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Modélisation Mathématique et Analyse Numérique

HOMOGENIZATION OF A MONOTONE PROBLEM IN A DOMAIN WITH OSCILLATING BOUNDARY

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Abstract. We study the asymptotic behaviour of the following nonlinear problem:

$$\begin{cases} -\mathrm{div}(a(Du_h)) + |u_h|^{p-2}u_h = f & \text{in } \Omega_h, \\ a(Du_h)\cdot\nu = 0 & \text{on } \partial\Omega_h, \end{cases}$$

in a domain Ω_h of $\mathbb{R}^n$ whose boundary $\partial\Omega_h$ contains an oscillating part with respect to h when h tends to ∞. The oscillating boundary is defined by a set of cylinders with axis $0x_n$ that are h^{-1}-periodically distributed. We prove that the limit problem in the domain corresponding to the oscillating boundary identifies with a diffusion operator with respect to x_n coupled with an algebraic problem for the limit fluxes.

Résumé. Nous étudions le comportement asymptotique du problème non linéaire monotone

$$\begin{cases} -\mathrm{div}(a(Du_h)) + |u_h|^{p-2}u_h = f & \text{dans } \Omega_h, \\ a(Du_h)\cdot\nu = 0 & \text{sur } \partial\Omega_h, \end{cases}$$

posé sur un ouvert Ω_h de $\mathbb{R}^n$ dont une partie de la frontière oscille avec h lorsque h tend vers ∞. Cette partie oscillante est constituée d'un ensemble de cylindres d'axe Ox_n distribués avec la période h^{-1}. Nous démontrons que dans le domaine correspondant à la partie oscillante, le problème limite couple un problème de diffusion en x_n et un problème algébrique pour les flux limites.

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INTRODUCTION

In this paper we study the asymptotic behaviour, as $h\,(\in \mathbb{N})$ diverges, of a monotone problem defined in a domain Ω_h of $\mathbb{R}^n$ $(n \geq 2)$, whose boundary contains an oscillating part depending on h.

The domain Ω_h is composed of two parts: a fixed part Ω^-, which is a parallelepiped with sides parallel to the coordinate planes, and a part Ω_h^+ that varies with h.

Keywords and phrases. Homogenization, nonlinear problem, oscillating boundary.

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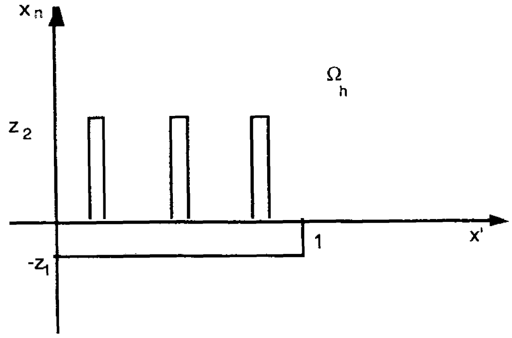

FIGURE 1.

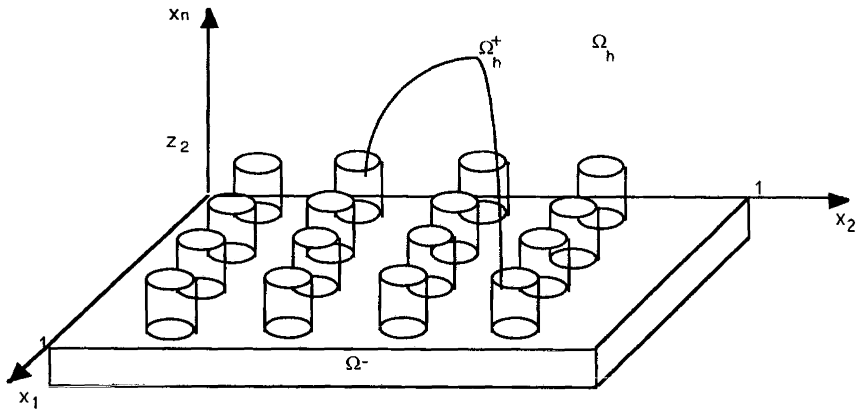

FIGURE 2.

The set Ω_h^+ is defined as follows: let C_h be a cylinder rescaled from a fixed one C by a h^{-1}-homothety in the first $n-1$ variables. Then Ω_h^+ is the union of such cylinders distributed with h^{-1}-periodicity in the first $n-1$ directions $x_1, \cdots, x_{n-1}$. The lower bases of these cylinders lie on the upper side Σ of Ω^- (see Figs. 1 and 2 for the case $n=2$ and $n=3$ respectively). Observe that the volume of the material included in Ω_h^+ does not converge to zero as h tends to $+\infty$.

We study the asymptotic behaviour of the solution u_h, as h diverges, of the following Neumann problem:

$$\begin{cases} -\operatorname{div}(a(Du_h)) + |u_h|^{p-2}u_h = f & \text{in } \Omega_h, \\ a(Du_h)\cdot\nu = 0 & \text{on } \partial\Omega_h, \end{cases} \tag{0.1}$$

where p is a given number in $]1,+\infty[$, f a given function in $L^{\frac{p}{p-1}}(\Omega)$, $a=(a_1,\cdots,a_n)$ a monotone continuous function from $\mathbb{R}^n$ to $\mathbb{R}^n$ satisfying usual growth conditions (see (1.2, 1.3)) and ν denotes the exterior unit normal to Ω_h.

We denote by Ω^+ the smallest parallelepiped containing the sets Ω_h^+ for every h and set $\Omega = \Omega^+ \cup \Omega^- \cup \Sigma$ (see Fig. 3). Moreover, we denote by $\widetilde{u_h}$ and $\widetilde{\partial u_h/\partial x_i}$ the zero extension to Ω of u_h and $\partial u_h/\partial x_i$ respectively.

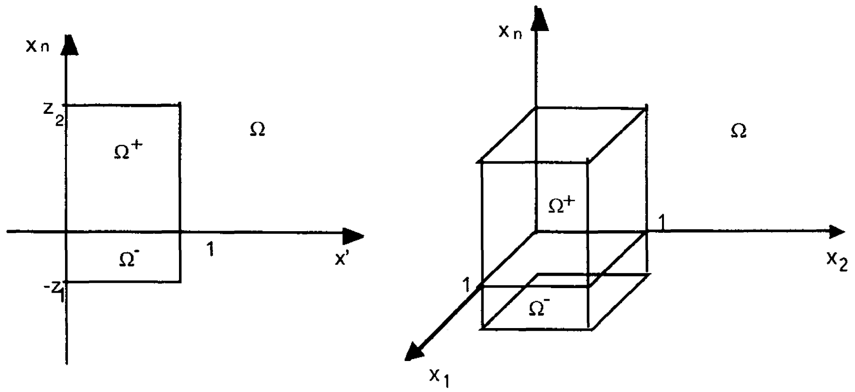

FIGURE 3.

In a nutshell, we prove the existence of a function u in $L^p(\Omega)\cap W^{1,p}(\Omega^-)$ with derivative with respect to x_n in $L^p(\Omega^+)$ and of $n-1$ functions $(d_1,\cdots,d_{n-1})$ in $(L^p(\Omega^+))^{n-1}$ such that

$$\begin{cases} \widetilde{u_h} \rightharpoonup |\omega| u & \text{weakly in } L^p(\Omega^+),\\ \dfrac{\partial \widetilde{u_h}}{\partial x_n} \rightharpoonup |\omega| \dfrac{\partial u}{\partial x_n} & \text{weakly in } L^p(\Omega^+),\\ u_h \rightharpoonup u & \text{weakly in } W^{1,p}(\Omega^-), \end{cases}$$

as h diverges,

$$\lim_{h\to+\infty}\int_{\Omega_h}(a(Du_h)Du_h+|u_h|^p)\,dx=|\omega|\int_{\Omega^+}\left(a_n\left(\frac{d_1}{|\omega|},\cdots,\frac{d_{n-1}}{|\omega|},\frac{\partial u}{\partial x_n}\right)\frac{\partial u}{\partial x_n}+|u|^p\right)dx$$
$$+\int_{\Omega^-}(a(Du)Du+|u|^p)\,dx$$

and $(u,d_1,\cdots,d_{n-1})$ is a weak solution of the following problem:

$$\begin{cases} -\frac{\partial}{\partial x_n}a_n\left(\dfrac{d_1}{|\omega|},\cdots,\dfrac{d_{n-1}}{|\omega|},\dfrac{\partial u}{\partial x_n}\right)+|u|^{p-2}u=f & \text{in } \Omega^+,\\ -\operatorname{div}(a(Du))+|u|^{p-2}u=f & \text{in } \Omega^-,\\ u^+=u^-,\quad |\omega|a_n\left(\dfrac{d_1}{|\omega|},\cdots,\dfrac{d_{n-1}}{|\omega|},\dfrac{\partial u^+}{\partial x_n}\right)=a_n(Du^-) & \text{on } \Sigma,\\ a_n\left(\dfrac{d_1}{|\omega|},\cdots,\dfrac{d_{n-1}}{|\omega|},\dfrac{\partial u}{\partial x_n}\right)=0 & \text{on the upper boundary of } \Omega,\\ a(Du)\cdot\nu=0 & \text{on } \partial\Omega^- - \Sigma,\\ a_i\left(\dfrac{d_1}{|\omega|},\cdots,\dfrac{d_{n-1}}{|\omega|},\dfrac{\partial u}{\partial x_n}\right)=0 & \text{in } \Omega^+,\quad \forall i\in 1,\cdots,n-1, \end{cases}$$

where u^- (resp. u^+) denotes the restriction of u to Ω^- (resp. Ω^+) and $|\omega|$ denotes the $(n-1)$-dimensional Lebesgue-measure of the section $\{(x_1,\cdots,x_n)\in C : x_n=0\}$ of the reference cylinder C (see Th. 1.2 and Cor. 1.3).

The limit behaviour of problem (0.1) with $a(\xi)=\xi$ is studied by Brizzi and Chalot in [5, 6] and, with a non-homogeneous Neumann boundary condition, by Gaudiello in [16].

The limit behaviour of problem (0.1) with $a(\xi) = |\xi|^{p-2}\xi$, p in $[2, +\infty[$, is also obtained, using a few arguments of Γ-convergence, by Corbo Esposito, Donato, Gaudiello and Picard in [8].

In the context of the asymptotic behaviour of thin plates or cylinders, similar limit problems are obtained in [19, 20].

The goal of the present paper is to achieve the limit process in (0.1) through usual monotonicity methods.

For general references about homogenization, we refer to [2–4, 11, 24]. For the homogenization of quasilinear operators in other periodic frameworks, we refer to [10, 15] for the case of a fixed domain, to [1, 9, 14] for the case of periodically perforated domains and to [7] for reinforcement problems by a layer with oscillating thickness.

If the Neumann boundary condition in Problem (0.1) is replaced by the homogeneous Dirichlet condition $u_h = 0$ on $\partial\Omega_h$, performing the limit process, as h diverges, becomes an easier task that is left to the reader (see *e.g.* [5, 13, 18, 21–23] for similar problems). In this case the limit problem reads as

$$\begin{cases} u \in W_0^{1,p}(\Omega), & \\ u = 0 & \text{in } \Omega^+, \\ -\operatorname{div}(a(Du)) + |u|^{p-2}u = f & \text{in } \Omega^-. \end{cases}$$

As far as this Dirichlet problem is concerned, the lower order term $|u_h|^{p-2}u_h$ may be removed in the whole analysis. By contrast, this term is in general necessary for the Neumann problem in order to derive an estimate on $\|u_h\|_{L^p(\Omega_h)}$ $(p > 1)$ independent of h, unless one has a Poincaré -Wirtinger inequality with a constant independent of h in $W^{1,p}(\Omega_h)$. This is still an open problem.

1. Statement of the problem and main results

Let z_1, z_2 be in $]0, +\infty[$, ω an open smooth subset of $\mathbb{R}^{n-1}$ such that $\omega \subset\subset]0, 1[^{n-1}$ $(n \geq 2)$. Let us introduce the following domains in $\mathbb{R}^n$:

$$\begin{cases} \Omega =]0, 1[^{n-1} \times] - z_1, z_2[, \\ \Omega^- =]0, 1[^{n-1} \times] - z_1, 0[, \quad \Omega^+ =]0, 1[^{n-1} \times]0, z_2[, \\ \Sigma =]0, 1[^{n-1} \times \{0\}, \\ \Omega_h = \Omega^- \cup \left(\bigcup_{k \in J_h} \left(\frac{1}{h}\omega + \frac{1}{h}k \right) \times [0, z_2[\right) \quad h \in \mathbb{N}, \\ J_h = \{k = (k_1, \cdots, k_{n-1}) \in \mathbb{N}^{n-1} \ : \ 0 \leq k_i \leq h-1, \ i = 1, \cdots, n-1\} \\ \Omega_h^+ = \Omega^+ \cap \Omega_h \quad h \in \mathbb{N}. \end{cases} \tag{1.1}$$

The generic point of $\mathbb{R}^n$ will be denoted by $x = (x_1, \cdots, x_{n-1}, x_n)$.

Let p be a given number in $]1, +\infty[$, f a given function in $L^{\frac{p}{p-1}}(\Omega)$ and $a = (a_1, \cdots, a_n)$ a monotone continuous function from $\mathbb{R}^n$ to $\mathbb{R}^n$ satisfying the following conditions:

$$\exists \alpha \in]0, +\infty[: \quad \alpha|\xi|^p \leq a(\xi)\xi \qquad \forall \xi \in \mathbb{R}^n, \tag{1.2}$$

$$\exists \beta, \gamma \in]0, +\infty[: \quad |a(\xi)| \leq \beta + \gamma|\xi|^{p-1} \quad \forall \xi \in \mathbb{R}^n. \tag{1.3}$$

Let us consider the following Neumann problem:

$$\begin{cases} -\operatorname{div}(a(Du_h)) + |u_h|^{p-2}u_h = f & \text{in } \Omega_h, \\ a(Du_h) \cdot \nu = 0 & \text{on } \partial\Omega_h, \end{cases} \tag{1.4}$$

where ν denotes the exterior unit normal to Ω_h. It is well known (see [17]) that problem (1.4) admits a unique weak solution u_h in $W^{1,p}(\Omega_h)$.

Our aim is to study the asymptotic behaviour of u_h as h diverges.

We recall that a function of $L^p(\Omega^+)$ with derivative with respect to x_n in $L^p(\Omega^+)$ admits a trace on Σ. Consequently, we introduce the space

$$V^p(\Omega) = \left\{ v \in L^p(\Omega) \ : \ v \in W^{1,p}(\Omega^-), \ \frac{\partial v}{\partial x_n} \in L^p(\Omega^+), \ v^+ = v^- \text{ on } \Sigma \right\}, \tag{1.5}$$

where v^- (resp. v^+) denotes the restriction of v to Ω^- (resp. Ω^+), provided with the norm:

$$\|v\|_{V^p(\Omega)} = \|v\|_{W^{1,p}(\Omega^-)} + \|v\|_{L^p(\Omega^+)} + \left\| \frac{\partial v}{\partial x_n} \right\|_{L^p(\Omega^+)} \qquad v \in V^p(\Omega).$$

We refer to Proposition 4.1 of [8] for the following properties of $V^p(\Omega)$:

Proposition 1.1. *$V^p(\Omega)$ is a Banach space and $W^{1,p}(\Omega)$ is dense in $V^p(\Omega)$ with continuous injection.*

Moreover, we recall that

$$\chi_{\Omega_h^+} \rightharpoonup |\omega| \quad \text{in } L^\infty(\Omega^+) \text{ weak } *, \tag{1.6}$$

where $|\omega|$ denotes the $(n-1)$-dimensional Lebesgue measure of ω and χ_A denotes the characteristic function of a set A.

In the sequel, $\widetilde{v}$ or $[v]^\sim$ denotes the zero-extension to Ω of any (vector) function v defined on a subset of Ω.

The main result of this paper is given in the following theorem:

Theorem 1.2. *Let u_h, h in $\mathbb{N}$, be the weak solution of problem (1.4) and $V^p(\Omega)$ the space defined in (1.5). Then, there exists u in $V^p(\Omega)$ such that*

$$\begin{cases} \widetilde{u_h} \rightharpoonup |\omega| u & \text{weakly in } L^p(\Omega^+), \\ \dfrac{\partial \widetilde{u_h}}{\partial x_n} \rightharpoonup |\omega| \dfrac{\partial u}{\partial x_n} & \text{weakly in } L^p(\Omega^+), \\ u_h \rightharpoonup u & \text{weakly in } W^{1,p}(\Omega^-); \end{cases} \tag{1.7}$$

an increasing sequence of positive integer numbers, still denoted by $\{h\}_{h\in\mathbb{N}}$, and $(d_1, \cdots, d_{n-1})$ in $(L^p(\Omega^+))^{n-1}$, depending possibly on the selected subsequence, such that

$$\widetilde{\frac{\partial u_h}{\partial x_i}} \rightharpoonup d_i \quad \text{weakly in } L^p(\Omega^+), \quad \forall i \in \{1, \cdots, n-1\}, \tag{1.8}$$

as h diverges, where $(u, d_1, \cdots, d_{n-1})$ is a weak solution of the following problem:

$$\begin{cases} -\dfrac{\partial}{\partial x_n} a_n\left(\dfrac{d_1}{|\omega|}, \cdots, \dfrac{d_{n-1}}{|\omega|}, \dfrac{\partial u}{\partial x_n}\right) + |u|^{p-2}u = f & \text{in } \Omega^+, \\ -\operatorname{div}(a(Du)) + |u|^{p-2}u = f & \text{in } \Omega^-, \\ u^+ = u^-, \quad |\omega| a_n\left(\dfrac{d_1}{|\omega|}, \cdots, \dfrac{d_{n-1}}{|\omega|}, \dfrac{\partial u^+}{\partial x_n}\right) = a_n(Du^-) & \text{on } \Sigma, \\ a_n\left(\dfrac{d_1}{|\omega|}, \cdots, \dfrac{d_{n-1}}{|\omega|}, \dfrac{\partial u}{\partial x_n}\right) = 0 & \text{on }]0,1[^{n-1} \times \{z_2\}, \\ a(Du) \cdot \nu = 0 & \text{on } \partial\Omega^- - \Sigma, \\ a_i\left(\dfrac{d_1}{|\omega|}, \cdots, \dfrac{d_{n-1}}{|\omega|}, \dfrac{\partial u}{\partial x_n}\right) = 0 & \text{in } \Omega^+, \quad \forall i \in 1, \cdots, n-1 \end{cases} \tag{1.9}$$

and the function u in $V^p(\Omega)$ satisfying problem (1.9) is unique.
Moreover, the energies converge in the sense that:

$$\lim_{h\to+\infty}\int_{\Omega_h}(a(Du_h)Du_h+|u_h|^p)\,\mathrm{d}x$$
$$=|\omega|\int_{\Omega^+}(a_n\left(\frac{d_1}{|\omega|},\cdots,\frac{d_{n-1}}{|\omega|},\frac{\partial u}{\partial x_n}\right)\frac{\partial u}{\partial x_n}+|u|^p)\,\mathrm{d}x+\int_{\Omega^-}(a(Du)Du+|u|^p)\,\mathrm{d}x$$
$$=\int_\Omega(|\omega|\chi_{\Omega^+}+\chi_{\Omega^-})\,fu\,\mathrm{d}x. \tag{1.10}$$

If a is monotone, there is a unique function u in $V^p(\Omega)$ satisfying problem (1.9) (see Step 10 of Sect. 2). Moreover, if a is strictly monotone, problem (1.9) admits a unique solution $(u,d_1,\cdots,d_{n-1})$ in $V^p(\Omega)\times(L^p(\Omega^+))^{n-1}$ (see Step 11 of Sect. 2). Consequently, convergence (1.8) holds for the whole sequence $\{u_h\}_{h\in\mathbb{N}}$ and Theorem 1.2 yields the following result:

Corollary 1.3. *Let u_h, h in $\mathbb{N}$, be the weak solution of problem (1.4) with a strictly monotone and $V^p(\Omega)$ the space defined in (1.5). Then,*

$$\begin{cases}|\omega|u & \text{weakly in } L^p(\Omega^+),\\ \dfrac{\partial\widetilde{u_h}}{\partial x_n}\rightharpoonup|\omega|\dfrac{\partial u}{\partial x_n} & \text{weakly in } L^p(\Omega^+),\\ \dfrac{\partial\widetilde{u_h}}{\partial x_i}\rightharpoonup d_i & \text{weakly in } L^p(\Omega^+),\quad \forall i\in\{1,\cdots,n-1\},\\ u_h\rightharpoonup u & \text{weakly in } W^{1,p}(\Omega^-),\end{cases}$$

as h diverges, where $(u,d_1,\cdots,d_{n-1})$ is the unique weak solution in $V^p(\Omega)\times(L^p(\Omega^+))^{n-1}$ of the problem (1.9). Moreover, the convergence of the energies (1.10) holds.

Remark 1.4. In the case $a(\xi)=\xi$, Corollary 1.3 is proved in [5, 6] by making use of a method introduced by Tartar in [25] (method of oscillating test functions).

The limit behaviour of problem (1.4) with $a(\xi)=\xi$ and with a non-homogeneous Neumann boundary condition is studied in [16]. In this case, an additional term may appear in the limit equation.

In the case $a(\xi)=|\xi|^{p-2}\xi$, with p in $[2,+\infty[$, Corollary 1.3 is also proved in [8] by following a method introduced by De Giorgi and Franzoni in [12] (Γ-convergence). In this case it results

$$d_1=\cdots=d_{n-1}=0\quad\text{a.e. in }\Omega^+$$

and limit problem (1.9) assumes the following formulation:

$$\begin{cases}-\dfrac{\partial}{\partial x_n}\left(\left|\dfrac{\partial u}{\partial x_n}\right|^{p-2}\dfrac{\partial u}{\partial x_n}\right)+|u|^{p-2}u=f & \text{in }\Omega^+,\\ -\mathrm{div}(|Du|^{p-2}Du)+|u|^{p-2}u=f & \text{in }\Omega^-,\\ u^+=u^-,\quad |\omega|\left|\dfrac{\partial u^+}{\partial x_n}\right|^{p-2}\dfrac{\partial u^+}{\partial x_n}=|Du^-|^{p-2}\dfrac{\partial u^-}{\partial x_n} & \text{on }\Sigma,\\ \dfrac{\partial u}{\partial x_n}=0 & \text{on }]0,1[^{n-1}\times\{z_2\},\\ |Du|^{p-2}Du\cdot\nu=0 & \text{on }\partial\Omega^--\Sigma.\end{cases}$$

□

The proof of Theorem 1.2 is performed in Section 2 with 12 steps. First, we give *a priori* norm-estimates for u_h, $|u_h|^{p-2}u_h$ and $a(Du_h)$. Then, by virtue of the particular shape of Ω_h and by making use of the method of the oscillating test functions, we identify the limit of $\widetilde{Du_h}$ in Ω^-, $\widetilde{\partial u_h/\partial x_n}$ in Ω^+ and $[a_1(Du_h)]^\sim, \cdots, [a_{n-1}(Du_h)]^\sim$ in Ω^+. Moreover, by a monotonicity argument, we identify the limit of $|\widetilde{u_h}|^{p-2}\widetilde{u_h}$ in Ω, $[a_n(Du_h)]^\sim$ in Ω^+, $a(Du_h)$ in Ω^- and obtain the last equation in (1.9). Finally, we pass to the limit in (1.4) and we conclude with some results about the uniqueness of the solution of problem (1.9).

2. Proof of the results

The proof of Theorem 1.2 will be performed in 12 steps.

Proof of Theorem 1.2. The variational formulation of problem (1.4) is given by

$$\begin{cases} \displaystyle\int_{\Omega_h} a(Du_h)Dv + |u_h|^{p-2}u_h v \,\mathrm{d}x = \int_{\Omega_h} fv \,\mathrm{d}x \quad \forall v \in W^{1,p}(\Omega_h), \\ u_h \in W^{1,p}(\Omega_h). \end{cases} \tag{2.1}$$

In the sequel, c will denote any positive constant independent of h.

Step 1. *A priori norm-estimate for u_h, $|u_h|^{p-2}u_h$ and $a(Du_h)$*

By choosing $v = u_h$ as test function in (2.1) and by making use of (1.2), it easily results

$$\|u_h\|_{W^{1,p}(\Omega_h)} \le c \quad \forall h \in \mathbb{N}. \tag{2.2}$$

From (2.2) it follows that

$$\|\, |u_h|^{p-2}u_h\|_{L^{\frac{p}{p-1}}(\Omega_h)} \le c \quad \forall h \in \mathbb{N}. \tag{2.3}$$

Moreover, (1.3) and (2.2) provide that

$$\|a(Du_h)\|_{\left(L^{\frac{p}{p-1}}(\Omega_h)\right)^n} \le c \quad \forall h \in \mathbb{N}. \tag{2.4}$$

By virtue of (2.2–2.4), there exists an increasing sequence of positive integer numbers, still denoted by $\{h\}_{h\in\mathbb{N}}$, u in $L^p(\Omega)$, $d = (d_1, \cdots, d_n)$ in $(L^p(\Omega))^n$, z in $L^{\frac{p}{p-1}}(\Omega)$ and $\eta = (\eta_1, \cdots, \eta_n)$ in $\left(L^{\frac{p}{p-1}}(\Omega)\right)^n$ satisfying the following convergences:

$$\widetilde{u_h} \rightharpoonup |\omega| u\chi_{\Omega^+} + u\chi_{\Omega^-} \quad \text{weakly in } L^p(\Omega), \tag{2.5}$$

$$\widetilde{Du_h} \rightharpoonup d \quad \text{weakly in } (L^p(\Omega))^n, \tag{2.6}$$

$$|\widetilde{u_h}|^{p-2}\widetilde{u_h} \rightharpoonup z \quad \text{weakly in } L^{\frac{p}{p-1}}(\Omega), \tag{2.7}$$

$$[a(Du_h)]^\sim \rightharpoonup \eta \quad \text{weakly in } \left(L^{\frac{p}{p-1}}(\Omega)\right)^n, \tag{2.8}$$

as h diverges. *A priori* u, d, z and η could depend on the selected subsequence.

In the sequel, $\{h\}_{h\in\mathbb{N}}$ will denote the previous selected subsequence of $\mathbb{N}$.

Step 2. *Identification of d on Ω^- and d_n on Ω^+*

Convergences (2.5, 2.6) provide that

$$d = Du \quad \text{a.e. in } \Omega^-. \tag{2.9}$$

Moreover, by following arguments identical to those used in Proposition 2.2 and Corollary 2.3 of [8], it is easy to prove that

$$d_n = |\omega| \frac{\partial u}{\partial x_n} \quad \text{a.e. in } \Omega^+ \tag{2.10}$$

and

$$u \in V^p(\Omega). \tag{2.11}$$

Step 3. *Identification of $\eta_1, \cdots, \eta_{n-1}$ on Ω^+*

This step is devoted to the proof of

$$\eta_i = 0 \quad \text{a.e. in } \Omega^+, \quad \forall i \in \{1, \cdots, n-1\} \cdot \tag{2.12}$$

For every i in $\{1, \cdots, n-1\}$, let $\{w_h^i\}_{h \in \mathbb{N}}$ be a sequence in $W^{1,\infty}(\Omega^+)$ satisfying the following conditions:

$$w_h^i \to x_i \quad \text{strongly in } L^\infty(\Omega^+) \text{ as } h \to +\infty, \tag{2.13}$$

$$Dw_h^i = 0 \quad \text{a.e. in } \Omega_h^+, \quad \forall h \in \mathbb{N}. \tag{2.14}$$

The existence of such sequences is proved in [8] Lemma 4.3.

By choosing $v = \varphi w_h^i$ and $v = \varphi x_i$, with φ in $C_0^\infty(\Omega^+)$, as test functions in (2.1), by virtue of (2.14) we obtain

$$\int_{\Omega^+} ([a(Du_h)]^\sim D\varphi w_h^i + |\widetilde{u_h}|^{p-2}\widetilde{u_h}\varphi w_h^i)\, \mathrm{d}x = \int_{\Omega^+} (\chi_{\Omega_h^+} f \varphi w_h^i)\mathrm{d}x \quad \forall \varphi \in C_0^\infty(\Omega^+), \tag{2.15}$$

$$\int_{\Omega^+} ([a(Du_h)]^\sim D(\varphi x_i) + |\widetilde{u_h}|^{p-2}\widetilde{u_h}\varphi x_i)\, \mathrm{d}x = \int_{\Omega^+} (\chi_{\Omega_h^+} f \varphi x_i)\mathrm{d}x \quad \forall \varphi \in C_0^\infty(\Omega^+), \tag{2.16}$$

for any h in $\mathbb{N}$ and every i in $\{1, \cdots, n-1\}$.

By passing to the limit, as h diverges, in (2.15, 2.16), convergences (1.6, 2.7, 2.8, 2.13) provide that

$$\int_{\Omega^+} (\eta D\varphi x_i + z\varphi x_i)\, \mathrm{d}x = \int_{\Omega^+} |\omega| f \varphi x_i \, \mathrm{d}x \quad \forall \varphi \in C_0^\infty(\Omega^+), \tag{2.17}$$

$$\int_{\Omega^+} (\eta D(\varphi x_i) + z\varphi x_i)\, \mathrm{d}x = \int_{\Omega^+} |\omega| f \varphi x_i \, \mathrm{d}x \quad \forall \varphi \in C_0^\infty(\Omega^+), \tag{2.18}$$

for every i in $\{1, \cdots, n-1\}$.

Statement (2.12) is obtained by subtracting (2.17) from (2.18).

Step 4. *Convergence of the energies*

This step is devoted to the proof of

$$\lim_{h\to+\infty}\int_{\Omega_h}(a(Du_h)Du_h+|u_h|^p)\,\mathrm{d}x=\int_{\Omega^+}\eta_n\frac{\partial u}{\partial x_n}\,\mathrm{d}x+\int_{\Omega^-}\eta Du\,\mathrm{d}x+\int_{\Omega}zu\,\mathrm{d}x. \tag{2.19}$$

By passing to the limit, as h diverges, in (2.1) with v in $W^{1,p}(\Omega)$, by virtue of (1.6, 2.7, 2.8, 2.12) we obtain

$$\int_{\Omega^+}\eta_n\frac{\partial v}{\partial x_n}\,\mathrm{d}x+\int_{\Omega^-}\eta Dv\,\mathrm{d}x+\int_{\Omega}zv\,\mathrm{d}x=\int_{\Omega}(|\omega|\chi_{\Omega^+}+\chi_{\Omega^-})\,fv\,\mathrm{d}x\quad\forall v\in W^{1,p}(\Omega). \tag{2.20}$$

Since $W^{1,p}(\Omega)$ is dense in $V^p(\Omega)$ (see Prop. 1.1), $v=u$ can be chosen as test function in (2.20). Consequently

$$\int_{\Omega^+}\eta_n\frac{\partial u}{\partial x_n}\,\mathrm{d}x+\int_{\Omega^-}\eta Du\,\mathrm{d}x+\int_{\Omega}zu\,\mathrm{d}x=\int_{\Omega}(|\omega|\chi_{\Omega^+}+\chi_{\Omega^-})\,fu\,\mathrm{d}x. \tag{2.21}$$

On the other hand, by choosing $v=u_h$ as test function in (2.1), by virtue of (2.5) we obtain

$$\lim_{h\to+\infty}\int_{\Omega_h}a(Du_h)Du_h+|u_h|^p\,\mathrm{d}x=\lim_{h\to+\infty}\int_{\Omega}f\widetilde{u_h}\,\mathrm{d}x=\int_{\Omega}f\left(|\omega|u\chi_{\Omega^+}+u\chi_{\Omega^-}\right)\,\mathrm{d}x. \tag{2.22}$$

Convergence (2.19) is obtained by comparing (2.21) with (2.22).

Step 5. *Monotone relation*

This step is devoted to the proof of

$$\int_{\Omega^+}\left(\eta_n\left(\frac{\partial u}{\partial x_n}-\tau_n\right)-a(\tau)(d-|\omega|\tau)\right)\mathrm{d}x+\int_{\Omega^-}(\eta-a(\tau))(Du-\tau)\,\mathrm{d}x+\int_{\Omega^+}\left(z-|\omega||v|^{p-2}v\right)(u-v)\,\mathrm{d}x$$
$$+\int_{\Omega^-}\left(z-|v|^{p-2}v\right)(u-v)\,\mathrm{d}x\geq 0\quad\forall\tau\in(L^p(\Omega))^n,\quad\forall v\in L^p(\Omega), \tag{2.23}$$

which will enable us to identify η, z and to derive the equation satisfied by u in Ω^+.

Let τ be in $(L^p(\Omega))^n$ and v in $L^p(\Omega)$.

Since the functions $a(\xi)$ and $|t|^{p-2}t$ are monotone, we obtain

$$(a(Du_h)-a(\tau))\,(Du_h-\tau)+\left(|u_h|^{p-2}u_h-|v|^{p-2}v\right)(u_h-v)\geq 0\quad\text{a.e. in }\Omega_h,\quad\forall h\in\mathbb{N},$$

from which it follows that

$$\int_{\Omega}([a(Du_h)]^{\sim}\widetilde{Du_h}-[a(Du_h)]^{\sim}\tau-a(\tau)\widetilde{Du_h}+a(\tau)\tau\chi_{\Omega_h})\,\mathrm{d}x$$
$$+\int_{\Omega}(|\widetilde{u_h}|^p-|\widetilde{u_h}|^{p-2}\widetilde{u_h}v-|v|^{p-2}v\widetilde{u_h}+|v|^p\chi_{\Omega_h})\,\mathrm{d}x\geq 0\quad\forall h\in\mathbb{N}. \tag{2.24}$$

By passing to the limit, as h diverges, in (2.24) and by making use of (1.3, 1.6, 2.5–2.9, 2.12, 2.19), we obtain

$$\int_{\Omega^+}\eta_n\frac{\partial u}{\partial x_n}\,\mathrm{d}x+\int_{\Omega^-}\eta Du\,\mathrm{d}x-\int_{\Omega^+}\eta_n\tau_n\,\mathrm{d}x-\int_{\Omega^-}\eta\tau\,\mathrm{d}x-\int_{\Omega^+}a(\tau)d\,\mathrm{d}x-\int_{\Omega^-}a(\tau)Du\,\mathrm{d}x+\int_{\Omega^+}|\omega|a(\tau)\tau\,\mathrm{d}x$$
$$+\int_{\Omega^-}a(\tau)\tau\,\mathrm{d}x+\int_{\Omega}zu\,\mathrm{d}x-\int_{\Omega}zv\,\mathrm{d}x-\int_{\Omega^+}|\omega||v|^{p-2}vu\,\mathrm{d}x-\int_{\Omega^-}|v|^{p-2}vu\,\mathrm{d}x+\int_{\Omega^+}|\omega||v|^p\,\mathrm{d}x+\int_{\Omega^-}|v|^p\,\mathrm{d}x\geq 0$$

and inequality (2.23) is proved.

Step 6. *Identification of z in Ω*

This step is devoted to the proof of

$$z = |\omega||u|^{p-2}u \quad \text{a.e. in } \Omega^+ \tag{2.25}$$

and

$$z = |u|^{p-2}u \quad \text{a.e. in } \Omega^-. \tag{2.26}$$

Let us remark that a typical nonlinear phenomenon occurs here: (2.7, 2.25) show that the $L^{\frac{p}{p-1}}(\Omega^+)$ - weak limit of $|\widetilde{u_h}|^{p-2}\widetilde{u_h}$ is $|\omega||u|^{p-2}u$ and not, as expected, $|\omega|^{p-1}|u|^{p-2}u$.

By choosing $\tau = \frac{1}{|\omega|}d\chi_{\Omega^+} + Du\chi_{\Omega^-}$ and $v = (u - t\varphi)\chi_{\Omega^+} + u\chi_{\Omega^-}$, with t in $(0, +\infty)$ and φ in $C_0^\infty(\Omega^+)$, in (2.23) and by recalling (2.10), we obtain

$$\int_{\Omega^+} \left(z - |\omega||u - t\varphi|^{p-2}(u - t\varphi)\right) t\varphi \, dx \geq 0 \quad \forall t \in (0, +\infty), \quad \forall \varphi \in C_0^\infty(\Omega^+). \tag{2.27}$$

By dividing (2.27) by t and by passing to the limit as t tends to zero, by virtue of the Lebesgue Theorem it follows that

$$\int_{\Omega^+} \left(z - |\omega||u|^{p-2}u\right) \varphi \, dx \geq 0 \quad \forall \varphi \in C_0^\infty(\Omega^+),$$

which implies (2.25). Statement (2.26) can be proved in the same way, by choosing $\tau = \frac{1}{|\omega|}d\chi_{\Omega^+} + Du\chi_{\Omega^-}$ and $v = u\chi_{\Omega^+} + (u - t\varphi)\chi_{\Omega^-}$, with t in $(0, +\infty)$ and φ in $C_0^\infty(\Omega^-)$, in (2.23).

Step 7. *Equation satisfied by d in Ω^+*

This step is devoted to the proof of

$$a_i\left(\frac{d_1}{|\omega|}, \cdots, \frac{d_{n-1}}{|\omega|}, \frac{\partial u}{\partial x_n}\right) = 0 \quad \text{a.e. in } \Omega^+, \quad \forall i \in \{1, \cdots, n-1\} \cdot \tag{2.28}$$

By choosing $\tau = (\tau_1, \cdots, \tau_{n-1}, \partial u/\partial x_n)\chi_{\Omega^+} + Du\chi_{\Omega^-}$ and $v = u$, with $\tau_1, \cdots, \tau_{n-1}$ in $L^p(\Omega^+)$, in (2.23) and by recalling (2.10), we obtain

$$-\int_{\Omega^+} \sum_{j=1}^{n-1} \left(a_j\left(\tau_1, \cdots, \tau_{n-1}, \frac{\partial u}{\partial x_n}\right)(d_j - |\omega|\tau_j)\right) dx \geq 0 \qquad \forall (\tau_1, \cdots, \tau_{n-1}) \in \left(L^p(\Omega^+)\right)^{n-1}. \tag{2.29}$$

Let i be fixed in $\{1, \cdots, n-1\}$. By choosing

$$\begin{cases} \tau_i &= \dfrac{d_i - t\varphi}{|\omega|}, \\ \tau_j &= \dfrac{d_j}{|\omega|} \qquad \forall j \in \{1, \cdots, n-1\} - \{i\}, \end{cases}$$

with t in $(0, +\infty)$ and φ in $C_0^\infty(\Omega^+)$, in (2.29), we obtain

$$\int_{\Omega^+} a_i\left(\frac{d_1}{|\omega|}, \cdots, \frac{d_i - t\varphi}{|\omega|}, \cdots, \frac{d_{n-1}}{|\omega|}, \frac{\partial u}{\partial x_n}\right) t\varphi \, dx \geq 0 \qquad \forall t \in (0, +\infty), \quad \forall \varphi \in C_0^\infty(\Omega^+). \tag{2.30}$$

By dividing (2.30) by t and by passing to the limit as t tends to zero, by virtue of the assumption on a and the Lebesgue Theorem it follows that

$$\int_{\Omega^+} a_i\left(\frac{d_1}{|\omega|},\dots,\frac{d_i}{|\omega|},\dots,\frac{d_{n-1}}{|\omega|},\frac{\partial u}{\partial x_n}\right)\varphi\,\mathrm{d}x \geq 0 \qquad \forall\varphi\in C_0^\infty(\Omega^+),$$

which implies (2.28).

Step 8. *Identification of η_n in Ω^+ and η in Ω^-*

This step is devoted to the proof of

$$\eta_n = |\omega| a_n\left(\frac{d_1}{|\omega|},\dots,\frac{d_{n-1}}{|\omega|},\frac{\partial u}{\partial x_n}\right) \quad \text{a.e. in } \Omega^+ \tag{2.31}$$

and

$$\eta = a(Du) \quad \text{a.e. in } \Omega^-. \tag{2.32}$$

By choosing $\tau = \left(\frac{d_1}{|\omega|},\dots,\frac{d_{n-1}}{|\omega|},\frac{d_n}{|\omega|}-t\varphi\right)\chi_{\Omega^+} + Du\chi_{\Omega^-}$ and $v=u$, with t in $(0,+\infty)$ and φ in $C_0^\infty(\Omega^+)$, in (2.23) and by recalling (2.10), we obtain

$$\int_{\Omega^+}\left(\eta_n - a_n\left(\frac{d_1}{|\omega|},\dots,\frac{d_{n-1}}{|\omega|},\frac{\partial u}{\partial x_n}-t\varphi\right)|\omega|\right)t\varphi\,\mathrm{d}x \geq 0 \qquad \forall t\in(0,+\infty),\quad \forall\varphi\in C_0^\infty(\Omega^+). \tag{2.33}$$

By dividing (2.33) by t and by passing to the limit as t tends to zero, by virtue of the assumption on a and the Lebesgue Theorem it follows that

$$\int_{\Omega^+}\left(\eta_n - a_n\left(\frac{d_1}{|\omega|},\dots,\frac{d_{n-1}}{|\omega|},\frac{\partial u}{\partial x_n}\right)|\omega|\right)\varphi\,\mathrm{d}x \geq 0 \qquad \forall\varphi\in C_0^\infty(\Omega^+),$$

which implies (2.31).

On the other hand, by choosing $\tau = \frac{1}{|\omega|}d\chi_{\Omega^+} + (Du - t\varphi)\chi_{\Omega^-}$ and $v=u$, with t in $(0,+\infty)$ and φ in $\left(C_0^\infty(\Omega^-)\right)^n$, in (2.23) and by recalling (2.10), it yields

$$\int_{\Omega^-}(\eta - a(Du-t\varphi))\,t\varphi\,\mathrm{d}x \geq 0 \qquad \forall t\in(0,+\infty),\quad \forall\varphi\in\left(C_0^\infty(\Omega^-)\right)^n. \tag{2.34}$$

By dividing (2.34) by t and by passing to the limit as t tends to zero, by virtue of the assumption on a and the Lebesgue Theorem it follows that

$$\int_{\Omega^-}(\eta - a(Du))\,\varphi\,\mathrm{d}x \geq 0 \quad \forall\varphi\in\left(C_0^\infty(\Omega^-)\right)^n,$$

which implies (2.32).

Step 9. *Equation satisfied by u and d*

By passing to the limit, as h diverges, in (2.1) with v in $W^{1,p}(\Omega)$ and by making use of (1.6, 2.7, 2.8, 2.11, 2.12, 2.25, 2.26, 2.31, 2.32), it follows that

$$\begin{cases} \displaystyle\int_{\Omega^+} |\omega| a_n\left(\frac{d_1}{|\omega|},\cdots,\frac{d_{n-1}}{|\omega|},\frac{\partial u}{\partial x_n}\right)\frac{\partial v}{\partial x_n}\,dx + \int_{\Omega^-} a(Du)Dv\,dx + \int_{\Omega^+}|\omega||u|^{p-2}uv\,dx + \int_{\Omega^-}|u|^{p-2}uv\,dx \\ \displaystyle = \int_\Omega (|\omega|\chi_{\Omega^+}+\chi_{\Omega^-})fv\,dx \qquad \forall v\in W^{1,p}(\Omega), \quad (u,d_1,\cdots,d_{n-1})\in V^p(\Omega)\times\left(L^p(\Omega^+)\right)^{n-1}. \end{cases} \tag{2.35}$$

Since $W^{1,p}(\Omega)$ is dense in $V^p(\Omega)$ (see Prop. 1.1), (2.35) implies that

$$\begin{cases} \displaystyle\int_{\Omega^+} |\omega| a_n\left(\frac{d_1}{|\omega|},\cdots,\frac{d_{n-1}}{|\omega|},\frac{\partial u}{\partial x_n}\right)\frac{\partial v}{\partial x_n}\,dx + \int_{\Omega^-} a(Du)Dv\,dx + \int_{\Omega^+}|\omega||u|^{p-2}uv\,dx + \int_{\Omega^-}|u|^{p-2}uv\,dx \\ \displaystyle = \int_\Omega (|\omega|\chi_{\Omega^+}+\chi_{\Omega^-})fv\,dx \qquad \forall v\in V^p(\Omega), \quad (u,d_1,\cdots,d_{n-1})\in V^p(\Omega)\times\left(L^p(\Omega^+)\right)^{n-1}. \end{cases} \tag{2.36}$$

Moreover, as proved in (2.28),

$$a_i\left(\frac{d_1}{|\omega|},\cdots,\frac{d_{n-1}}{|\omega|},\frac{\partial u}{\partial x_n}\right) = 0 \quad \text{a.e. in } \Omega^+, \quad \forall i\in\{1,\cdots,n-1\}\cdot \tag{2.37}$$

Step 10. *Uniqueness of u*

This step is devoted to prove that there exists a unique function u in $V^p(\Omega)$ satisfying problem (2.36, 2.37).

Let $(u, d_1,\cdots,d_{n-1})$ and $(\overline{u},\overline{d_1},\cdots,\overline{d_{n-1}})$ two solutions in $V^p(\Omega)\times(L^p(\Omega^+))^{n-1}$ of problem (2.36, 2.37).

By subtracting the equation satisfied by $(\overline{u},\overline{d_1},\cdots,\overline{d_{n-1}})$ from the equation satisfied by $(u,d_1,\cdots,d_{n-1})$, we obtain

$$\begin{aligned} &\int_{\Omega^+}|\omega|\left(a_n\left(\frac{d_1}{|\omega|},\cdots,\frac{d_{n-1}}{|\omega|},\frac{\partial u}{\partial x_n}\right) - a_n\left(\frac{\overline{d_1}}{|\omega|},\cdots,\frac{\overline{d_{n-1}}}{|\omega|},\frac{\partial \overline{u}}{\partial x_n}\right)\right)\frac{\partial v}{\partial x_n}\,dx + \int_{\Omega^-}(a(Du)-a(D\overline{u}))Dv\,dx \\ &\qquad + \int_{\Omega^+}|\omega|\left(|u|^{p-2}u - |\overline{u}|^{p-2}\overline{u}\right)v\,dx + \int_{\Omega^-}\left(|u|^{p-2}u-|\overline{u}|^{p-2}\overline{u}\right)v\,dx = 0 \qquad \forall v\in V^p(\Omega) \end{aligned} \tag{2.38}$$

and

$$a_i\left(\frac{d_1}{|\omega|},\cdots,\frac{d_{n-1}}{|\omega|},\frac{\partial u}{\partial x_n}\right) - a_i\left(\frac{\overline{d_1}}{|\omega|},\cdots,\frac{\overline{d_{n-1}}}{|\omega|},\frac{\partial \overline{u}}{\partial x_n}\right) = 0 \quad \text{a.e. in } \Omega^+, \quad \forall i\in\{1,\cdots,n-1\}\cdot \tag{2.39}$$

Equation (2.39) imply that

$$\int_{\Omega^+}|\omega|\left[\sum_{i=1}^{n-1}\left(\left(a_i\left(\frac{d_1}{|\omega|},\cdots,\frac{d_{n-1}}{|\omega|},\frac{\partial u}{\partial x_n}\right) - a_i\left(\frac{\overline{d_1}}{|\omega|},\cdots,\frac{\overline{d_{n-1}}}{|\omega|},\frac{\partial \overline{u}}{\partial x_n}\right)\right)\left(\frac{d_i}{|\omega|}-\frac{\overline{d_i}}{|\omega|}\right)\right)\right]dx = 0. \tag{2.40}$$

By adding (2.40) to (2.38) with $v = u - \overline{u}$, it follows that

$$\int_{\Omega^+} |\omega| \Bigg[\left(a\left(\frac{d_1}{|\omega|}, \cdots, \frac{d_{n-1}}{|\omega|}, \frac{\partial u}{\partial x_n}\right) - a\left(\frac{\overline{d_1}}{|\omega|}, \cdots, \frac{\overline{d_{n-1}}}{|\omega|}, \frac{\partial \overline{u}}{\partial x_n}\right)\right) \left(\left(\frac{d_1}{|\omega|}, \cdots, \frac{d_{n-1}}{|\omega|}, \frac{\partial u}{\partial x_n}\right) - \left(\frac{\overline{d_1}}{|\omega|}, \cdots, \frac{\overline{d_{n-1}}}{|\omega|}, \frac{\partial \overline{u}}{\partial x_n}\right)\right)\Bigg] \mathrm{d}x + \int_{\Omega^-} (a(Du) - a(D\overline{u}))\,(Du - D\overline{u})\, \mathrm{d}x$$
$$+ \int_{\Omega^+} |\omega| \left(|u|^{p-2}u - |\overline{u}|^{p-2}\overline{u}\right)(u - \overline{u})\, \mathrm{d}x + \int_{\Omega^-} \left(|u|^{p-2}u - |\overline{u}|^{p-2}\overline{u}\right)(u - \overline{u})\, \mathrm{d}x = 0. \quad (2.41)$$

Since $a(\xi)$ and $|t|^{p-2}t$ are monotone functions, (2.41) gives that

$$\int_{\Omega^+} \Bigg[\left(a\left(\frac{d_1}{|\omega|}, \cdots, \frac{d_{n-1}}{|\omega|}, \frac{\partial u}{\partial x_n}\right) - a\left(\frac{\overline{d_1}}{|\omega|}, \cdots, \frac{\overline{d_{n-1}}}{|\omega|}, \frac{\partial \overline{u}}{\partial x_n}\right)\right) \left(\left(\frac{d_1}{|\omega|}, \cdots, \frac{d_{n-1}}{|\omega|}, \frac{\partial u}{\partial x_n}\right) - \left(\frac{\overline{d_1}}{|\omega|}, \cdots, \frac{\overline{d_{n-1}}}{|\omega|}, \frac{\partial \overline{u}}{\partial x_n}\right)\right)\Bigg] \mathrm{d}x = 0 \quad (2.42)$$

and

$$\int_{\Omega} \left(|u|^{p-2}u - |\overline{u}|^{p-2}\overline{u}\right)(u - \overline{u})\, \mathrm{d}x = 0. \quad (2.43)$$

Since $|t|^{p-2}t$ is strictly monotone, from (2.43) it follows that

$$u = \overline{u} \quad \text{a.e. in } \Omega.$$

Step 11. *Uniqueness of the solution of problem (2.36, 2.37) with a strictly monotone*

This step is devoted to a proof that problem (2.36, 2.37) admits a unique solution, if a is strictly monotone. Let $(u, d_1, \cdots, d_{n-1})$ and $(\overline{u}, \overline{d_1}, \cdots, \overline{d_{n-1}})$ two solutions in $V^p(\Omega) \times (L^p(\Omega^+))^{n-1}$ of problem (2.36, 2.37). Step 10 provides that

$$u = \overline{u} \quad \text{a.e. in } \Omega.$$

Moreover, if a is strictly monotone, from (2.42) it follows that

$$d_1 = \overline{d_1}, \cdots, d_{n-1} = \overline{d_{n-1}} \quad \text{a.e. in } \Omega^+.$$

Step 12. *Conclusion: End of proof of Theorem 1.2 and Corollary 1.3*

First, let us observe that the particular shape of Ω_h^+ provides that (see [6, 8])

$$\widetilde{\frac{\partial u_h}{\partial x_n}} = \frac{\partial \widetilde{u_h}}{\partial x_n} \quad \text{a.e. in } \Omega^+, \quad \forall h \in \mathbb{N}. \quad (2.44)$$

Then, convergences (1.7, 1.8) follow from (2.5, 2.6, 2.9–2.11, 2.44).

The limit problem (1.9) is given (in a weak formulation) by (2.36, 2.37) of Step 9.

The convergence of the energies (1.10) is obtained by passing to the limit, as h diverges, in (2.1) with $v = u_h$ as test function, by making use of convergence (2.5) and by choosing $v = u$ as test function in (2.36).