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SEMI-GLOBAL $C^1$ SOLUTION AND EXACT BOUNDARY CONTROLLABILITY FOR REDUCIBLE QUASILINEAR HYPERBOLIC SYSTEMS

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Abstract. By means of a result on the semi global $C^1$ solution, we establish the exact boundary controllability for the reducible quasilinear hyperbolic system if the $C^1$ norm of initial data and final state is small enough.

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1 Introduction

We consider the exact boundary control for the following reducible quasilinear hyperbolic system

\[
\begin{align*}
\frac{\partial r}{\partial t} + \lambda(r,s) \frac{\partial r}{\partial x} &= 0, \\
\frac{\partial s}{\partial t} + \mu(r,s) \frac{\partial s}{\partial x} &= 0
\end{align*}
\]

with the initial data

\[
r(0,x) = r_0(x), \quad s(0,x) = s_0(x), \quad 0 \leq x \leq 1
\]

and the nonlinear boundary feedback controls

\[
\begin{align*}
s &= g(t,r) + v(t) \quad \text{at} \quad x = 0, \\
r &= f(t,s) + u(t) \quad \text{at} \quad x = 1
\end{align*}
\]

Without loss of generality, we may assume that

\[
f(t,0) \equiv g(t,0) \equiv 0
\]
Since we consider only $C^1$ solution, we assume that the coefficients $\lambda, \mu$ and the nonlinear feedback laws $f, g$ are all $C^1$ functions on the domain under consideration. Moreover, we suppose that the system is strictly hyperbolic and

$$\lambda(r, s) < 0 < \mu(r, s).$$

(1.5)

Also we assume that the initial data $(r_0, s_0)$ and the input control $(u, v)$ are all $C^1$ functions satisfying the compatibility conditions

$$\begin{align*}
  \begin{cases}
    s_0(0) = g(0, r_0(0)) + v(0), \\
    r_0(1) = f(0, s_0(1)) + u(0)
  \end{cases}
\end{align*}$$

(1.6)

and

$$\begin{align*}
  \begin{cases}
    \mu(r_0(0), s_0(0)) s_0'(0) = -\frac{\partial g}{\partial t}(0, r_0(0)) + \frac{\partial g}{\partial r}(0, r_0(0)) \lambda(r_0(0), s_0(0)) r_0'(0) - v'(0), \\
    \lambda(r_0(1), s_0(1)) r_0'(1) = -\frac{\partial f}{\partial t}(0, s_0(1)) + \frac{\partial f}{\partial s}(0, s_0(1)) \mu(r_0(1), s_0(1)) r_0'(1) - u'(0).
  \end{cases}
\end{align*}$$

(1.7)

As in [5], we will consider the following exact boundary controllability:

Given initial data $r_0, s_0 \in C^1[0, 1]$ and final data $r_T, s_T \in C^1[0, 1]$, can we find a time $T > 0$ and boundary input controls $u, v \in C^1[0, T]$ such that the mixed initial-boundary value problem (1.1)-(1.3) admits a unique $C^1$ solution $(r(t, x), s(t, x))$ verifying the final condition

$$r(T, x) = r_T(x), \quad s(T, x) = s_T(x), \quad \forall 0 \leq x \leq 1?$$

(1.8)

First of all, because of the finite speed of the wave propagation, the exact boundary controllability of hyperbolic system requires that the controllability time $T$ must be greater than a given constant. On the other hand, following the local existence theorem of $C^1$ solution (see [6]), there exists a constant $\delta > 0$ such that the mixed initial-boundary value problem (1.1)-(1.3) admits a unique $C^1$ solution $(r(t, x), s(t, x))$ on the domain

$$D_\delta = \{(t, x)| 0 \leq t \leq \delta, \quad 0 \leq x \leq 1\}.$$ 

(1.9)

But this $C^1$ solution may blow up in a finite time (see Ref. [4]). So, the mixed initial-boundary value problem (1.1)-(1.3) has no global $C^1$ solution in general. We even don’t know if the life span of the $C^1$ solution could be greater than a given $T > 0$. In order to avoid this difficulty, in [5] the authors considered the linearly degenerate case:

$$\lambda(r, s) \equiv \lambda(s), \quad \mu(r, s) \equiv \mu(r).$$

(1.10)

In that case, the global existence of $C^1$ solution $(r(t, x), s(t, x))$ and the global exact boundary controllability for the system (1.1)-(1.3) were actually proved.

In this paper, we consider the general case that system (1.1) is not necessary to be linearly degenerate. We first give suitable conditions on the initial data $(r_0, s_0)$ and the input control $(u, v)$ such that for a given $T > 0$, the mixed initial-boundary value problem (1.1)-(1.3) admits a unique $C^1$ solution $(r(t, x), s(t, x))$ on the domain

$$D_T = \{(t, x)| 0 \leq t \leq T, \quad 0 \leq x \leq 1\}.$$ 

(1.11)

We will refer to this solution as a semi-global $C^1$ solution.

Let $(r(t, x), s(t, x))$ be a local $C^1$ solution to the mixed initial-boundary value problem (1.1)-(1.3) on the domain $D_\delta$ with $0 < \delta \leq T$. In order to extend this local $C^1$ solution up to the domain $D_T$, it suffices to
establish the following uniform \textit{a priori} estimate: for any given $\delta$ with $0 < \delta \leq T$,

$$\|(r(t, \cdot), s(t, \cdot))\|_{C^1[0, 1]} \leq C(T), \quad \forall 0 \leq t \leq \delta,$$

where $C$ is a positive constant depending on $T$, but independent of $\delta$.

In Section 2, we will prove the existence and uniqueness of semi-global $C^1$ solution to the mixed initial-boundary value problem (1.1)--(1.3), provided that the $C^1$ norm of initial data $(r_0, s_0)$ and the boundary control $(u, v)$ is small enough. In Section 3, using a similar approach as that in [5], we will establish the local exact boundary controllability for the system (1.1)--(1.3).

There is a number of publications concerning the exact controllability and uniform stabilization for linear hyperbolic systems (see [7, 8, 9] and the references therein). Furthermore, the exact boundary controllability for semilinear wave and plate equations were also established in [10] and [3]. However, only a little is known concerning quasilinear hyperbolic systems. We mention that M. Cirinà [1, 2] considered the local exact boundary controllability for quasilinear hyperbolic systems with linear boundary controls. In order to obtain the semi-global solution, the author of [1, 2] needed very strong conditions on the coefficients of the system (globally bounded and globally Lipschitz continuous). This is a grave restriction to the application. Since except [2] there is little results on the semi-global $C^1$ solution to quasilinear hyperbolic systems in the literature, we hope that the discussion in this paper on the semi-global $C^1$ solution to quasilinear hyperbolic systems would also promote a systematic investigation in this area.

2. Existence and Uniqueness of Semi-global $C^1$ Solution

In this section, we will give the existence and uniqueness of semi-global $C^1$ solution to the mixed initial-boundary value problem (1.1)--(1.3). The main result is the following

\textbf{Theorem 2.1.} For a given $T > 0$, the mixed initial-boundary value problem (1.1)--(1.3) admits a unique semi-global $C^1$ solution $(r(t, x), s(t, x))$ on the domain $D_T$ defined in (1.11), provided that the $C^1$ norms $\|(r_0, s_0)\|_{C^1[0, 1]}$ and $\|(u, v)\|_{C^1[0, T]}$ are small enough. Moreover the $C^1$ norm of the solution $(r(t, x), s(t, x))$ can be arbitrarily small.

\textbf{Proof.} Following the local existence theorem of $C^1$ solution (see [6]), the mixed initial-boundary value problem (1.1)--(1.3) admits a unique local $C^1$ solution $(r(t, x), s(t, x))$ on the domain $D_\delta$. In order to obtain the semi-global $C^1$ solution on the domain $D_T$, it is sufficient to prove that, for any given $\delta$ with $0 < \delta \leq T$, the local $C^1$ solution $(r(t, x), s(t, x))$ on $D_\delta$ satisfies the following uniform \textit{a priori} estimate

$$\|(r(t, \cdot), s(t, \cdot))\|_{C^1[0, 1]} \leq C(T), \quad \forall 0 \leq t \leq \delta,$$

where $C$ is a positive constant independent of $\delta$.

We first assume that there exists a constant $M > 0$ such that

$$\|(r(t, \cdot), s(t, \cdot))\|_{C^0[0, 1]} \leq M, \quad \forall 0 \leq t \leq \delta. \quad \text{(2.2)}$$

We will justify this assumption at the end of the proof. Let

$$T_1 = \min_{|\mu|, |\lambda| \leq M} \left\{ \frac{1}{\mu}, \frac{1}{|\lambda|} \right\} > 0. \quad \text{(2.3)}$$

For any given point $(t_0, x_0) \in D_{T_1}$, we consider the $\lambda$-characteristic $x = x_1(t)$ passing through $(t_0, x_0)$:

$$\frac{dx_1(t)}{dt} = \lambda(r(t, x_1(t)), s(t, x_1(t))), \quad x_1(t_0) = x_0. \quad \text{(2.4)}$$
We distinguish two cases: (a) The $\lambda$-characteristic $x = x_1(t)$ intersects the $x$-axis at a point $(0, \alpha)$. Then, since $r$ is the corresponding Riemann invariant, we have

$$|r(t_0, x_0)| = |r(0, \alpha)| = |r_0(\alpha)| \leq \|r_0\|_{C^0[0,1]}.$$  \hspace{1cm} (2.5)

(b) The $\lambda$-characteristic $x = x_1(t)$ intersects the right boundary of $D_{T_1}$ at a point $(\tilde{t}_0, 1)$. Then, we consider the $\mu$-characteristic $x = x_2(t)$ passing through $(\tilde{t}_0, 1)$:

$$\frac{dx_2(t)}{dt} = \mu(r(t, x_2(t)), s(t, x_2(t))), \quad x_2(\tilde{t}_0) = 1.$$  \hspace{1cm} (2.6)

By virtue of the choice of $T_1$, the $\mu$-characteristic $x = x_2(t)$ must intersect the $x$-axis at a point $(0, \beta)$. Since $r$ and $s$ are the corresponding Riemann invariants respectively, we have

$$r(t_0, x_0) = r(\tilde{t}_0, 1), \quad s(\tilde{t}_0, 1) = s_0(\beta).$$  \hspace{1cm} (2.7)

On the other hand, using the boundary condition (1.3) we have

$$r(\tilde{t}_0, 1) = f(\tilde{t}_0, s_0(\beta)) + u(\tilde{t}_0).$$  \hspace{1cm} (2.8)

Then, noting (1.4) we get

$$|r(t_0, x_0)| \leq |f(\tilde{t}_0, s_0(\beta))| + |u(\tilde{t}_0)|
\leq \max_{|s| \leq M} \left| \frac{\partial f}{\partial s}(\tilde{t}_0, s) \right| |s_0(\beta)| + |u(\tilde{t}_0)|
\leq A_0\|s_0\|_{C^0[0,1]} + \|u\|_{C^0[0,T]} = (A_0 + 1)a_0,$$  \hspace{1cm} (2.9)

where

$$\left\{ \begin{array}{l} a_0 = \max \left\{ \| (r_0, s_0) \|_{C^0[0,1]}, \| (u, v) \|_{C^0[0,T]} \right\}, \\
A_0 = \max_{0 \leq t \leq T, |r|, |s| \leq M} \left\{ \left| \frac{\partial f}{\partial s}(t, s) \right|, \left| \frac{\partial g}{\partial r}(t, r) \right| \right\}. \end{array} \right.$$  \hspace{1cm} (2.10)

Similarly, we have

$$|s(t_0, x_0)| \leq (A_0 + 1)a_0.$$  \hspace{1cm} (2.11)

Combining (2.9) and (2.11) we get

$$\|(r(t, \cdot), s(t, \cdot))\|_{C^0[0,1]} \leq (A_0 + 1)a_0, \quad \forall 0 \leq t \leq T_1.$$  \hspace{1cm} (2.12)

Repeating the previous procedure with the new initial data $(r(T_1, x), s(T_1, x))$ on $t = T_1$, we obtain

$$\|(r(t, \cdot), s(t, \cdot))\|_{C^0[0,1]} \leq (A_0 + 1)^2a_0, \quad \forall T_1 \leq t \leq 2T_1.$$  \hspace{1cm} (2.13)

In this way, after at most $N \leq \lfloor T/T_1 \rfloor + 1$ iterations, we arrive at the estimate

$$\|(r(t, \cdot), s(t, \cdot))\|_{C^0[0,1]} \leq (A_0 + 1)^Na_0, \quad \forall (N - 1)T_1 \leq t \leq \delta.$$  \hspace{1cm} (2.14)

Then, collecting the estimates (2.12)–(2.14), we obtain

$$\|(r(t, \cdot), s(t, \cdot))\|_{C^0[0,1]} \leq (A_0 + 1)^Na_0 \leq C_1(T)a_0, \quad \forall 0 \leq t \leq \delta,$$  \hspace{1cm} (2.15)
where $C_1(T)$ is independent of $\delta$. Moreover, since $a_0$ can be taken to be small enough, (2.15) also verifies the validity of (2.2).

Now we estimate the $C^0$ norm of $\frac{\partial r}{\partial x}$ and $\frac{\partial s}{\partial x}$. First let us recall the well-known Lax transformation [4]:

$$
U = e^{k(r,s)} \frac{\partial r}{\partial x}, \quad V = e^{k(r,s)} \frac{\partial s}{\partial x},
$$

(2.16)

where the functions $h, k$ are given by

$$
\begin{align*}
\frac{\partial h}{\partial s} &= \frac{1}{\lambda - \mu} \frac{\partial \lambda}{\partial s}, & h(0, 0) &= 0, \\
\frac{\partial k}{\partial r} &= \frac{1}{\mu - \lambda} \frac{\partial \mu}{\partial r}, & k(0, 0) &= 0.
\end{align*}
$$

(2.17)

A straightforward computation shows that along the $\lambda$-characteristic $x = x_1(t)$, $U$ satisfies the following Riccati’s equation

$$
\frac{dU}{dt} = -\frac{\partial \lambda}{\partial r} e^{-k(r,s)} U^2,
$$

(2.18)

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \lambda(r, s) \frac{\partial}{\partial x}$, and along the $\mu$-characteristic $x = x_2(t)$, $V$ satisfies the following Riccati’s equation

$$
\frac{dV}{dt} = -\frac{\partial \mu}{\partial s} e^{-k(r,s)} V^2
$$

(2.19)

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \mu(r, s) \frac{\partial}{\partial x}$.

We next define some constants:

$$
b_0 = \max \{\|r_0^t, s_0^t\|_{C^0[0, 1]}, \|u^t, v^t\|_{C^0[0, \tau]}\},
$$

(2.20)

$$
A_1 = \max_{0 \leq t \leq \tau, |r|, |s| \leq M} \left\{ \left| \frac{\partial f}{\partial t} (t, s) \right|, \left| \frac{\partial g}{\partial t} (t, r) \right| \right\},
$$

(2.21)

$$
M_1 = \max_{|r|, |s| \leq M} \left\{ e^{\|h(r,s)\|}, e^{\|k(r,s)\|} \right\},
$$

(2.22)

$$
M_2 = \max_{|r|, |s| \leq M} \left\{ \left| \frac{\partial \lambda}{\partial r} (r, s) \right|, \left| \frac{\partial \mu}{\partial s} (r, s) \right| \right\},
$$

(2.23)

$$
M_3 = \max_{|r|, |s| \leq M} \left\{ \left| \lambda(r, s) \right|, \left| \mu(r, s) \right| \right\},
$$

(2.24)

$$
M_4 = \max_{|r|, |s| \leq M} \left\{ \frac{1}{\lambda(r, s)}, \frac{1}{\mu(r, s)} \right\}. \quad (2.25)
$$

As in the previous stage, we distinguish two cases:

(a) The $\lambda$-characteristic $x = x_1(t)$ passing through $(t_0, x_0) \in D_{\tau_1}$ intersects the $x$-axis at a point $(0, \alpha)$. Then solving (2.18), we get

$$
U(t_0, x_0) = \frac{U(0, \alpha)}{1 + \int_{t_0}^{\tau_1} \frac{\partial \lambda}{\partial r} e^{-h(r, x_1(r)), \alpha} \, dr}. \quad (2.26)
$$

From the definition (2.16) and noting (1.2), we have

$$
U(0, \alpha) = e^{-h(r_0(\alpha), s_0(\alpha))} r_0'(\alpha).
$$
Then

\[ |U(0, \alpha)| \leq M_1 |r_0(\alpha)| \leq M_1 b_0. \] (2.27)

Thus

\[ 1 + U(0, \alpha) \int_0^{t_0} \frac{\partial \lambda}{\partial r} e^{-h(r, x_1(r)), s(r, x_1(r))} dr \geq 1 - T_1 M_1^2 M_2 b_0 \geq \frac{1}{2}, \] (2.28)

provided that \( b_0 \) is small enough. It follows from (2.26)–(2.28) that

\[ |U(t_0, x_0)| \leq 2M_1 b_0. \] (2.29)

(b) The \( \lambda \)-characteristic \( x = x_1(t) \) passing through \((t_0, x_0) \in D_{T_1}\) intersects the right boundary of \( D_{T_1}\) at a point \((\tilde{t}_0, 1)\). Then by virtue of the choice of \( T_1\), the \( \mu \)-characteristic \( x = x_2(t) \) passing through \((\tilde{t}_0, 1)\) must intersect the \( x \)-axis at a point \((0, \beta)\). Therefore, solving (2.18) and (2.19) respectively, we get

\[ V(\tilde{t}_0, 1) = \frac{V(0, \beta)}{1 + V(0, \beta) \int_0^{t_0} \frac{\partial \mu}{\partial s} e^{-h(r, x_2(r)), s(r, x_2(r))} dr}. \] (2.30)

and

\[ U(t_0, x_0) = \frac{U(\tilde{t}_0, 1)}{1 + U(\tilde{t}_0, 1) \int_0^{t_0} \frac{\partial \lambda}{\partial r} e^{-h(r, x_1(r)), s(r, x_1(r))} dr}. \] (2.31)

Similarly to (2.29), we get

\[ |V(\tilde{t}_0, 1)| \leq 2M_1 b_0. \] (2.32)

On the other hand, differentiating the boundary condition (1.3) at the end \( x = 1 \) with respect to \( t \), we have

\[ \frac{\partial \tau}{\partial x} = - \frac{1}{\lambda} \left( \frac{\partial f}{\partial t} + u' \right) + \frac{\mu}{\lambda} \frac{\partial f}{\partial s} \frac{\partial s}{\partial x} \quad \text{on} \quad x = 1. \] (2.33)

Thus, noting (2.16), we obtain that

\[ U(\tilde{t}_0, 1) = - e^{h(t, 1)} \left( \frac{\partial f}{\partial t}(\tilde{t}_0, s(\tilde{t}_0, 1)) + u'(\tilde{t}_0) \right) + \frac{\mu}{\lambda} \frac{\partial f}{\partial s}(\tilde{t}_0, s(\tilde{t}_0, 1)) e^{-h} V(\tilde{t}_0, 1). \] (2.34)

It follows from (2.32) and (2.34) that

\[ |U(\tilde{t}_0, 1)| \leq M_1 M_4 (A_1 + b_0) + A_0 M_1^2 M_3 M_4 |V(\tilde{t}_0, 1)| \leq M_1 M_4 (A_1 + b_0 + 2A_0 M_1^2 M_3 b_0). \] (2.35)

Moreover, noting (1.4) and (2.15), it is easy to see that \( A_1 \to 0 \) as \( a_0 \to 0 \). Therefore, similarly to (2.28), we have

\[ 1 + U(\tilde{t}_0, 1) \int_{t_0}^{\tilde{t}_0} \frac{\partial \lambda}{\partial r} e^{-h(r, x_1(r)), s(r, x_1(r))} dr \geq \frac{1}{2}, \] (2.36)

provided that \( a_0, b_0 \) are small enough. Noting (2.35) and (2.36), it follows from (2.31) that

\[ |U(t_0, x_0)| \leq 2M_1 M_4 (A_1 + b_0 + 2A_0 M_1^2 M_3 b_0), \] (2.37)
provided that \( a_0, b_0 \) are small enough. Since \( M_4 > 1 \), the estimate (2.37) remains true for the two cases.

In a similar way, we can prove that \( |V(t_0, x_0)| \) satisfies also the estimate (2.37). Then, noting (2.16), we obtain

\[
\left\| \left( \frac{\partial r}{\partial x}(t, \cdot), \frac{\partial s}{\partial x}(t, \cdot) \right) \right\|_{C^0[0,1]} \leq 2M_4^2M_3(b_0 + 2A_0M_4^2M_3b_0)
\]

(2.38)

for all \( t \) with \( 0 \leq t \leq T_1 \). In particular, we see that the \( C^1 \) norm of \((r(T_1, x), s(T_1, x))\) can be sufficiently small as \( a_0, b_0 \to 0 \). Thus we can repeat the previous procedure with the new initial data \((r(T_1, x), s(T_1, x))\) on \( t = T_1 \).

After at most \([T/T_1] + 1\) iterations, we obtain the following uniform \( a \ priori \) estimate

\[
\left\| \left( \frac{\partial r}{\partial x}(t, \cdot), \frac{\partial s}{\partial x}(t, \cdot) \right) \right\|_{C^0[0,1]} \leq C_2(T; a_0, b_0), \quad \forall 0 \leq t \leq \delta,
\]

(2.39)

where \( C_2(T; a_0, b_0) \) is a positive constant independent of \( \delta \) and can be sufficiently small as \( a_0, b_0 \to 0 \). The combination of (2.15) and (2.39) gives the uniform \( a \ priori \) estimate (2.1). The proof is then completed.

### 3. Local Exact Boundary Controllability

Now we can precisate the framework of the exact boundary controllability. First of all, for a given constant \( M > 0 \), we put

\[
\lambda_{\text{max}} = \max_{|r|,|s| \leq M} \lambda(r, s), \quad \mu_{\text{min}} = \min_{|r|,|s| \leq M} \mu(r, s)
\]

(3.1)

and we define the time \( T_0 \) by

\[
T_0 = \max \left\{ -\frac{1}{\lambda_{\text{max}}}, \frac{1}{\mu_{\text{min}}} \right\}.
\]

(3.2)

By Theorem 2.1, for any given \( T \geq T_0 \), the mixed initial-boundary value problem (1.1)–(1.3) admits a unique semi-global \( C^1 \) solution on the domain \( D_T \), and the \( C^1 \) norm of the solution can be sufficiently small, provided that the \( C^1 \) norm of the initial data \((r_0, s_0)\) and the boundary control \((u, v)\) is small enough.

Using the same idea as in the proof of Theorem 2.1, we can get without any difficulty the following preliminary result.

**Lemma 3.1.** The Cauchy problem (1.1)–(1.2) admits a unique global \( C^1 \) solution \((r(t, x), s(t, x))\) on the maximal determinate domain enclosed by the \( \lambda \)-characteristic passing through \((0, 0)\), the \( \mu \)-characteristic passing through \((0, 1)\) and the \( x \)-axis, provided that the \( C^1 \) norm of the initial data \(||(r_0, s_0)||_{C^1[0,1]}\) is small enough. Moreover, the \( C^1 \) norm of the solution \((r(t, x), s(t, x))\) can be arbitrarily small.

Now we give the local exact boundary controllability for the system (1.1)–(1.3).

**Theorem 3.1.** Let \( T > T_0 \). For any given initial data \((r_0, s_0)\) and any given final state \((r_T, s_T)\) with \( C^1 \) norm small enough, there exist two boundary controls \( u, v \) with \( C^1 \) norm small enough such that the mixed initial-boundary value problem (1.1)–(1.3) admits a unique semi-global \( C^1 \) solution \((r(t, x), s(t, x))\) on the domain \( D_T \), which satisfies the final condition (1.8).

**Proof.** First, thanks to Lemma 3.1, if the \( C^1 \) norm of the initial data \((r_0, s_0)\) is small enough, the Cauchy problem (1.1)–(1.2) admits a unique global \( C^1 \) solution \((r_d(t, x), s_d(t, x))\) on the maximal determinate domain \( \Omega_d \) enclosed by the \( x \)-axis, the \( \lambda \)-characteristic \( x = x_1(t) \) passing through \( A = (0, 1) \) and the \( \mu \)-characteristic
$x = x_2(t)$ passing through $O = (0,0)$. The $C^1$ norm of the solution $(r_d(t, x), s_d(t, x))$ can be arbitrarily small. Moreover, it is easy to see that the two characteristics intersect at a point $D = (t_d, x_d)$ with

$$t_d \leq \frac{1}{\mu_{\min} - \lambda_{\max}}. \quad (3.3)$$

Next, for the given final data $(r_T, s_T)$ with $C^1$ norm small enough, the backward Cauchy problem for the system (1.1) with the final data $(r_u(t, x), s_u(t, x))$ on the maximal determinate domain $\Omega_u$ enclosed by the $\lambda$-characteristic $x = y_1(t)$ passing through $C = (T, 0)$, the $\mu$-characteristic $x = y_2(t)$ passing through $B = (T, 1)$ and the segment $BC$. The two characteristics intersect at a point $U = (t_u, x_u)$ with

$$t_u \geq T - \frac{1}{\mu_{\min} - \lambda_{\max}}. \quad (3.4)$$

Noting (3.2), it follows from (3.3) and (3.4) that

$$t_u - t_d \geq T - \frac{2}{\mu_{\min} - \lambda_{\max}} \geq T - T_0. \quad (3.5)$$

In particular, the subdomains $\Omega_d, \Omega_u$ are disjointed provided that $T > T_0$.

Finally, let $\Omega_t$ be the subdomain enclosed by the characteristics $x = x_2(t), x = y_1(t)$ and the segments $DU, OC$, and $\Omega_r$ be the subdomain enclosed by the characteristics $x = x_1(t), x = y_2(t)$ and the segments $DU, AB$. The domain $D_T$ is then divided into four subdomains $\Omega_d, \Omega_u, \Omega_l$ and $\Omega_r$. Moreover, since $T > T_0$, we know (see Appendix in [5]) that the angle between the segment $DU$ and the characteristic $x = x_1(t)$ (resp. $x = x_2(t), x = y_1(t)$ and $x = y_2(t)$) is less than $\pi$. Thus we can consider the following mixed initial-boundary value problem on the subdomain $\Omega_t$:

$$\begin{cases}
\frac{\partial r}{\partial x} + \frac{1}{\lambda(r, s)} \frac{\partial r}{\partial t} = 0, \\
\frac{\partial s}{\partial s} + \frac{1}{\mu(r, s)} \frac{\partial s}{\partial t} = 0
\end{cases} \quad (3.6)$$

with the boundary conditions

$$\begin{cases}
r = r_d(t, x_2(t)) \text{ on } x = x_2(t), \quad 0 \leq t \leq t_d, \\
s = s_u(t, y_1(t)) \text{ on } x = y_1(t), \quad t_u \leq t \leq T
\end{cases} \quad (3.7)$$

and the initial data

$$r(t, x_3(t)) = r_m(t), \quad s(t, x_3(t)) = s_m(t), \quad t_d \leq t \leq t_u, \quad (3.8)$$

where $x = x_3(t)$ denotes the equation of the segment $DU$.

We notice that Theorem 2.1 applies well to problem (3.6)-(3.8). In fact, if the $C^1$ norm of the initial data $(r_0, s_0)$ and the final data $(r_T, s_T)$ is small enough, the $C^1$ norm of the boundary value $(r_d(t, x_2(t)), s_u(t, y_1(t)))$ is also small. In order to solve problem (3.6)-(3.8), the initial data $(r_m(t), s_m(t))$ should be small in $C^1$ norm and satisfy suitable compatibility conditions. Observing that $r_d(t, x)$ (resp. $s_d(t, x), r_u(t, x)$ and $s_u(t, x)$) is constant along the characteristic $x = x_1(t)$ (resp. $x = x_2(t), x = y_1(t)$ and $x = y_2(t))$, we get

$$\begin{cases}
r_d(t_d, x_1(t_d)) = r_0(1), \quad s_d(t_d, x_2(t_d)) = s_0(0), \\
r_u(t_u, y_1(t_u)) = r_T(0), \quad s_u(t_u, y_2(t_u)) = s_T(1).
\end{cases} \quad (3.9)$$
Then noting that the $C^0$ compatibility asks

\[
\begin{cases}
    r_u(t_u, y_1(t_u)) = r_m(t_u), & s_d(t_d, x_2(t_d)) = s_m(t_d), \\
    r_d(t_d, x_1(t_d)) = r_m(t_d), & s_u(t_u, y_2(t_u)) = s_m(t_u),
\end{cases}
\]  

(3.10)

we deduce the $C^0$ compatibility conditions:

\[
\begin{cases}
    r_m(t_u) = r_T(0), & s_m(t_d) = s_0(0), \\
    r_m(t_d) = r_0(1), & s_m(t_u) = s_T(1).
\end{cases}
\]  

(3.11)

Next, differentiating $r_m(t)$, $s_m(t)$ with respect to $t$ and noting that the segment $DU$ is described by the equation $x = x_3(t)$

\[x_3(t) = x_u + \Delta(t - t_u), \quad t_d \leq t \leq t_u,\]  

(3.12)

where

\[\Delta = \frac{x_u - x_d}{t_u - t_d},\]  

(3.13)

we obtain that

\[
\begin{cases}
    r_m'(t) = (\Delta - \lambda(r_m(t), s_m(t))) \frac{\partial r}{\partial x}(t, x_3(t)), \\
    s_m'(t) = (\Delta - \mu(r_m(t), s_m(t))) \frac{\partial s}{\partial x}(t, x_3(t)).
\end{cases}
\]  

(3.14)

Then, noting (3.11), we obtain the $C^1$ compatibility conditions:

\[
\begin{cases}
    r_m'(t_d) = (\Delta - \lambda(r_0(1), s_0(0))) \frac{\partial r_d}{\partial x}(t_d, x_d), \\
    s_m'(t_u) = (\Delta - \mu(r_T(0), s_T(1))) \frac{\partial s_u}{\partial x}(t_u, x_u).
\end{cases}
\]  

(3.15)

Similarly, on the subdomain $\Omega_r$, we consider the mixed initial-boundary value problem for system (3.6) with the boundary conditions

\[
\begin{cases}
    s = s_d(t, x_1(t)) \quad \text{on} \quad x = x_1(t), \quad 0 \leq t \leq t_d, \\
    r = r_u(t, y_2(t)) \quad \text{on} \quad x = y_2(t), \quad t_u \leq t \leq T
\end{cases}
\]  

(3.16)

and the same initial data as in (3.8). This time, except the $C^0$ compatibility conditions (3.11), we need the following $C^1$ compatibility conditions:

\[
\begin{cases}
    r_m'(t_u) = (\Delta - \lambda(r_T(0), s_T(1))) \frac{\partial r_u}{\partial x}(t_u, x_u), \\
    s_m'(t_d) = (\Delta - \mu(r_0(1), s_0(0))) \frac{\partial s_d}{\partial x}(t_d, x_d).
\end{cases}
\]  

(3.17)

Taking (3.11, 3.15, 3.17) into account, we can choose the initial data $(r_m(t), s_m(t))$ as the Hermite interpolation on the interval $[t_d, t_u]$, which is uniquely determined by the values $r_0(1), s_0(0), r_T(0), s_T(1)$ and the derivatives $\frac{\partial r_d}{\partial x}(t_d, x_d), \frac{\partial s_d}{\partial x}(t_d, x_d), \frac{\partial r_u}{\partial x}(t_u, x_u), \frac{\partial s_u}{\partial x}(t_u, x_u)$. Since the $C^1$ norm of $(r_m(t), s_m(t))$ can be sufficiently small,
applying Theorem 2.1 we can find \((r_l(t,x), s_l(t,x))\) and \((r_r(t,x), s_r(t,x))\) which solve the problem (3.6)-(3.8) and the problem (3.6, 3.8, 3.16) on the subdomains \(\Omega_l, \Omega_r\) respectively.

Finally, taking \((r(t,x), s(t,x))\) as the collection of the solutions on the four subdomains \(\Omega_d, \Omega_u, \Omega_l, \Omega_r\), and defining the boundary controls \(u, v\) by

\[
\begin{align*}
  v(t) &= s_l(t, 0) - g(t, r_l(t, 0)) \quad \text{at } x = 0, \\
  u(t) &= r_r(t, 1) - f(t, s_r(t, 1)) \quad \text{at } x = 1,
\end{align*}
\]

we check easily that \((r(t,x), s(t,x))\) solves the problem (1.1)-(1.3) and satisfies the final condition (1.8). The proof is thus achieved.

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References