\right)\mathrm{d}\sigma \\ &= \sum_{i=0}^{M-1} (v_h)_{*u_h}(m_{i+\frac{1}{2}})(u_{h*}(m_i) - u_{h*}(m_{i+1})). \end{aligned}$$

As $(v_h)_{*u}$ is continuous on each interval, values $(v_h)_{*u_h}(m_{i+\frac{1}{2}})$ are well defined. Thus, the variational formulation can be rewritten as:

$$\begin{cases} \{\text{Find}\}\mathrm{u_h} \in \mathrm{V_h} \text{ such that} \\ \displaystyle\int_0^1 u'_h(x) v'_h(x)\,\mathrm{d}x - \lambda \sum_{i=1}^{M-1} (v_h)_{*u_h}(m_{i+\frac{1}{2}})(u_{h*}(m_i) - u_{h*}(m_{i+1})) - \int_0^1 f(x) v_h(x)\,\mathrm{d}x = 0 \\ \hfill \text{for all } v_h \in V_h. \end{cases}$$

The previous variational problem amounts to solving a finite dimensional system associated with a basis $\{\varphi_k\}_{k=1}^N$ of V_h:

$$(\mathcal{P}_{v0}^h)' \begin{cases} \text{Find } u_h = \sum_{k=1}^{N} u_k\varphi_k \in V_h \text{ such that} \\ \int_0^1 u_h'(x)\varphi_k'(x)\,\mathrm{d}x - \lambda \sum_{i=0}^{M-1} (\varphi_k)_{*u_h}(m_{i+\frac{1}{2}})(u_{h*}(m_i) - u_{h*}(m_{i+1})) - \int_0^1 f(x)\varphi_k(x)\,\mathrm{d}x = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for all } k = 1, 2, \dots N \end{cases}$$

The key of this result is the linearity in the test function v_h proved in Lemma 2.4.

Values m_i and $(\varphi_k)_{*u}(m_{i+\frac{1}{2}})$ can be easily computed by using the mesh values u_i and a suitable function η: $\{0, 1, 2, \dots N+1\} \to \{0, 1, 2, \dots M\}$ such that $t_{\eta(i)} = u_i$ (η is well defined since there are not repeated values t).

6.1. The symmetric case

The biggest difficults to write the nonlinear system $(\mathcal{P}_{v0}^h)'$ is the explicit expression of the measures and rearrangements as function of the values $\{u_0, u_1, u_2, \dots u_N, u_{N+1}\}$, the mesh $\{x_0, x_1, x_2, \dots x_N, x_{N+1}\}$ and the permutation η. Even though, we can compute them by using formulas (23-24-28-29-30), we shall have "nice" equations only in very particular cases.

Thanks to the qualitative properties of the solution, if f is a positive symmetric function ($f(x) = f(1-x)$), we can suppose that u has not plateau and that it is positive, symmetric respect to the middle point in the interval $(0,1)$ (*i.e.* $u(x) = u(1-x)$), which is its unique maximum, and the minimum is reached on the boundary. Furthermore, we are going to consider a spatial mesh $\Delta\,\{0 = x_0 < x_1 < \dots < x_{N+1} = 1\}$ with equidistant points, (*i.e.* $h = x_i - x_{i-1}$ for all $i = 1, \dots N+1$) and N odd (so, the middle point of the interval is $x_{\frac{N+1}{2}}$ and the discrete solution has no plateau). Then, $M = \dfrac{N+1}{2}$ and

$$\eta(i) = \begin{cases} i & 0 \le i \le \dfrac{N+1}{2} \\ N - i + 1 & \dfrac{N+1}{2} \le i \le N+1 \end{cases}$$

Now, the relative rearrangements of the piecewise linear basis related to mesh Δ can be exactly computed. More precisely we have

Proposition 6.1. *i) If* $1 \le i < \dfrac{N+1}{2}$, *then*

$$\varphi_{i*u}(s) = \begin{cases} \dfrac{X_{i-1}(u_*(s)) - x_{i-1}}{2h} & \text{if } s \in (m_i, m_{i-1}) \\ -\dfrac{X_i(u_*(s)) - x_{i+1}}{2h} & \text{if } s \in (m_{i+1}, m_i) \\ 0 & \text{otherwise.} \end{cases}$$

ii) If $\dfrac{N+1}{2} < i \le N$, *then*

$$\varphi_{i*u}(s) = \begin{cases} -\dfrac{X_i(u_*(s)) - x_{i+1}}{2h} & \text{if } s \in (m_{\eta(i)}, m_{\eta(i+1)}) \\ \dfrac{X_{i-1}(u_*(s)) - x_{i-1}}{2h} & \text{if } s \in (m_{\eta(i-1)}, m_{\eta(i)}) \\ 0 & \text{otherwise.} \end{cases}$$

iii) If $i = \dfrac{N+1}{2}$, *then*

$$\varphi_{i*u}(s) = \begin{cases} \dfrac{X_{i-1}(u_*(s)) - x_{i-1}}{h} & \text{if } s \in (m_i, m_{i-1}) \\ 0 & \text{otherwise.} \end{cases}$$

□

By using this expressions in $(\mathcal{P}^h_{v0})'$ we obtain the *linear* problem

$$(\mathcal{P}_l) \begin{cases} \text{Find } u \in V_h \text{ such that} \\ u_0 = 0 \\ \displaystyle\int_{x_{i-1}}^{x_{i+1}} u'(x)\varphi_i'(x)\,\mathrm{d}x - \lambda\frac{1}{4}\left[u_{i-1} - u_{i+1}\right] - \int_{x_{i-1}}^{x_{i+1}} f(x)\varphi_i(x)\,\mathrm{d}x = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for all } i = 1, \dots \dfrac{N-1}{2} \\ \displaystyle\int_{x_{i-1}}^{x_{i+1}} u'(x)\varphi_i'(x)\,\mathrm{d}x - \lambda\frac{1}{4}\left[u_{i-1} - 2u_i + u_{i+1}\right] - \int_{x_{i-1}}^{x_{i+1}} f(x)\varphi_i(x)\,\mathrm{d}x = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } i = \dfrac{N+1}{2} \\ \displaystyle\int_{x_{i-1}}^{x_{i+1}} u'(x)\varphi_i'(x)\,\mathrm{d}x - \lambda\frac{1}{4}\left[u_{i+1} - u_{i-1}\right] - \int_{x_{i-1}}^{x_{i+1}} f(x)\varphi_i(x)\,\mathrm{d}x = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for all } i = \dfrac{N+3}{2}, \dots N, \\ u_{N+1} = 0 \end{cases}$$

which can be reduced to a *linear* system whose matrix reads

$$\mathcal{A}_h = \left(\begin{array}{cccc|c|cccc} \frac{2}{h} & -\frac{1}{h}+\frac{\lambda}{4} & & & 0 & 0 & \cdots & 0 & 0 \\ -\frac{1}{h}-\frac{\lambda}{4} & \frac{2}{h} & -\frac{1}{h}+\frac{\lambda}{4} & & 0 & 0 & \cdots & 0 & 0 \\ & \ddots & \ddots & \ddots & \vdots & & & \vdots & \vdots \\ & & -\frac{1}{h}-\frac{\lambda}{4} & \frac{2}{h} & -\frac{1}{h}+\frac{\lambda}{4} & 0 & \cdots & 0 & 0 \\ \hline 0 & \cdots & 0 & -\frac{1}{h}-\frac{\lambda}{4} & \frac{2}{h}+\frac{\lambda}{2} & -\frac{1}{h}-\frac{\lambda}{4} & \cdots & 0 & 0 \\ \hline 0 & \cdots & 0 & 0 & -\frac{1}{h}+\frac{\lambda}{4} & \frac{2}{h} & -\frac{1}{h}-\frac{\lambda}{4} & & \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 0 & 0 & & -\frac{1}{h}+\frac{\lambda}{4} & \frac{2}{h} & -\frac{1}{h}-\frac{\lambda}{4} \\ 0 & \cdots & 0 & 0 & 0 & & & -\frac{1}{h}+\frac{\lambda}{4} & \frac{2}{h} \end{array}\right)$$

This matrix is a M-matrix if $0 < \lambda < \frac{4}{h}$ (see [61]) and satisfies a discrete maximum principle (if $\mathbf{b} > \mathbf{0}$, then the solution of $\mathcal{A}_h\mathbf{u} = \mathbf{b}$ satisfies $\mathbf{u} > \mathbf{0}$) according to the results for the continuous problem.

6.2. The nonsymmetric case

When f is not symmetric, we cannot expect a symmetric solution. Then, the nonlinear system $(\mathcal{P}_{v0}^h)'$ is solved by using a fixed point algorithm.

- Given $u_h^{(0)} \in \hat{V}_h = \left\{v \in C^0[0,1]\colon v(0) = c_0,\ v(1) = c_1,\ v|_{[x_\imath, x_{\imath+1}]} \in \mathbb{P}_1, \text{ for all } \imath = 0, 1, \dots N\right\}$, for $l = 1, 2, \dots$, we find $u_h^{(l)} = c_0\varphi_0 + \sum_{k=1}^{N} u_k^{(l)}\varphi_k + c_1\varphi_{N+1} \in \hat{V}_h$ such that

$$\left\{\begin{array}{l} \displaystyle\int_0^1 (u_h^{(l)})'(x)\varphi_k'(x)\,\mathrm{d}x = \\ \displaystyle\lambda \sum_{\imath=0}^{M-1} (\varphi_k)_{*u_h^{(l-1)}}(m_{\imath+\frac{1}{2}}^{(l-1)})(u_{h*}^{(l-1)}(m_\imath^{(l-1)}) - u_{h*}^{(l-1)}(m_{\imath+1}^{(l-1)})) + \int_0^1 f(x)\varphi_k(x)\,\mathrm{d}x \\ \hfill \text{for all } k = 1, 2, \dots N. \end{array}\right. \tag{31}$$

In order to compute relative and decreasing rearrangements of φ_k and $u_h^{(l-1)}$ in each iteration, we must find the table (the index $(l-1)$ is drooped)

$\mathbf{t}$	$\mathbf{m}$	β^k	$\bar{\Phi}^k$
t_0	m_0	β_0^k	ϕ_0^k
t_1	m_1	β_1^k	ϕ_1^k
$\vdots$	$\vdots$	$\vdots$	$\vdots$
t_M	m_M	β_M^k	ϕ_M^k

$$\beta_\imath^k = \int_{\left\{u_h^{(l-1)} > u_{h*}^{(l-1)}(m_\imath)\right\}} \varphi_k(x)\,\mathrm{d}x \qquad\qquad \phi_\imath^k = \frac{\beta_{\imath+1}^k - \beta_\imath^k}{m_{\imath+1} - m_\imath} \quad \imath = 0, \dots M-1$$

where the two final columns correspond to a generic basis function φ_k.

The measures $\beta_\imath^k$ are exactly computed by a numerical quadrature formula of low order (for instance the middle point or the trapezoidal rules). Furthermore, if $u_h^{(l-1)}$ is without flat regions, the measures are continuous functions and $m_\imath = m_{u_h^{(l-1)}}(t_\imath)$, $\beta_\imath^k = m_{u_h^{(l-1)}}^{\varphi_k}(t_\imath)$. So, from (28), we have $\phi_\imath^k = (\varphi_k)_{*u_h^{(l-1)}}\left(\dfrac{m_\imath + m_{\imath+1}}{2}\right)$. Finally, we have $u_{h*}^{(l-1)}(m_\imath) = t_\imath$ and the rightside in (31) can be evaluated.

The same technique works for solving the full problem:

$$(\mathcal{P}_v^h)' \left\{\begin{array}{l} \text{Find } u_h = \displaystyle\sum_{k=1}^{N} u_k\varphi_k \in V_h \text{ such that} \\ \displaystyle\int_0^1 u_h'(x)\varphi_k'(x)\,\mathrm{d}x - \lambda \sum_{\imath=0}^{M-1} (\varphi_k)_{*u_h}(m_{\imath+\frac{1}{2}})(u_{h*}(m_\imath) - u_{h*}(m_{\imath+1})) - \int_0^1 F(u_h)(x)\varphi_k(x)\,\mathrm{d}x = \\ \hfill = \displaystyle\int_0^1 f(x)\varphi_k(x)\,\mathrm{d}x \qquad \text{for all } k = 1, 2, \dots N \end{array}\right.$$

where $F(u_h)(x) = a(x)\left[F_0^2 - \displaystyle\int_{m_{u_h}(0)}^{m_{u_h}(u_{h+}(x))} p'(u_{h*})(s)u_{h*}'(s)b_{*u_h}(s)\,\mathrm{d}s\right]_+^{1/2}$.

For each fixed point iteration we must find $u^{(l)} \in \hat{V}_h$ the solution of

$$\begin{cases} \displaystyle\int_0^1 (u_h^{(l)})'(x)\varphi_k'(x)\,\mathrm{d}x = \int_0^1 F(u_h^{(l-1)})(x)\varphi_k(x)\,\mathrm{d}x + \\ \displaystyle +\lambda\sum_{i=0}^{M-1}(\varphi_k)_{*u_h^{(l-1)}}(m_{i+\frac{1}{2}}^{(l-1)})(u_{h*}^{(l-1)}(m_i^{(l-1)}) - u_{h*}^{(l-1)}(m_{i+1}^{(l-1)})) + \int_0^1 f(x)\varphi_k(x)\,\mathrm{d}x \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } k = 1,2,\ldots N. \end{cases}$$

The relative rearrangement $b_{*u_h^{(l-1)}}$ in the new term $F(u_h^{(l-1)})$ can be approached inside the intervals $[m_{i+1}, m_i]$ by w_i where

$$w_i = \frac{m_{i+1}^b - m_i^b}{m_{i+1} - m_i}, \qquad\qquad m_i^b = \int_{\left\{u_h^{(l-1)} > u_{h*}^{(l-1)}(m_i)\right\}} b(x)\,\mathrm{d}x, \quad i = 0,\ldots M-1,$$

just by adding two columns in the above table.

6.3. Numerical results

Firstly, we have tested the accuracy and the influence of the mesh on several problems for the simplified model ($a(x) = 0$). More precisely, given a data function f, we compute the solution u of

$$\begin{cases} -u''(x) - \lambda u_*'(m_u(u(x))) = f(x) \text{ in } (0,1) \\ u(0) = c_0 \qquad u(1) = c_1 \end{cases}$$

in the cases:

Test 1: $f(x) = \pi^2\sin(\pi x) + \dfrac{\lambda\pi}{2}|\cos(\pi x)|$ (the exact solution is $u(x) = \sin(\pi x)$) and,

Test 2: $f(x) = 2 + \dfrac{\lambda}{2}\sqrt{1 - 4x(1-x)}$ (the exact solution is $u(x) = x(1-x)$).

These problems have been solved by using five meshes indexed by their number of points, N, in view of to see the influence of the nonlocal term in the approximation order.

Table 1. Errors in L^∞-norm.

N	11	21	31	41	81	91
Test 1	4.28×10^{-3}	1.08×10^{-3}	4.82×10^{-4}	2.71×10^{-4}	6.76×10^{-5}	5.35×10^{-5}
Test 2	1.11×10^{-3}	2.76×10^{-4}	1.23×10^{-4}	6.91×10^{-5}	1.73×10^{-5}	1.36×10^{-5}

The table 1 shows the same approximation order as in the linear problem (Test 1) but there is not exactly computation when the solution belongs to $\mathbb{P}_2$ (Test 2).

Next, a more general case is considered, where f and, consequently, the solution u are nonsymmetric:

$$\text{Test 3: } f(x) = -2 - \lambda\begin{cases} \dfrac{4(1-\sqrt{0.25 + 4x(x-0.5)}) - 3}{2} & \text{if } x \in (0, 0.5) \\ \dfrac{-1 - 2\sqrt{0.25 + 4x(x-0.5)}}{8} & \text{if } x \in (0.5, 1). \end{cases}, u(x) = x(x - 0.5).$$

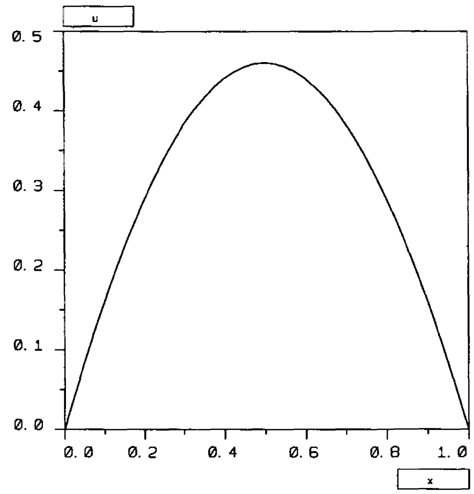

FIGURE 3. Solution.

Now, the analyse in Section 6.1 is no more valid, but the difficulty coming from the nonlocal terms can be solved by using the fixed point method. Errors in L^∞-norm are repported here below.

N	11	21	31	41	81	91
Test 3	2.40×10^{-3}	1.18×10^{-3}	9.34×10^{-4}	6.72×10^{-4}	3.56×10^{-4}	3.26×10^{-4}

In order to test the influence of the approximation of the relative rearrangement of a given function b, we take $f(x) = \pi^2 \cos(\pi x) - \lambda\, g(\cos(\pi x)) \sin(\pi x)$, $b(x) = \sin(\pi x)$ and we solve

$$\begin{cases} -u''(x) - \lambda g(u(x)) b_{*u}(m_u(u(x))) = f(x) \text{ in } (0,1) \\ u(0) = c_0 \qquad u(1) = c_1. \end{cases}$$

The convergence of fixed point iterations is not assured since the relative rearrangement does not satisfy a Lipschitz condition related to u. Nevertheless, in some particular cases, it can be proved that

$$\|b_{*u} - b_{*v}\| \leq L(b)\|u - v\|$$

(see [46] for details).

We pose $g(u) = u$ in Test 4 and $g(u) = u^2$ in Test 5 (in both cases the exact solution is $u(x) = \cos(\pi x)$) and we find the same order of approximation as in the linear problem:

N	11	21	31	41	81	91
Test 4	8.40×10^{-4}	2.10×10^{-4}	9.33×10^{-5}	5.26×10^{-5}	1.32×10^{-5}	1.04×10^{-5}
Test 5	2.77×10^{-3}	6.93×10^{-4}	3.10×10^{-4}	1.74×10^{-4}	4.36×10^{-5}	3.44×10^{-5}

Finally, we solve $(\mathcal{P}_v)$ with $f(x) = 0$, $b(x) = x^2$, $a(x) = 1$, $p(x) = \dfrac{x^2}{2}$ and $F_0 = 4$. The computed solution is given in Figure 3.

Remark 6.1. Analogous discretization and approximation techniques can be employed for $2 - D$ and $3 - D$ nonsymmetric problems (*see* [5] for $(\mathcal{PS})$).

Acknowledgements. We wish to thank A. Miranville for having corrected the English.

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