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## A CONVERGENCE RESULT FOR AN ITERATIVE METHOD FOR THE EQUATIONS OF A STATIONARY QUASI-NEWTONIAN FLOW WITH TEMPERATURE DEPENDENT VISCOSITY (\*)

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**Abstract** — *We study a system of equations describing the stationary and incompressible flow of a quasi-Newtonian fluid with temperature dependent viscosity and with a viscous heating. An algorithm which decouples the calculation of the temperature  $T$  and the velocity and the pressure  $(v, p)$  is presented. It consists in solving iteratively a problem with a nonlinear Stokes's operator for  $v$  and  $p$  and the Poisson's equation with right-hand side in  $L^1$  for  $T$ . We prove, using the method of pseudomonotonicity and under a regularity assumption of Meyers type that the mapping defined by this scheme is a contraction for sufficiently small data.* © Elsevier, Paris

**Résumé** — *On étudie un système modélisant l'écoulement d'un fluide quasi-Newtonien stationnaire incompressible avec une viscosité dépendant de la température et en tenant compte des effets d'échauffement visqueux. On présente un algorithme découplant le calcul du couple vitesse-pression et de la température. Il s'agit de résoudre itérativement un problème concernant un opérateur de Stokes non linéaire en vitesse et pression, à température donnée, puis une équation de Poisson à second membre  $L^1$  en température, à vitesse donnée. On montre à l'aide de la méthode de pseudo-monotonie et sous une hypothèse de régularité de type Meyers que l'application définie par ce schéma est contractante pour des données suffisamment petites.* © Elsevier, Paris

### 1. INTRODUCTION

We consider equations describing the incompressible quasi-Newtonian fluid flow with temperature dependant viscosity. Existence for such problem of a weak solution has been recently proved by Baranger and Mikelic, (see [3]), using Schauder fixed point theorem; uniqueness of this solution was left as an open problem.

In numerical simulations one usually uses an iterative decoupled algorithm: here, it will consist in solving iteratively a problem with a nonlinear Stokes problem for  $v$  and  $p$  and the Poisson's equation with right-hand side in  $L^1$  for  $T$ .

We prove in this paper, for small data and under a Meyers's type regularity property of the  $r$ -Stokesian operator, that this simple algorithm is convergent to the unique weak solution of the problem. In fact, we prove that the operator defined from the iterative method is a contraction and use Banach fixed-point theorem.

Some similar problems, but in the simpler case of two scalar elliptic equations coupling the Laplacian and the heat equation, have been studied by Howinson *et al.* (see [7]) with uniqueness result for sufficiently small data and sufficiently regular solution, (see also [4]). We will adapt the functional framework and some ideas from [3] in proving existence.

Let us consider a bounded domain  $\Omega$  in  $\mathbb{R}^N$ ,  $N = 2$  or  $3$ , with a regular boundary  $\Gamma$ , and an incompressible quasi-Newtonian fluid flowing in  $\Omega$ , with temperature dependent viscosity and with a viscous heating. We consider

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the steady case and neglect inertia effects.  $T$  being the temperature,  $v$  the velocity and  $p$  the pressure of the fluid, we consider the following problem  $(\mathcal{P})$ , (see [3] for a derivation of the model from the basic principles of continuum mechanics):

$$(\mathcal{P}) \left\{ \begin{array}{l} -\operatorname{div} [\mu(T, |D(v)|) D(v)] + \nabla p = f \quad \text{in } \Omega, \\ \operatorname{div} v = 0 \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \Gamma, \\ -k \Delta T + \rho c_p(T) v \nabla T = \mu(T, |D(v)|) |D(v)|^2 \quad \text{in } \Omega, \\ T = \tau_0 \quad \text{on } \Gamma. \end{array} \right.$$

where  $D(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ ,  $c_p(\cdot)$  is a bounded continuous function on  $\mathbb{R}$ ,  $k$  is a positive constant,  $\rho$  is the constant density of the fluid,

$$\tau_0 \in L^\infty(\Gamma) \cap \bigcap_{q < \frac{N}{N-1}} W^{1-1/q, q}(\Gamma); \quad \tau_0 > C_0 > 0 \quad (\text{a.e.}) \text{ on } \Gamma \quad (1.1)$$

This is more realistic than the assumption:  $\tau_0 \in H^{1/2}(\Gamma)$ , (see [3]).

Furthermore, this assumption on the boundary data ensures the existence of an extension of  $\tau_0$ , which we will denote by  $\bar{\tau}_0$ , such that:  $\bar{\tau}_0 \in W^{1, q}(\Omega)$ ,  $\forall q < N'$ , owing to the isomorphism between  $W^{1-1/q, q}(\Gamma)$  and  $W^{1, q}(\Omega)/\ker \gamma$ ,  $\gamma$  being the trace operator on  $\Gamma$  (see [1], Theorem 7.53).

$\mu$  is supposed continuous on  $\mathbb{R}^2$  and satisfies the following properties:  $\forall s_1, s_2 \in \mathbb{R}, \forall \xi \in \mathbb{R}_{sym}^{N^2}$

$$|\mu(s_1, |\xi|) - \mu(s_2, |\xi|)| \leq K_1 \beta(|s_1 - s_2|) |\xi|^{r-2}, \quad 1 < r \leq 2, \quad (1.2)$$

$$\text{where } : \beta \in C_b(\mathbb{R}), \quad \beta \geq 0 \text{ and } \beta(0) = 0, \quad (1.3)$$

$$[\mu(s, |\xi|) \xi - \mu(s, |\eta|) \eta] : (\xi - \eta) \geq K_2 |\xi - \eta|^2 \{|\xi| + |\eta|\}^{r-2}, \quad (1.4)$$

$$\forall s \in \mathbb{R}, \forall \xi, \eta \in \mathbb{R}_{sym}^{N^2},$$

$$|[\mu(s, |\xi_1|) \xi_1 - \mu(s, |\xi_2|) \xi_2] : \eta| \leq K_3 |\eta| |\xi_1 - \xi_2|^{r-1}, \quad (1.5)$$

$$\forall \xi_1, \xi_2, \eta \in \mathbb{R}_{sym}^{N^2}.$$

We remark that a classical exemple of viscosity is the product of an Arrhenius law:  $\lambda(T) = C \exp \frac{K}{T}$  and a power law  $\nu(|D(v)|) = \nu_0 |D(v)|^{r-2}$  (see [2]), the above conditions being satisfied in that case.

Now, for studying problem  $(\mathcal{P})$ , we define the following functional spaces: For the velocity  $v$ , since we have to solve a  $r$ -Stokes monotone problem:

$$V_r = \{v \in [W_0^{1, r}(\Omega)]^N / \operatorname{div} v = 0 \text{ in } \Omega\} \quad (1.6)$$

and for the temperature  $T$ , since we have a Poisson equation with a right-hand side in  $L^1(\Omega)$  :

$$W_N = \bigcap_{1 \leq q < \frac{N}{N-1}} W_0^{1, q}(\Omega) \quad (1.7)$$

We say that  $(v, T)$ , with  $v \in V_r$ ,  $T - \bar{\tau}_0 \in W_N$ ,  $T > C_0$  (a.e.) in  $\Omega$ ,  $f \in L^r(\Omega)$ , is a weak solution of problem  $(\mathcal{P})$  if:

$$\int_{\Omega} \mu(T, |D(v)|) D(v) : D(\varphi) = \int_{\Omega} f\varphi, \quad \forall \varphi \in V_r; \tag{1.8}$$

$$k \int_{\Omega} \nabla T \nabla \xi - \rho \int_{\Omega} v C_p(T) \nabla \xi = \int_{\Omega} \mu(T, |D(v)|) |D(v)|^2 \xi, \tag{1.9}$$

$$\forall \xi \in W_0^{1,\infty}(\Omega), \quad \text{where } C_p(T) = \int_0^T c_p(s) ds.$$

**2. THE FIXED POINT ALGORITHM**

We introduce the following decoupled algorithm:

We start by  $T^0 = \bar{\tau}_0$ , and  $(v^0, p^0) =$  the solution in  $V_r \times L^r(\Omega)$  of the Stokes problem, (see [12]):

$$\begin{cases} -\operatorname{div} [\mu(\bar{\tau}_0, |D(v^0)|) D(v^0)] + \nabla p^0 = f & \text{in } \Omega \\ \operatorname{div} v^0 = 0 & \text{in } \Omega \\ v^0 = 0 & \text{on } \Gamma. \end{cases}$$

For  $T^n, v^n, p^n$  given, we search for  $T^{n+1}, v^{n+1}, p^{n+1}$  weak solutions in  $W_N \times V_r \times L^r(\Omega)$  of the following homogeneous problem:

$$(\mathcal{P}_{n+1}) \begin{cases} -\operatorname{div} [\mu(T^n + \bar{\tau}_0, |D(v^{n+1})|) D(v^{n+1})] + \nabla p^{n+1} = f & \text{in } \Omega \\ -k\Delta(T^{n+1} + \bar{\tau}_0) + \rho c_p(T^{n+1} + \bar{\tau}_0) v^{n+1} \nabla(T^{n+1} + \bar{\tau}_0) \\ \quad = \mu(T^n + \bar{\tau}_0, |D(v^{n+1})|) |D(v^{n+1})|^2 & \text{in } \Omega \end{cases}$$

We define, from this algorithm, the following fixed point operator:

$$\Phi : V_r \times W_N \rightarrow V_r \times W_N$$

$$(u, T_u) \mapsto (v, T_v) = \Phi(u, T_u) \text{ solution of :}$$

$$\begin{cases} -\operatorname{div} [\mu(T_u + \bar{\tau}_0, |D(v)|) D(v)] + \nabla p_v = f \text{ in } \Omega, \text{ and :} \\ -k\Delta(T_v + \bar{\tau}_0) + \rho c_p(T_v + \bar{\tau}_0) v \nabla(T_v + \bar{\tau}_0) = \mu(T_u + \bar{\tau}_0, |D(v)|) |D(v)|^2 \text{ in } \Omega. \end{cases} \tag{2.1}$$

where  $p_v \in L^r(\Omega)$  is the pressure associated to  $v$  and is unique up to a constant.

In order to prove that  $\Phi$  is a contracting mapping and hence, to state a convergence theorem for the algorithm  $(\mathcal{P}_{n+1})$ , we describe a Meyers's type regularity property of the  $r$ -Stokesian operator used in the first step of  $(\mathcal{P}_{n+1})$ , i.e. solution of the  $r$ -Stokes problem:

$$(\mathcal{S}_r) \begin{cases} -\operatorname{div} [\mu_r(T, |D(v)|) D(v)] + \nabla p = f & \text{in } \Omega \\ \operatorname{div} v = 0 & \text{in } \Omega \\ v = 0 & \text{on } \Gamma, \end{cases}$$

where  $\mu_r(\dots) := \mu(\dots)$  satisfies assumptions (1.2)-(1.5). We can formulate this property as follows:

There exists  $\gamma^* > r$  such that: for  $f \in L^\gamma(\Omega)$  with  $\frac{1}{\gamma} = \frac{1}{\gamma^*} + \frac{1}{N^r}$  we have, for each  $v$  solution of the  $r$ -Stokes problem  $(\mathcal{P}_r)$ :

$$D(v) \in L^p(\Omega), \quad \forall r < p \leq \gamma^*, \quad \text{and} \quad \|D(v)\|_{L^\gamma(\Omega)} \leq C \|f\|_{L^\gamma(\Omega)} \quad (2.2)$$

the constant  $C$  depending only on the data.

Such a regularity result has been proved in [11] for second order equation. See [13] for the case of the  $r$ -Stokesian operator.

For technical reason, we introduce:

$$\gamma_0 = \begin{cases} r & \text{if } N = 2 \\ \frac{3(r-1)}{2r-3} r & \text{if } N = 3. \end{cases} \quad (2.3)$$

We can state:

**THEOREM 2.1:** *Assume (1.1)-(1.5),  $\frac{N}{2} < r \leq 2$ , and that the exponent  $\gamma^*$  in (2.2) satisfies:  $\gamma^* > \gamma_0$ , where  $\gamma_0$  is given by (2.3). Then there exists a constant  $\bar{C}$ , depending only on the data, such that: if  $\|f\|_{L^\gamma(\Omega)} \leq \bar{C}$ , with  $\frac{1}{\gamma} = \frac{1}{\gamma^*} + \frac{1}{N^r}$ , then the fixed point iteration is a contraction.*

**COROLLARY 2.1:** *Under the previous assumptions, Problem  $(\mathcal{P})$  has a unique weak solution and the fixed point algorithm  $(\mathcal{P}_n)$  is convergent.*

### 3. PROOF OF THEOREM 2.1

The proof is based on four propositions:

**PROPOSITION 3.1:** *Under the assumptions of theorem 2.1, the fixed point operator  $\bar{\Psi}$  is well defined.*

*Proof:* Let us prove existence and uniqueness of a weak solution of  $(\mathcal{P}_{n+1})$ :

The solution  $v^{n+1}$  of the  $r$ -Stokes problem in  $(\mathcal{P}_{n+1})$  exists in  $V_r$ , is unique owing to the assumptions (1.2)-(1.4); and there exists a corresponding pressure  $p^{n+1}$  unique up to a constant, in  $L^r(\Omega)$  (see [12]).

Furthermore, we obtain easily, taking  $v^{n+1}$  as a test-function in the first equation of  $(\mathcal{P}_{n+1})$ , using (1.4) and the Poincaré's inequality:

$$\|D(v^{n+1})\|_{L^r(\Omega)^{N^2}} \leq \left(\frac{C(\Omega)}{K_2}\right)^{\frac{1}{r-1}} \|f\|_{L^{\frac{1}{r-1}}(\Omega)} = C(\Omega, r, f). \quad (3.1)$$

In the second equation in  $(\mathcal{P}_{n+1})$ , the right-hand side is in  $L^1(\Omega)$  since  $v$  is in  $V_r$  and since  $\mu$  satisfies (1.2), (1.3). So we do not have a sufficient regularity for using the classical variational formulatin for this problem. Adapting an idea of [3], we decompose this equation in two simpler ones:

Firstly:

$$\begin{cases} -k \Delta T_1^{n+1} = \mu(T^n + \bar{\tau}_0, |D(v^{n+1})|) |D(v^{n+1})|^2 \text{ in } \Omega. \\ T_1^{n+1} = 0 \text{ on } \Gamma. \end{cases} \quad (3.2)$$

Then, we can apply the results on Poisson's equation with right-hand side in  $L^1$ , (see for example [5]) and we obtain existence and uniqueness of a solution to (3.2)

$$T_1^{n+1} \in W_0^{1,q}(\Omega), \quad \forall 1 \leq q < \frac{N}{N-1} = N',$$

and we have the estimate  $\|T_1^{n+1}\|_{W_0^{1,q}(\Omega)} \leq C(\Omega, N, r, \tau_0)$ ,  $\forall 1 \leq q < N'$

In fact, for  $N = 3$ , we can use some results from [10] (see Theorem 12.1) to get that the solution of (3.2) lies in  $W_0^{1,N}(\Omega)$

Indeed, using the first equation of  $(\mathcal{P}_{n+1})$ , we can write formally the right hand side of (3.2) as follows

$$\operatorname{div} \{ [\mu(T^n + \bar{\tau}_0, |D(v^{n+1})|) D(v^{n+1}) - p^{n+1} I] v^{n+1} \} + f v^{n+1},$$

where

$$[\mu(T^n + \bar{\tau}_0, |D(v^{n+1})|) D(v^{n+1}) - p^{n+1} I] v^{n+1} \in L^N \quad \text{and} \quad f v^{n+1} \in W^{-1,N}(\Omega)$$

This can be easily seen using Holder's inequality with exponents  $p = \frac{(N-1)r}{N(r-1)}$ ,  $p' = \frac{(N-1)r}{N-r}$ , (Note that  $p > 1$  for  $r < N$ ) Indeed, we obtain, with (1.2)-(1.3)

$$\begin{aligned} \int_{\Omega} |\mu(T^n + \bar{\tau}_0, |D(v^{n+1})|) D(v^{n+1}) v^{n+1}|^N &\leq C \int_{\Omega} \{ |D(v^{n+1})|^{r-1} |v^{n+1}| \}^N \\ &\leq \|D(v^{n+1})\|_{L^r}^{N(r-1)} \|v^{n+1}\|_{L^{r'}}^N \end{aligned}$$

$\leq \|D(v^{n+1})\|_{L^r}^{rN}$ , by Poincaré's inequality and Sobolev Imbedding Theorem,  $\leq C(\Omega, r, f)$ , by (3.1)

For  $f v^{n+1}$ , it is easy to see that  $\forall \varphi \in W_0^{1,N}(\Omega) (\subset L^p(\Omega), \forall p < \infty)$ ,

$$\int_{\Omega} f v^{n+1} \varphi \leq C \|f\|_{L^r} \|v^{n+1}\|_{L^{r'}} \|\varphi\|_{L^r(\frac{r}{r'})}$$

Secondly

$$\begin{cases} -k\Delta(T_2^{n+1} + \bar{\tau}_0) + \rho c_p(T_1^{n+1} + T_2^{n+1} + \bar{\tau}_0) v^{n+1} \nabla(T_1^{n+1} + T_2^{n+1} + \bar{\tau}_0) = 0 \text{ in } \Omega \\ T_2^{n+1} = 0 \text{ on } \Gamma \end{cases} \quad (3.3)$$

We have, since  $c_p$  is bounded  $\forall T \in H^1(\Omega)$ ,

$$\begin{aligned} \left| \int_{\Omega} v^{n+1} c_p(T_1^{n+1} + \bar{\tau}_0 + T) \nabla(T_1^{n+1} + \bar{\tau}_0 + T) \right| &\leq C \|v^{n+1}\|_{L^{r'}} \|T_1^{n+1} + \bar{\tau}_0 + T\|_{W^{1,r^*}} \\ &\leq C \|v^{n+1}\|_{L^{r'}} \|T_1^{n+1} + \bar{\tau}_0 + T\|_{W_N} \text{ since } (r^*)' = \frac{Nr}{Nr - N + r} < \frac{N}{N-1}, \text{ for } r > \frac{N}{2}, \end{aligned}$$

and:

$$\begin{aligned} \forall \varphi \in H_0^1(\Omega), \quad \left| \int_{\Omega} v^{n+1} T \nabla \varphi \right| &\leq C \|\varphi\|_{H_0^1} \|v^{n+1}\|_{L^r} \|T\|_{L^2\left(\frac{r}{2}\right)}, \\ &\leq C \|\varphi\|_{H_0^1} \|v^{n+1}\|_{W^{1,r}} \|T\|_{H^1}, \text{ since } 2\left(\frac{r}{2}\right)' = \frac{6r}{5r-6} < 2^* = 6, \text{ for } r > \frac{N}{2}, \end{aligned}$$

this for  $N = 3$ ; obtaining a same estimate for  $N = 2$  being more easy due to Sobolev Imbedding Theorem.

Then, we can apply results of pseudomonotone operators theory, (see [9]), to get existence and uniqueness of a solution  $T_2^{n+1}$  in  $H_0^1(\Omega)$  to problem (3.3) and that:  $\|T_2^{n+1}\|_{H_0^1(\Omega)} \leq C$ , where  $C$  depends only on the coefficients of the equation and the data. So, by (3.1),  $C$  depends only on the data.

Note that if  $c_p(T^{n+1} + \bar{\tau}_0)$  is replaced by  $c_p(T^n + \bar{\tau}_0)$  in the algorithm, then we can deduce existence and uniqueness of a solution of (3.3) in  $H_0^1(\Omega)$  directly from the results of linear elliptic equations with unbounded coefficients (see [8]) since the coefficient  $v^{n+1}$  satisfies:  $\|v^{n+1}\|_{L^{p/2}(\Omega)} \leq C + \infty$ , with  $p = 2r > N$ .

Finally, taking:  $T^{n+1} = T_1^{n+1} + T_2^{n+1}$ , we obtain a unique weak solution of  $(\mathcal{P}_{n+1})$ , which satisfies:

$$\|T^{n+1}\|_{W_N} \leq C(\Omega, N, r, \tau_0). \quad (3.4)$$

We conclude that the mapping  $\Phi$  is well defined. ■

**PROPOSITION 3.2:** *If the iterative method converges to  $(v_0, T_0)$ , then  $(v_0, T_0 + \bar{\tau}_0)$  is a weak solution of  $(\mathcal{P})$ .*

*Proof:* From the estimates (3.1) and (3.4), we deduce that there exists a subsequence, still denoted by the same symbol, such that:

— firstly:  $v^n \rightarrow v_0$  in  $V_r$  weak. So, by Rellich's Compactness Theorem,

$$v^n \rightarrow v_0 \text{ in } L^p(\Omega) \text{ strong, for } 1 \leq p < r^* = \frac{Nr}{N-r} \text{ if } r < N, \quad (3.5)$$

for all  $p < \infty$ , if  $r = N$

— secondly:  $T^n \rightarrow T_0$  in  $W_0^{1,q}(\Omega)$  weak,  $\forall 1 \leq q < N'$ . Then:

$$T^n \rightarrow T_0 \text{ in } L^m(\Omega) \text{ strong for } 1 \leq m < (N')^* = \frac{N}{N-2}, \text{ if } N = 3, \quad (3.6)$$

for all  $m < \infty$  if  $N = 2$ , and  $T^n \rightarrow T_0$  a.e. in  $\Omega$ .

Let us now show that:  $(v^n) \xrightarrow[n \rightarrow \infty]{} v_0$  in  $V_r$  strong:

We have by (1.8):

$$\int_{\Omega} \mu(T^n + \bar{\tau}_0, |D(v^{n+1})|) D(v^{n+1}) : D(\psi) = \int_{\Omega} f\psi, \quad \forall \psi \in V_r, \quad (3.7)$$

and taking:  $\psi = \varphi - v^{n+1}$ :

$$\int_{\Omega} \mu(T^n + \bar{\tau}_0, |D(v^{n+1})|) D(v^{n+1}) : D(\varphi - v^{n+1}) = \int_{\Omega} f(\varphi - v^{n+1}). \quad (3.8)$$

But (1.4) gives:

$$\int_{\Omega} [\mu(T^n + \bar{\tau}_0, |D(\varphi)|) D(\varphi) - \mu(T^n + \bar{\tau}_0, |D(v^{n+1})|) D(v^{n+1})] : D(\varphi - v^{n+1}) \geq 0.$$

Then:

$$\int_{\Omega} \mu(T^n + \bar{\tau}_0, |D(\varphi)|) D(\varphi) : D(\varphi - v^{n+1}) \geq \int_{\Omega} f(\varphi - v^{n+1}). \quad (3.9)$$

Then, passing to the limit in this inequality, using the continuity of  $\mu$ , the a.e. convergence of  $T^n$  to  $T_0$  and the weak convergence of  $v^n$  to  $v_0$ , we get:

$$\int_{\Omega} \mu(T_0 + \bar{\tau}_0, |D(\varphi)|) D(\varphi) : D(\varphi - v_0) \geq \int_{\Omega} f(\varphi - v_0). \quad (3.10)$$

Now, by a usual procedure from Minty's lemma (taking first  $\varphi = v_0 + \alpha\psi$ , with  $\alpha > 0$ , in (3.10), then letting  $\alpha \rightarrow 0$ , and taking  $\psi = -\varphi$ ), we obtain:

$$\int_{\Omega} \mu(T_0 + \bar{\tau}_0, |D(v_0)|) D(v_0) : D(\varphi) = \int_{\Omega} f\varphi, \quad \forall \varphi \in V_r. \quad (3.12)$$

So in particular:  $\int_{\Omega} \mu(T_0 + \bar{\tau}_0, |D(v_0)|) |D(v_0)|^2 = \int_{\Omega} f v_0$ ; and, with (3.7):

$$\begin{aligned} & \int_{\Omega} \mu(T^n + \bar{\tau}_0, |D(v^{n+1})|) |D(v^{n+1})|^2 - \int_{\Omega} \mu(T_0 + \bar{\tau}_0, |D(v_0)|) |D(v_0)|^2 \\ &= \left| \int_{\Omega} f(v^{n+1} - v_0) \right| \xrightarrow{n \rightarrow \infty} 0. \end{aligned} \quad (3.13)$$

Furthermore, we have:

$$\begin{aligned} & \int_{\Omega} [\mu(T^n + \bar{\tau}_0, |D(v_0)|) D(v_0) - \mu(T^n + \bar{\tau}_0, |D(v^{n+1})|) D(v^{n+1})] : D(v_0 - v^{n+1}) \\ &= \int_{\Omega} [\mu(T^n + \bar{\tau}_0, |D(v_0)|) - \mu(T_0 + \bar{\tau}_0, |D(v_0)|)] D(v_0) : D(v_0 - v^{n+1}) \\ &\quad + \int_{\Omega} \mu(T_0 + \bar{\tau}_0, |D(v_0)|) D(v_0) : D(v_0 - v^{n+1}) \\ &\quad - \int_{\Omega} \mu(T^n + \bar{\tau}_0, |D(v^{n+1})|) D(v^{n+1}) : D(v_0 - v^{n+1}) \\ &= \int_{\Omega} [\mu(T^n + \bar{\tau}_0, |D(v_0)|) - \mu(T_0 + \bar{\tau}_0, |D(v_0)|)] D(v_0) : D(v_0 - v^{n+1}), \end{aligned}$$



by (3.7) and (3.12). This, with condition (1.4), gives:

$$\begin{aligned}
& K_2 \int_{\Omega} |D(v_0 - v^{n+1})|^{2r} \{|D(v_0)| + |D(v^{n+1})|\}^{r-2} \\
& \leq \left| \int_{\Omega} [\mu(T^n + \bar{\tau}_0, |D(v_0)|) - \mu(T_0 + \bar{\tau}_0, |D(v_0)|)] D(v_0) : D(v_0 - v^{n+1}) \right| \\
& \leq K_1 \int_{\Omega} \beta(|T^n - T_0|) |D(v_0)|^{r-1} |D(v_0 - v^{n+1})|, \text{ by (1.2)} \\
& \leq C \|D(v_0 - v^{n+1})\|_{(L^r(\Omega))^{N^2}} \left[ \int_{\Omega} \beta(|T^n - T_0|)^r |D(v_0)|^r \right]^{\frac{1}{r}}, \tag{3.14}
\end{aligned}$$

by Holder's inequality (with  $\frac{1}{r'} + \frac{1}{r} = 1$ ). But, we have (for  $r < 2$ ):

$$\begin{aligned}
& \int_{\Omega} |D(v_0 - v^{n+1})|^r \\
& = \int_{\Omega} |D(v_0 - v^{n+1})|^{r \{ |D(v_0)| + |D(v^{n+1})| \}^{\frac{r-2}{2}} \{ |D(v_0)| + |D(v^{n+1})| \}^{\frac{2-r}{2} r}}, \\
& \leq \left[ \int_{\Omega} |D(v_0 - v^{n+1})|^{2r} \{|D(v_0)| + |D(v^{n+1})|\}^{r-2} \right]^{\frac{r}{2}} \left[ \int_{\Omega} \{|D(v_0)| + |D(v^{n+1})|\}^r \right]^{\frac{2-r}{2}}.
\end{aligned}$$

and since:

$$\int_{\Omega} \{|D(v_0)| + |D(v^{n+1})|\}^r \leq 2^{r-1} (\|D(v_0)\|_{L^r}^r + \|D(v^{n+1})\|_{L^r}^r) \leq C,$$

by (3.1), then we get (for  $r \leq 2$ ):

$$\|D(v_0 - v^{n+1})\|_{(L^r(\Omega))^{N^2}}^r \leq C \left[ \int_{\Omega} |D(v_0 - v^{n+1})|^{2r} \{|D(v_0)| + |D(v^{n+1})|\}^{r-2} \right]^{\frac{r}{2}}.$$

This gives:

$$\begin{aligned}
& \|D(v_0 - v^{n+1})\|_{(L^r(\Omega))^{N^2}}^2 \\
& \leq C \int_{\Omega} |D(v_0 - v^{n+1})|^{2r} \{|D(v_0)| + |D(v^{n+1})|\}^{r-2}, \tag{3.15} \\
& \leq C \|D(v_0 - v^{n+1})\|_{(L^r(\Omega))^{N^2}} \left[ \int_{\Omega} \beta(|T^n - T_0|)^r |D(v_0)|^r \right]^{\frac{1}{r}}, \text{ by (3.14)}.
\end{aligned}$$

Therefore:

$$\|D(v_0 - v^{n+1})\|_{(L^r(\Omega))^{N^2}} \leq C \left[ \int_{\Omega} \beta(|T^n - T_0|)^{r'} |D(v_0)|^r \right]^{\frac{1}{r}}. \tag{3.16}$$

Since  $\beta$  is bounded, then we have:

$$\forall n, \quad |\beta(|T^n - T_0|)|^{r'} |D(v_0)|^r \leq C |D(v_0)|^r := g \text{ a.e. in } \Omega, \text{ with } g \in L^1(\Omega).$$

Then, using Lebesgue's Dominated Convergence Theorem and the continuity of  $\beta$  (we have:  $T^n \rightarrow T_0$  a.e.), we deduce from (3.16):  $\|D(v_0 - v^{n+1})\|_{L^r(\Omega)^{N^2}} \xrightarrow[n \rightarrow \infty]{} 0$ . Consequently,

$$(v^n) \xrightarrow[n \rightarrow \infty]{} v_0 \text{ in } V_r \text{ strong.} \tag{3.17}$$

For  $(T^n)$ , we have, by (3.13):  $\forall \xi \in W_0^{1,\infty}(\Omega)$ ,

$$\int_{\Omega} \mu(T^n + \bar{\tau}_0, |D(v^{n+1})|) |D(v^{n+1})|^2 \xi \xrightarrow[n \rightarrow \infty]{} \int_{\Omega} \mu(T_0 + \bar{\tau}_0, |D(v_0)|) |D(v_0)|^2 \xi,$$

and by (3.17) and (3.6):

$$\int_{\Omega} v^{n+1} C_p(T^{n+1} + \bar{\tau}_0) \nabla \xi \xrightarrow[n \rightarrow \infty]{} \int_{\Omega} v_0 C_p(T_0 + \bar{\tau}_0) \nabla \xi.$$

Indeed:

$$\begin{aligned} & \rho \int_{\Omega} \{v^{n+1} C_p(T^{n+1} + \bar{\tau}_0) - v_0 C_p(T_0 + \bar{\tau}_0)\} \nabla \xi \\ & \leq C \left\{ \int_{\Omega} |v^{n+1} - v_0| |C_p(T^{n+1} + \bar{\tau}_0)| + \int_{\Omega} |v_0| |C_p(T^{n+1} + \bar{\tau}_0) - C_p(T_0 + \bar{\tau}_0)| \right\} \\ & \leq C \{ \|v^{n+1} - v_0\|_{L^r} \|T^{n+1} + \bar{\tau}_0\|_{L^r} + \|v_0\|_{L^r} \|T^{n+1} - T_0\|_{L^r} \} \xrightarrow[n \rightarrow \infty]{} 0, \end{aligned}$$

by (3.6) and (3.17), since we have:  $r' = \frac{r}{r-1} < \frac{N}{N-2}$ , for  $r > \frac{N}{2}$ .

Furthermore, by (3.6) and the Sobolev Imbedding:  $W_0^{1,\infty}(\Omega) \subset W_0^{1,q'}(\Omega)$ ,  $\forall q < \frac{N}{N-1}$ , we have:  $\forall \xi \in W_0^{1,\infty}(\Omega)$ ,

$$k \int_{\Omega} \nabla(T^{n+1} + \bar{\tau}_0) \nabla \xi \xrightarrow[n \rightarrow \infty]{} k \int_{\Omega} \nabla(T_0 + \bar{\tau}_0) \nabla \xi.$$

So, by uniqueness of the limit, we obtain:

$$\begin{aligned} & k \int_{\Omega} \nabla(T_0 + \bar{\tau}_0) \nabla \xi - \rho \int_{\Omega} v_0 C_p(T_0 + \bar{\tau}_0) \nabla \xi \\ & = \int_{\Omega} \mu(T_0 + \bar{\tau}_0, |D(v_0)|) |D(v_0)|^2 \xi; \quad \forall \xi \in W_0^{1,\infty}(\Omega). \end{aligned} \tag{3.18}$$

Furthermore, the assumption on  $\tau_0$  implies that the limit  $T_0 + \bar{\tau}_0 \geq C_0 > 0$  a.e. in  $\Omega$ , (see [3], [6]).

This, (3.12) and (3.18) imply that  $(v_0, T_0 + \bar{\tau}_0)$  is a weak solution of  $(\mathcal{P})$ .

There exists a corresponding pressure  $p_0$  in  $L^{r'}(\Omega)$ , convergence of  $(v^n)$  giving that of  $(p^n)$  in  $W^{-1, r'}(\Omega)$ .

■ In the sequel, for simplicity, we will take  $c_p(T) = 1$ , this function being of secondary importance in the obtaining of the following estimates, since it is bounded.

PROPOSITION 3.3: *Under the assumptions of theorem 2.1, the velocities satisfy the following estimate:*

$$\|D(v_1 - v_2)\|_{L^r} \leq C \|f\|_{L^{r'}(\Omega)}^{\frac{r}{r'}} \|T_{u_1} - T_{u_2}\|_{W_N},$$

where:  $(v_1, T_{v_1}) = \phi(u_1, T_{u_1})$  and  $(v_2, T_{v_2}) = \phi(u_2, T_{u_2})$ ,  $C$  depending only on the data:  $\Omega, N, r, \tau_0, f$ .

*Proof:* We easily get from the definition of  $\phi$  :

$$\begin{aligned} & \int_{\Omega} [\mu(T_{u_1} + \bar{\tau}_0, |D(v_1)|) D(v_1) - \mu(T_{u_1} + \bar{\tau}_0, |D(v_2)|) D(v_2)] : D(v_1 - v_2) \\ &= - \int_{\Omega} [\mu(T_{u_1} + \bar{\tau}_0, |D(v_2)|) - \mu(T_{u_2} + \bar{\tau}_0, |D(v_2)|)] D(v_2) : D(v_1 - v_2). \end{aligned} \quad (3.19)$$

Therefore, by (1.4) and (1.2):

$$\begin{aligned} & K_2 \int_{\Omega} |D(v_1 - v_2)|^2 \{ |D(v_1)| + |D(v_2)| \}^{r-2} \\ & \leq K_1 \int_{\Omega} \beta(|T_{u_1} - T_{u_2}|) |D(v_2)|^{r-1} |D(v_1 - v_2)|, \\ & \leq K_1 \|D(v_1 - v_2)\|_{L^r} \left( \int_{\Omega} \beta(|T_{u_1} - T_{u_2}|)^r |D(v_2)|^r \right)^{\frac{1}{r}}. \end{aligned}$$

Then, similarly as in estimate (3.16), we obtain:

$$\|D(v_1 - v_2)\|_{L^r(\Omega)^{N^2}} \leq C \left( \int_{\Omega} |T_{u_1} - T_{u_2}|^r |D(v_2)|^r \right)^{\frac{1}{r}}. \quad (3.20)$$

And, by the Meyers's regularity property of the  $r$ -Stokes problem, using Hölder's inequality, we obtain:

$$\|D(v_1 - v_2)\|_{L^r(\Omega)^{N^2}} \leq C \|D(v_2)\|_{L^{r^*}(\Omega)^{N^2}}^{\frac{r}{r^*}} \|T_{u_1} - T_{u_2}\|_{L^{r^* - r}(\Omega)}^{\frac{r\gamma^*}{r^*}}.$$

Hence, by (2.2):

$$\|D(v_1 - v_2)\|_{L^r(\Omega)^{N^2}} \leq C \|f\|_{L^{r'}(\Omega)}^{\frac{r}{r^*}} \|T_{u_1} - T_{u_2}\|_{L^{r^* - r}(\Omega)}^{\frac{r\gamma^*}{r^*}}. \quad (3.21)$$

Then, in order to have an estimate of  $\|T_{u_1} - T_{u_2}\|_{L^{\frac{r\gamma^*}{\gamma^* - r}}(\Omega)}$  with  $r > \frac{N}{2}$ , we need to add, for  $N = 3$ , the following regularity assumption:  $\gamma^* > \gamma_0$ , where  $\gamma_0 = \frac{N(r-1)}{2r-N}r$ , which is a necessary and sufficient condition to have:  $\frac{r\gamma^*}{\gamma^* - r} < \frac{N}{N-2}$ . This, with (3.21) gives Proposition 3.3. ■

*Remark 3.1:* The method used in the previous step does not allow us to prove Proposition 3.3 in the case  $r > 2$ , under a natural assumption on  $\mu$ , that is:

$$[\mu(s, |\xi|) \xi - \mu(s, |\eta|) \eta] : (\xi - \eta) \geq K_4 |\xi - \eta|^r.$$

Indeed, (3.19) and (1.2) would give:

$$\begin{aligned} K_4 \int_{\Omega} |D(v_1 - v_2)|^r &\leq K_1 \int_{\Omega} \beta(|T_{u_1} - T_{u_2}|) |D(v_2)|^{r-1} |D(v_1 - v_2)|, \\ &\leq C \|D(v_1 - v_2)\|_{L^r} \left( \int_{\Omega} |T_{u_1} - T_{u_2}|^r |D(v_2)|^r \right)^{\frac{1}{r}}. \end{aligned}$$

So:

$$\|D(v_1 - v_2)\|_{L^r(\Omega)}^{r-1} \leq C \left( \int_{\Omega} |T_{u_1} - T_{u_2}|^r |D(v_2)|^r \right)^{\frac{1}{r}}$$

Finally, we would get, by Hölder's inequality and for  $\gamma^* > \gamma_0$ ,

$$\|D(v_1 - v_2)\|_{L^r(\Omega)} \leq C \|f\|_{L^r(\Omega)} \|T_{u_1} - T_{u_2}\|_{W_N}^{\frac{1}{r-1}}.$$

Because of the exponent  $\frac{1}{r-1} < 1$ , for  $r > 2$ , we can not deduce from this estimate that  $\Phi$  is a contracting mapping in that case.

**PROPOSITION 3.4:** *Under the assumptions of theorem 2.1, the temperatures satisfy the following estimate:*

$$\begin{aligned} \|T_{v_1} - T_{v_2}\|_{W_N} &\leq C \{ \|f\|_{L^r(\Omega)}^{\frac{2r}{r-1}} + \|f\|_{L^r(\Omega)}^{\frac{r}{r-1}} \} \|T_{u_1} - T_{u_2}\|_{W_N} \\ &\quad + C \|f\|_{L^r(\Omega)} \|T_{v_1} - T_{v_2}\|_{W_N}, \end{aligned}$$

where the constant  $C$  depends only of the data:  $\Omega, N, r, \tau_0, f$ .

*Proof:*  $(T_{v_1} - T_{v_2})$  is a solution of the equation:

$$\begin{aligned} -k\Delta(T_{v_1} - T_{v_2}) &= \{ \mu(T_{u_1} + \bar{\tau}_0, |D(v_1)|) |D(v_1)|^2 - \mu(T_{u_2} + \bar{\tau}_0, |D(v_2)|) |D(v_2)|^2 \} \\ &\quad - \rho \{ v_1 \nabla(T_{v_1} + \bar{\tau}_0) - v_2 (\nabla T_{v_2} + \bar{\tau}_0) \}. \end{aligned} \tag{3.22}$$

We get, from the definition of  $\phi$  :

$$\begin{aligned} & \int_{\Omega} \{ \mu(T_{u_1} + \bar{\tau}_0, |D(v_1)|) |D(v_1)|^2 - \mu(T_{u_2} + \bar{\tau}_0, |D(v_2)|) |D(v_2)|^2 \} \\ &= \int_{\Omega} \mu(T_{u_2} + \bar{\tau}_0, |D(v_2)|) D(v_2) : D(v_1 - v_2) . \end{aligned}$$

Then:

$$\begin{aligned} & \left| \int_{\Omega} \{ \mu(T_{u_1} + \bar{\tau}_0, |D(v_1)|) |D(v_1)|^2 - \mu(T_{u_2} + \bar{\tau}_0, |D(v_2)|) |D(v_2)|^2 \} \right| \\ & \leq C \int_{\Omega} |D(v_2)|^{r-1} |D(v_1 - v_2)|, \text{ by (1.2) - (1.3) ,} \\ & \leq C \|D(v_2)\|_{L^r(\Omega)^{N^2}}^{\frac{r}{r-1}} \|D(v_1 - v_2)\|_{L^r(\Omega)^{N^2}} , \\ & \leq \|f\|_{L^r(\Omega)}^{2\frac{r}{r-1}} \|T_{u_1} - T_{u_2}\|_{W_N}, \text{ by (2.2) and Proposition 3.3 .} \end{aligned} \tag{3.23}$$

Furthermore,

$$\begin{aligned} & \rho \left| \int_{\Omega} v_1 \nabla(T_{v_1} + \bar{\tau}_0) - v_2 \nabla(T_{v_2} + \bar{\tau}_0) \right| \\ & \leq \rho \int_{\Omega} |(v_1 - v_2) \nabla(T_{v_1} + \bar{\tau}_0)| + \rho \int_{\Omega} |v_2 (\nabla T_{v_1} - \nabla T_{v_2})| \\ & \leq C \|v_1 - v_2\|_{L^{Nr/N-r}} \|\nabla(T_{v_1} + \bar{\tau}_0)\|_{L^{Nr/Nr-N+r}} \\ & + C \|v_2\|_{L^{Nr/N-r}} \|\nabla T_{v_1} - \nabla T_{v_2}\|_{L^{Nr/Nr-N+r}} \text{ (for } r < N \text{)} ; \\ & \leq C \|D(v_1 - v_2)\|_{L^r} \|T_{v_1} + \bar{\tau}_0\|_{W_N} + C \|D(v_2)\|_{L^r} \|T_{v_1} - T_{v_2}\|_{W_N} , \end{aligned}$$

by Poincaré's inequality and Sobolev imbedding theorem (Recall that:  $\frac{Nr}{Nr-N+r} < \frac{N}{N-1}$ , for  $r > \frac{N}{2}$ ). Then, by Proposition 3.3, estimates (2.2) and (3.4), we obtain:

$$\begin{aligned} & \rho \left| \int_{\Omega} \{ v_1 \nabla(T_{v_1} + \bar{\tau}_0) - v_2 \nabla(T_{v_2} + \bar{\tau}_0) \} \right| \\ & \leq C \|f\|_{L^r(\Omega)}^{\frac{r}{r-1}} \|T_{u_1} - T_{u_2}\|_{W_N} + C \|f\|_{L^r(\Omega)} \|T_{v_1} - T_{v_2}\|_{W_N} \end{aligned} \tag{3.24}$$

Then, (3.22)-(3.24) imply that:  $T_{v_1} - T_{v_2}$  is a solution of the equation:  $-A(T_{v_1} - T_{v_2}) = F$ , where  $F \in L^1(\Omega)$  and consequently the following estimate holds (see [5]):

$$\|T_{v_1} - T_{v_2}\|_{W^{1,q}(\Omega)} \leq C \|F\|_{L^1(\Omega)}, \quad \forall q < \frac{N}{N-1}.$$

This, with estimates (3.23) and (3.24) gives Proposition 3.4.

*End of proof of Theorem 2.1:* We can now deduce that there exists a closed ball  $B_R$  nonempty in  $V_r \times W_N$  such that:  $\phi(B_R) \subset B_R$ ; and  $\phi$  is a contracting mapping on  $B_R$ , for  $r > \frac{N}{2}$  and  $\|f\|_{L^r}$  sufficiently small:

By the definition of  $v^0$  and  $T^0$ , we can easily choose  $R > 0$  such that:  $\|D(v^0)\|_{L^r(\Omega)} + \|\tau_0\|_{L^\infty(\Gamma)} \leq R$ , and consequently  $(v^0, T^0) \in B_R$ .

Our aim is to prove that there exists  $\delta$ ,  $0 < \delta < 1$ , such that:

$$\|(v_1, T_{v_1}) - (v_2, T_{v_2})\|_{V_r \times W_N} \leq \delta \|(u_1, T_{u_1}) - (u_2, T_{u_2})\|_{V_r \times W_N}.$$

Using Proposition 3.4, we obtain that if  $\|f\|_{L^r(\Omega)}$  is sufficiently small, that is:  $C \max \{ \|f\|_{L^r(\Omega)}^{2r/r'}, \|f\|_{L^r(\Omega)}, \|f\|_{L^r(\Omega)}^{r/r'} \} < \bar{\delta} < \frac{1}{2}$ , then:

$$(1 - \bar{\delta}) \|T_{v_1} - T_{v_2}\|_{W_N} \leq \bar{\delta} \|T_{u_1} - T_{u_2}\|_{W_N}.$$

Finally, taking:  $\delta = \frac{\bar{\delta}}{1 - \bar{\delta}}$ , we get:

$$\|T_{v_1} - T_{v_2}\|_{W_N} \leq \delta \|T_{u_1} - T_{u_2}\|_{W_N}, \quad \text{with } 0 < \delta < 1. \tag{3.25}$$

Analogously, in proposition 3.3, if  $f$  is sufficiently small, then:

$$\|D(v_1 - v_2)\|_{L^r} \leq \delta \|T_{u_1} - T_{u_2}\|_{W_N} \tag{3.26}$$

Finally, (3.25) and (3.26) imply that  $\phi$  is a contraction mapping, for  $r > \frac{N}{2}$ ,  $f$  sufficiently small and, for  $N = 3$ ,  $v$  sufficiently regular:  $D(v) \in L^{\gamma^*}$ ;  $\gamma^* > \gamma_0$ . This gives Theorem 2.1.

Then, under the above assumptions, we can apply the Banach fixed-point theorem to get that  $\phi$  admits a unique fixed point  $(v_0, T_0)$  in  $B_R$ . Furthermore, there exists a corresponding pressure  $p$  unique up to a constant. Then, the algorithm  $(\mathcal{P}_n)$  converges to this solution. Since, a solution of  $(\mathcal{P})$  corresponds to a fixed point of  $\phi$ , then, using Proposition 3.2, we obtain that  $(v_0, T_0 + \bar{\tau}_0)$  is the unique weak solution of problem  $(\mathcal{P})$ . Therefore, Corollary 2.1 is proved.

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