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Modélisation Mathématique et Analyse Numérique

ON THE DERIVATION OF HOMOGENEOUS HYDROSTATIC EQUATIONS

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Abstract. In this paper we study the derivation of homogeneous hydrostatic equations starting from 2D Euler equations, following for instance [2,9]. We give a convergence result for convex profiles and a divergence result for a particular inflexion profile.

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1. INTRODUCTION

We will consider the following classical homogeneous hydrostatic model

$$\partial_t u_1 + u_1 \partial_x u_1 + u_2 \partial_y u_1 + \partial_x p = 0, \tag{1}$$

$$\partial_x u_1 + \partial_y u_2 = 0, \tag{2}$$

$$\partial_y p = -g\rho, \tag{3}$$

$$u_2 = 0 \qquad \text{on } \partial\Omega \tag{4}$$

where n is the outer normal of $\Omega = \mathbb{T} \times [0,1]$. We moreover assume that g and ρ are given constants, independent on t and x, such that up to a slight change in the definition of the pressure, (3) can be replaced by

$$\partial_y p = 0. \tag{5}$$

The word "homogeneous" refers to the fact that ρ is a constant in the domain. The case $\Omega = \mathbb{R} \times [0,1]$ is similar and can be treated using the same methods. This system has been investigated in [2] where local existence of solutions under a convexity assumption is in particular proved. Namely

Theorem 1.1 ([2]). *Let $s > 5$, and let $(u_1^0, u_2^0) \in H^s(\Omega)$ be a given divergence free vector field, tangent to $\partial\Omega$, with $\int_\Omega u_1^0 = 0$ (which can always been assumed up to a change of variables). Let us assume moreover that*

$$|\partial_{yy}^2 u_1^0(x,y)| \geq \sigma_0 \tag{6}$$

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for some $\sigma_0 > 0$*, and that there exists a constant* k *such that*

$$\partial_y u_1^0(x,0) = k \qquad \text{and} \qquad \partial_y u_1^0(x,1) = k+1 \qquad \forall x. \tag{7}$$

Then there exists $T > 0$ *and a solution* $(u_1(t,x,y), u_2(t,x,y))$ *of (1, 2, 4, 5) on* $[0,T[$*, with initial data* (u_1^0, u_2^0)*, such that, for every* $T' < T$*,* $(u_1, u_2) \in L^\infty([0,T'], H^s(\Omega))$*.*

As in [2,3], and following [9] we consider system (1, 2, 4, 5) as a "geometrical limit" of the incompressible Euler equations in a thin domain. More precisely we consider

$$\partial_t u_1^\varepsilon + u_1^\varepsilon \partial_x u_1^\varepsilon + u_2^\varepsilon \partial_y u_1^\varepsilon + \partial_x p^\varepsilon = 0, \tag{8}$$

$$\partial_t u_2^\varepsilon + u_1^\varepsilon \partial_x u_2^\varepsilon + u_2^\varepsilon \partial_y u_2^\varepsilon + \partial_y p^\varepsilon = 0, \tag{9}$$

$$\partial_x u_1^\varepsilon + \partial_y u_2^\varepsilon = 0, \tag{10}$$

$$u_2^\varepsilon = 0 \quad \text{on } \partial\Omega \tag{11}$$

in $\Omega_\varepsilon = \mathbb{T} \times [0,\varepsilon]$. The usual change of velocity $\tilde{u}_1^\varepsilon(t,x,y) = u_1^\varepsilon(t,x,\varepsilon y)$, $\tilde{u}_2^\varepsilon(t,x,y) = \varepsilon^{-1} u_2^\varepsilon(t,x,\varepsilon y)$ leads to (after dropping the tildes)

$$\partial_t u_1^\varepsilon + u_1^\varepsilon \partial_x u_1^\varepsilon + u_2^\varepsilon \partial_y u_1^\varepsilon + \partial_x p^\varepsilon = 0, \tag{12}$$

$$\varepsilon^2 \left(\partial_t u_2^\varepsilon + u_1^\varepsilon \partial_x u_2^\varepsilon + u_2^\varepsilon \partial_y u_2^\varepsilon\right) + \partial_y p^\varepsilon = 0, \tag{13}$$

$$\partial_x u_1^\varepsilon + \partial_y u_2^\varepsilon = 0, \tag{14}$$

$$u_2^\varepsilon = 0 \quad \text{on } \partial\Omega \tag{15}$$

in $\Omega = \mathbb{T} \times [0,1]$. Formally as $\varepsilon > 0$ goes to 0, systems (12–15) goes to (1, 2, 4, 5). However to prove that solutions of (12–15) converge to solutions of (1, 2, 4, 5) is not straightforward and appears to be false in some cases. This problem is deeply linked to stability properties of time independent shear layers flows. It is well known since Lord Rayleigh [10] that the stability of such flows depends on the presence of inflexion points in the tangential velocity profile. Roughly speaking, shear layers $(u(y),0)$ are stable if u is convex and may be instable if u has an inflexion point. Stability has rigorously been proved by Arnold [1] for general 2D time independent flow using Lyapounov and Hamiltonian techniques and more recently investigated in the time dependent case in [6,7], using a direct energy approach. In this paper we use energy methods derived from [6,7] (see in particular [7] for the link with the work of Arnold) to prove the following convergence result

Theorem 1.2. *Under the assumptions of Theorem 1.1, for every* $\varepsilon > 0$*, there exists* $T^\varepsilon > 0$ *and a solution* $(u_1^\varepsilon(t,x,y), u_2^\varepsilon(t,x,y))$ *of (12–15) on* $[0,T^\varepsilon[$*, with initial data* (u_1^0, u_2^0)*. Moreover for every* $T' < T$*,* $T^\varepsilon > T'$ *for* ε *small enough, and* $(u_1^\varepsilon, u_2^\varepsilon)$ *are uniformly bounded in* $L^\infty([0,T'], H^{s'}(\Omega))$ *with respect to* ε *(for* ε *small enough), for some* $s' \le s$*. Last, for every* $T' < T$*, as* $\varepsilon \to 0$*,*

$$(u_1^\varepsilon, u_2^\varepsilon) \to (u_1, u_2) \quad \text{in} \quad L^\infty([0,T'[, H^{s'}(\Omega)). \tag{16}$$

This theorem justifies in particular completely the formal limit, under the convexity assumption (6).

When there is an inflexion point in the velocity profile, the convergence may not hold. For the sake of completeness we recall the following Theorem, proved in [8] using techniques of [6].

Theorem 1.3. *For every* s *and* N *arbitrarily large, there exists a time independent smooth solution* $(u(y),0)$ *of (1, 2, 4, 5), a constant* $\sigma_0 > 0$*, a sequence of times* T^ε *with* $\lim_{\varepsilon\to 0} T^\varepsilon = 0$*, and smooth solutions* $(u_1^\varepsilon(t,x,y), u_2^\varepsilon(t,x,y))$ *of (12–15) such that*

$$\|(u_1^\varepsilon(0,x,y), u_2^\varepsilon(0,x,y)) - (u(y),0)\|_{H^s(\mathbb{T}\times[0,1])} \le \varepsilon^N \tag{17}$$

and

$$\|(u_1^\varepsilon(T^\varepsilon,x,y),u_2^\varepsilon(T^\varepsilon,x,y))-(u(y),0)\|_{L^\infty(\mathbb{T}\times[0,1])}\geq\sigma_0 \tag{18}$$

and

$$\|(u_1^\varepsilon(T^\varepsilon,x,y),u_2^\varepsilon(T^\varepsilon,x,y))-(u(y),0)\|_{L^2(\mathbb{T}\times[0,1])}\geq\sigma_0. \tag{19}$$

Such sequences of solutions of (12–15) does not converge to the formal limit system (1,2,4,5) in sup-norm, even for short time. An example of such a profile $u(y)$ is given in [6]. It has of course an inflexion point in it. The theorem is in fact, up to time and space rescalings, a nonlinear instability theorem. We refer to [4,5] for another approach.

2. Proof of the convergence theorem

Notice that usual energy estimates on (12–15) lead to control $\int|u_1^\varepsilon|^2+\varepsilon^2\int|u_2^\varepsilon|^2$ which appears to be unsufficient in the limit $\varepsilon\to 0$ since we lose any control on $\int|u_2^\varepsilon|^2$. The main difficulty is therefore to obtain estimates on the linearized version of (12–15) which are uniform in ε. Once we get such estimates, it is routine work to prove a convergence theorem like Theorem 1.2. Therefore we will focus on the construction of such a norm in Section 2.2, on higher order derivatives in Section 2.3 and only sketch the end of the proof of Theorem 1.2.

2.1. **Preliminaries**

Let $\mathbf{u}^\varepsilon=(u_1^\varepsilon,u_2^\varepsilon)$ and $\mathbf{u}=(u,v)$. Let us introduce the vorticity ω^ε and the stream function Ψ^ε of u_1^ε and u_2^ε after rescaling. System (12–15) is equivalent to (20–23)

$$\partial_t\omega^\varepsilon+(\mathbf{u}^\varepsilon.\nabla)\omega^\varepsilon=0, \tag{20}$$
$$\mathbf{u}^\varepsilon=\nabla^\perp\Psi^\varepsilon, \tag{21}$$
$$\varepsilon^2\partial_{xx}^2\Psi^\varepsilon+\partial_{yy}^2\Psi^\varepsilon=\omega^\varepsilon, \tag{22}$$
$$\Psi^\varepsilon=0 \qquad \text{for } y=0,1. \tag{23}$$

Notice that *a priori* we only get that Ψ^ε is constant on $y=0$ and $y=1$ and equals some time dependent constants C_0 and C_1. However, up to the addition of a constant to Ψ^ε we can assume $C_0=0$, and up to a Galilean change of variables we can assume

$$\int_{\mathbb{T}\times[0,1]}u_1^\varepsilon=0, \tag{24}$$

which leads then to $C_1=0$. Hence as (24) is an assumption of Theorem 1.2 we can assume (23).

We also remark that the limit system (1,2,4,5) can be rewritten (under the same assumption (24))

$$\partial_t\omega+(\mathbf{u}.\nabla)\omega=0, \tag{25}$$
$$\mathbf{u}=\nabla^\perp\Psi, \tag{26}$$
$$\partial_{yy}^2\Psi=\omega, \tag{27}$$
$$\Psi=0 \qquad \text{for } y=0,1. \tag{28}$$

2.2. Weighted estimates

Let us first prove uniform estimates on the linearized version of (20–23) in $(\mathbf{v}^\varepsilon, \theta^\varepsilon)$, where $\mathbf{v}^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon)$:

$$\partial_t \theta^\varepsilon + (\mathbf{u}^\varepsilon.\nabla)\theta^\varepsilon + (\mathbf{v}^\varepsilon.\nabla)\omega^\varepsilon = 0, \tag{29}$$

$$\mathbf{v}^\varepsilon = \nabla^\perp \Phi^\varepsilon, \tag{30}$$

$$\varepsilon^2 \partial_{xx}^2 \Phi^\varepsilon + \partial_{yy}^2 \Phi^\varepsilon = \theta^\varepsilon, \tag{31}$$

$$\Phi^\varepsilon = 0 \qquad \text{for } y = 0, 1. \tag{32}$$

Following the strategy of [6, 7] we introduce

$$(N^\varepsilon)^2(\mathbf{v}, \theta) = \int |v_1|^2 + \varepsilon^2 |v_2|^2 + g^\varepsilon |\theta|^2, \tag{33}$$

where g^ε will be chosen carefully. Notice that this energy is deeply linked to Arnold's approach of stability for stationary flows.

Lemma 2.1. *Let us assume that there exists g^ε and a constant $\bar{u}$ such that*

$(H1')$ $\qquad |\partial_t g^\varepsilon| + |\nabla g^\varepsilon.\nabla^\perp \Psi^\varepsilon| \le C g^\varepsilon,$

$(H2')$ $\qquad |g^\varepsilon \partial_x \Delta_\varepsilon \Psi^\varepsilon - \partial_x \Psi^\varepsilon| + \varepsilon^{-1} |g^\varepsilon \partial_y \Delta_\varepsilon \Psi^\varepsilon - \partial_y \Psi^\varepsilon - \bar{u}| \le C\sqrt{g}$

for some constant C independent on ε, then there exists a constant C_0 independent on ε such that every solution $(v^\varepsilon, \Phi^\varepsilon)$ of (29–32) satisfies

$$\partial_t N^\varepsilon \le C_0 N^\varepsilon. \tag{34}$$

Proof. Let us drop all the ε indices. We have

$$\partial_t \nabla^\perp \Phi + (\nabla^\perp \Psi.\nabla)\nabla^\perp \Phi + (\nabla^\perp \Phi.\nabla)\nabla^\perp \Psi = -\begin{pmatrix} \partial_x q \\ \varepsilon^{-2} \partial_y q \end{pmatrix},$$

where q is some linearized pressure and

$$\partial_t \Delta_\varepsilon \Phi + (\nabla^\perp \Psi.\nabla)\Delta_\varepsilon \Phi + (\nabla^\perp \Phi.\nabla)\Delta_\varepsilon \Psi = 0.$$

First

$$\begin{aligned} -\frac{1}{2}\partial_t \int g|\theta|^2 &= \int g\Delta_\varepsilon \Phi (\nabla^\perp \Psi.\nabla)\Delta_\varepsilon \Phi + \int g\Delta_\varepsilon \Phi (\nabla^\perp \Phi.\nabla)\Delta_\varepsilon \Psi - \frac{1}{2}\int \partial_t g |\Delta_\varepsilon \Phi|^2 \\ &= I_1 + I_2 + I_3. \end{aligned}$$

The first right-hand side term equals

$$I_1 = \int g(\nabla^\perp \Psi.\nabla)\frac{|\Delta_\varepsilon \Phi|^2}{2} = -\int \nabla.(g\nabla^\perp \Psi)\frac{|\Delta_\varepsilon \Phi|^2}{2}$$

which is bounded by CN^2 using (H1'). Similarly, $|I_3| \le CN^2$ by (H1'). On the other side,

$$-\frac{1}{2}\partial_t \int |v_1|^2 + \varepsilon^2 |v_2|^2 = \int \nabla_\varepsilon^\perp \Phi (\nabla^\perp \Psi.\nabla)\nabla_\varepsilon^\perp \Phi + \nabla_\varepsilon^\perp \Phi (\nabla^\perp \Phi.\nabla)\nabla_\varepsilon^\perp \Psi = I_4 + I_5.$$

We have

$$I_4 = \int (\nabla^\perp \Psi . \nabla) \frac{|\nabla_\varepsilon^\perp \Phi|^2}{2} = 0$$

since $\nabla^\perp \Psi$ is divergence free. Next

$$\begin{aligned} I_2 &= \int g \Delta_\varepsilon \Phi v_1 \partial_x \Delta_\varepsilon \Psi + \int g \Delta_\varepsilon \Phi v_2 \partial_y \Delta_\varepsilon \Psi \\ &= \int \Delta_\varepsilon \Phi \nabla^\perp \Phi \nabla \Psi + \int \Delta_\varepsilon \Phi v_1 (g \partial_x \Delta_\varepsilon \Psi - \partial_x \Psi) + \int \Delta_\varepsilon \Phi v_2 (g \partial_y \Delta_\varepsilon \Psi - \partial_y \Psi - \bar{u}) + \int \Delta_\varepsilon \Phi v_2 \bar{u} \\ &= \int \Delta_\varepsilon \Phi (\nabla^\perp \Phi . \nabla) \Psi + \int \Delta_\varepsilon \Phi \nabla^\perp \Phi (g \nabla \Delta_\varepsilon \Psi - \nabla \Psi - (0, \bar{u})) \end{aligned}$$

since

$$\int \Delta_\varepsilon \Phi v_2 \bar{u} = 0.$$

The last two terms are bounded by CN^2 using (H2') and the first integral equals

$$- \int \nabla_\varepsilon \Phi (\nabla^\perp \Phi . \nabla) \nabla_\varepsilon \Psi$$

which cancels with I_5, which ends the proof. □

It is then easy to check the following lemma

Lemma 2.2. *Let us assume that Ψ^ε and $\partial_t \Psi^\varepsilon$ are bounded sequences of $L^\infty([0,T], H^s)$ with $s > 5$. Let us assume that there exists two constants $\bar{u}$ and $C_1 > 0$ such that*

$$(H) \qquad C_1^{-1} \le \partial_y \Psi^\varepsilon + \bar{u}, \qquad C_1^{-1} \le \frac{\partial_y \Psi^\varepsilon + \bar{u}}{\partial^3_{yyy} \Psi^\varepsilon} \le C_1$$

for every x, y, $0 \le t \le T$ and every $0 \le \varepsilon \le 1$. Then

$$g^\varepsilon = \frac{\partial_y \Psi^\varepsilon + \bar{u}}{\partial^3_{yyy} \Psi^\varepsilon}$$

satisfies (H1') and (H2').

2.3. Higher order estimates

Let us define for $s \ge 1$,

$$N_s^2(\theta) = \sum_{\alpha + \beta \le s} \int |\partial_x^\alpha \partial_y^\beta \theta|^2$$

with the convention $N_0 = N^\varepsilon$.

Lemma 2.3. *Let us assume that there exists constants $C_{\alpha,\beta}$ such that*

$$\|\partial_x^\alpha \partial_y^\beta \mathbf{u}^\varepsilon\|_{L^\infty(\Omega)} \le C_{\alpha,\beta} \tag{35}$$

for $0 < \varepsilon \le 1$. Then there exists a constant C_s independent on ε such that every solution $(v^\varepsilon, \Phi^\varepsilon)$ of (29–32) satisfies

$$\partial_t N_s^2 \le C_s N_s^2 + C_s \varepsilon^{-2} N_{s-1}^2.$$

Proof. Notice, using the divergence free condition, that $\|\partial_x^\alpha \partial_y^\beta v_1\|_{L^2} \leq CN_{\alpha+\beta-1}$. However we only get $\|\partial_x^\alpha \partial_y^\beta v_2\|_{L^2} \leq C\varepsilon^{-1} N_{\alpha+\beta-1}$. We have for $\alpha + \beta \geq 1$,

$$\partial_t \partial_x^\alpha \partial_y^\beta \theta + (\mathbf{u}.\nabla)\partial_x^\alpha \partial_y^\beta \theta = \mathcal{R},$$

where

$$\mathcal{R} = - \sum_{\alpha' \leq \alpha, \beta' \leq \beta, 1 \leq \alpha'+\beta' \leq \alpha+\beta} (\partial_x^{\alpha'} \partial_y^{\beta'} \mathbf{u}.\nabla)\partial_x^{\alpha-\alpha'} \partial_y^{\beta-\beta'} \theta \\ - \sum_{\alpha' \leq \alpha, \beta' \leq \beta, 0 \leq \alpha'+\beta' \leq \alpha+\beta} (\partial_x^{\alpha-\alpha'} \partial_y^{\beta-\beta'} \mathbf{v}.\nabla)\partial_x^{\alpha'} \partial_y^{\beta'} \omega.$$

Notice that $\int \partial_x^\alpha \partial_y^\beta \theta (\mathbf{u}.\nabla)\partial_x^\alpha \partial_y^\beta \theta$ vanishes. Moreover by a crude bound,

$$\|\mathcal{R}\|_{L^2}^2 \leq CN_s^2 + \varepsilon^{-2} N_{s-1}^2,$$

which ends the proof of the lemma.

2.4. End of the proof

We will only sketch the end of the proof since the following arguments have been developed and written down with full details in nearby contexts elsewhere [6, 7].

The next step is to prove a lemma like Lemma 2.3 for the following nonlinear equation

$$\partial_t \theta^\varepsilon + (\mathbf{u}^\varepsilon.\nabla)\theta^\varepsilon + (\mathbf{v}^\varepsilon.\nabla)\omega^\varepsilon + (\mathbf{v}^\varepsilon.\nabla)\theta^\varepsilon = \mathcal{R}^\varepsilon \tag{36}$$

where $\mathcal{R}^\varepsilon$ is a given source term, which is a straightforward adaptation of the former proof.

We then construct an approximate solution of (12,13,14,15) starting from a solution of the limit system, which is easy but lengthly. The last step of the proof is to use the bounds on the nonlinear equation (36) on the difference between the true solution and the approximate one.

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