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A convergence result for an iterative method for the equations of a stationary quasi-newtonian flow with temperature dependent viscosity


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A CONVERGENCE RESULT FOR AN ITERATIVE METHOD FOR THE EQUATIONS OF A STATIONARY QUASI-NEWTONIAN FLOW WITH TEMPERATURE DEPENDENT VISCOSITY (*)

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Abstract — We study a system of equations describing the stationary and incompressible flow of a quasi-Newtonian fluid with temperature dependent viscosity and with a viscous heating. An algorithm which decouples the calculation of the temperature $T$ and the velocity and the pressure $(v,p)$ is presented. It consists in solving iteratively a problem with a nonlinear Stokes's operator for $v$ and $p$ and the Poisson's equation with right-hand side in $L^1$ for $T$. We prove, using the method of pseudomonotonicity and under a regularity assumption of Meyers type that the mapping defined by this scheme is a contraction for sufficiently small data. © Elsevier, Paris

Résumé — On étudie un système modélisant l'écoulement d'un fluide quasi-Newtonien stationnaire incompressible avec une viscosité dépendant de la température et en tenant compte des effets d'échauffement visqueux. On présente un algorithme découplant le calcul du couple vitesse-pression et de la température $T$, il s'agit de résoudre itérativement un problème concernant un opérateur de Stokes non linéaire en vitesse et pression, à température donnée, puis une équation de Poisson à second membre $L^1$ en température, à vitesse donnée. On montre à l'aide de la méthode de pseudo-monotonicité et sous une hypothèse de régularité de type Meyers que l'application définie par ce schéma est contractante pour des données suffisamment petites. © Elsevier, Paris

1. INTRODUCTION

We consider equations describing the incompressible quasi-Newtonian fluid flow with temperature dependant viscosity. Existence for such problem of a weak solution has been recently proved by Baranger and Mikelic, (see [3]), using Schauder fixed point theorem; uniqueness of this solution was left as an open problem.

In numerical simulations one usually uses an iterative decoupled algorithm: here, it will consist in solving iteratively a problem with a nonlinear Stokes problem for $v$ and $p$ and the Poisson's equation with right-hand side in $L^1$ for $T$.

We prove in this paper, for small data and under a Meyers's type regularity property of the $r$-Stokesian operator, that this simple algorithm is convergent to the unique weak solution of the problem. In fact, we prove that the operator defined from the iterative method is a contraction and use Banach fixed-point theorem.

Some similar problems, but in the simpler case of two scalar elliptic equations coupling the Laplacian and the heat equation, have been studied by Howinson et al. (see [7]) with uniqueness result for sufficiently small data and sufficiently regular solution, (see also [4]). We will adapt the functional framework and some ideas from [3] in proving existence.

Let us consider a bounded domain $\Omega$ in $\mathbb{R}^N$, $N = 2$ or $3$, with a regular boundary $\Gamma$, and an incompressible quasi-Newtonian fluid flowing in $\Omega$, with temperature dependent viscosity and with a viscous heating. We consider

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the steady case and neglect inertia effects. \( T \) being the temperature, \( v \) the velocity and \( p \) the pressure of the fluid, we consider the following problem \((\mathcal{P})\), (see [3] for a derivation of the model from the basic principles of continuum mechanics):

\[
\begin{aligned}
\{ - \text{div} \left[ \mu(T, |D(v)|) D(v) \right] + \nabla p &= f \quad \text{in } \Omega, \\
\text{div} \, v &= 0 \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \Gamma, \\
-v \Delta T + \rho c_p(T) v \nabla T &= \mu(T, |D(v)|) |D(v)|^2 \quad \text{in } \Omega, \\
T &= T_0 \quad \text{on } \Gamma.
\end{aligned}
\]

where \( D(u) = \frac{1}{2} (\nabla u + \nabla u^T) \), \( c_p(\cdot) \) is a bounded continuous function on \( \mathbb{R} \), \( k \) is a positive constant, \( \rho \) is the constant density of the fluid,

\[
\tau_0 \in L^\infty(\Gamma) \cap \bigcap_{q < N-1} W^{1, q}(\Omega); \quad \tau_0 > C_0 > 0 \text{ (a.e.) on } \Gamma \quad (1.1)
\]

This is more realistic than the assumption: \( \tau_0 \in H^{1/2}(\Gamma) \), (see [3]).

Furthermore, this assumption on the boundary data ensures the existence of an extension of \( \tau_0 \), which we will denote by \( \tilde{\tau}_0 \), such that: \( \tilde{\tau}_0 \in W^{1,q}(\Omega), \forall q < N' \), owing to the isomorphism between \( W^{1, -1/q, q}(\Gamma) \) and \( W^{1,q}(\Omega)/\ker \gamma \), \( \gamma \) being the trace operator on \( \Gamma \) (see [1], Theorem 7.53).

\( \mu \) is supposed continuous on \( \mathbb{R}^2 \) and satisfies the following properties: \( \forall s_1, s_2 \in \mathbb{R}, \forall \xi \in \mathbb{R}^N \)

\[
|\mu(s_1, |\xi|) - \mu(s_2, |\xi|) | \leq K_1 \beta(|s_1 - s_2|) |\xi| r - 2, \quad 1 < r \leq 2,
\]

where: \( \beta \in C_{\infty}(\mathbb{R}), \quad \beta \geq 0 \) and \( \beta(0) = 0 \),

\[
[\mu(s, |\xi|) - \mu(s, |\eta|) |\eta|] : (\xi - \eta) \geq K_2 (|\xi| + |\eta|)^{r-2},
\]

\( \forall s \in \mathbb{R}, \forall \xi, \eta \in \mathbb{R}^N \),

\[
|\mu(s, |\xi|) |\xi| - \mu(s, |\xi_2|) |\xi_2| : |\eta| \leq K_3 |\xi_1 - \xi_2|^{r-1},
\]

\( \forall \xi_1, \xi_2, \eta \in \mathbb{R}^N \).

We remark that a classical example of viscosity is the product of an Arrhenius law: \( \lambda(T) = C \exp \frac{-K}{T} \) and a power law \( \nu(|D(v)|) = v_0 |D(v)|^{r-2} \) (see [2]), the above conditions being satisfied in that case.

Now, for studying problem \((\mathcal{P})\), we define the following functional spaces: For the velocity \( v \), since we have to solve a \( \tau \)-Stokes monotone problem:

\[
V_{\tau} = \{ v \in \left[ W^{1, (N)}_0(\Omega) \right]^N / \text{div } v = 0 \text{ in } \Omega \} \quad (1.6)
\]

and for the temperature \( T \), since we have a Poisson equation with a right-hand side in \( L^1(\Omega) \):

\[
W_N = \bigcap_{1 \leq q < \frac{N}{N-1}} W^{1,q}_0(\Omega) \quad (1.7)
\]

M² AN Modélisation mathématique et Analyse numérique
Mathematical Modelling and Numerical Analysis
We say that \((v, T), \text{ with } u \in V_r, T \in W_N, T > C_0 \text{ (a.e.) in } \Omega, f \in L^r(\Omega),\) is a weak solution of problem \((\mathcal{P})\) if:

\[
\int_{\Omega} \mu(T, |D(v)|) D(v) : D(\varphi) = \int_{\Omega} f \varphi, \quad \forall \varphi \in V_r; \tag{1.8}
\]

\[
k \int_{\Omega} \nabla T \nabla \xi - \rho \int_{\Omega} v C_p(T) \nabla \xi = \int_{\Omega} \mu(T, |D(v)|) |D(v)|^2 \xi, \quad \forall \xi \in W_0^{1,\infty}(\Omega), \quad \text{where } C_p(T) = \int_0^T c_p(s) \, ds. \tag{1.9}
\]

2. THE FIXED POINT ALGORITHM

We introduce the following decoupled algorithm:

We start by \(T^0 = \tilde{\tau}_0\), and \((v^0, p^0) = \) the solution in \(V_r \times L^r(\Omega)\) of the Stokes problem, (see [12]):

\[
\begin{cases}
- \text{div} \left[ \mu(T^0, |D(v^0)|) D(v^0) \right] + \nabla p^0 = f & \text{in } \Omega \\
\text{div } v^0 = 0 & \text{in } \Omega \\
v^0 = 0 & \text{on } \Gamma.
\end{cases}
\]

For \(T^n, v^n, p^n\) given, we search for \(T^{n+1}, v^{n+1}, p^{n+1}\) weak solutions in \(W_N \times V_r \times L^r(\Omega)\) of the following homogeneous problem:

\[
(\mathcal{P}_{n+1}) \begin{cases}
- \text{div} \left[ \mu(T^n + \tilde{\tau}_0, |D(v^n+1)|) D(v^n+1) \right] + \nabla p^{n+1} = f & \text{in } \Omega \\
-k A(T^n + \tilde{\tau}_0 + p c_p(T^n + \tilde{\tau}_0) v^n+1 \nabla (T^n+1 + \tilde{\tau}_0) \\
= \mu(T^n + \tilde{\tau}_0, |D(v^n+1)|) |D(v^n+1)|^2 & \text{in } \Omega
\end{cases}
\]

We define, from this algorithm, the following fixed point operator:

\[
\Phi : V_r \times W_N \to V_r \times W_N
\]

\[(u, T_u) \mapsto (v, T_v) = \Phi(u, T_u) \text{ solution of:}
\]

\[
\begin{cases}
- \text{div} \left[ \mu(T_u + \tilde{\tau}_0, |D(v)|) D(v) \right] + \nabla p_v = f \text{ in } \Omega, \quad \text{and:} \\
-k A(T_v + \tilde{\tau}_0 + p c_p(T_v + \tilde{\tau}_0) v \nabla (T_v + \tilde{\tau}_0) \\
= \mu(T_u + \tilde{\tau}_0, |D(v)|) |D(v)|^2 \text{ in } \Omega. \tag{2.1}
\end{cases}
\]

where \(p_v \in L^r(\Omega)\) is the pressure associated to \(v\) and is unique up to a constant.

In order to prove that \(\Phi\) is a contracting mapping and hence, to state a convergence theorem for the algorithm \((\mathcal{P}_{n+1})\), we describe a Meyers’s type regularity property of the \(r\)-Stokesian operator used in the first step of \((\mathcal{P}_{n+1})\), i.e. solution of the \(r\)-Stokes problem:

\[
(\mathcal{P}_r) \begin{cases}
- \text{div} \left[ \mu(T, |D(v)|) D(v) \right] + \nabla p = f & \text{in } \Omega \\
\text{div } v = 0 & \text{in } \Omega \\
v = 0 & \text{on } \Gamma,
\end{cases}
\]

where \(\mu(\ldots) \coloneqq \mu(\ldots)\) satisfies assumptions (1.2)-(1.5). We can formulate this property as follows:

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There exists $\gamma^* > r$ such that: for $f \in L^r(\Omega)$ with $\frac{1}{\gamma} = \frac{1}{r} + \frac{1}{N}$, we have, for each $v$ solution of the $r$-Stokes problem $(\mathcal{P}_r)$:

$$D(v) \in L^p(\Omega), \quad \forall r < p \leq \gamma^*, \quad \text{and} \quad \|D(v)\|_{L^r(\Omega)} \leq C\|f\|_{L^r(\Omega)}$$  \hspace{1cm} (2.2)

the constant $C$ depending only on the data.

Such a regularity result has been proved in [11] for second order equation. See [13] for the case of the $r$-Stokesian operator.

For technical reason, we introduce:

$$\gamma_0 = \begin{cases} 
\gamma_0 = \frac{r}{2} & \text{if } N = 2 \\
\frac{3(r-1)}{2r-3} r & \text{if } N = 3 .
\end{cases}$$  \hspace{1cm} (2.3)

We can state:

**THEOREM 2.1:** Assume (1.1)-(1.5), $\frac{N}{2} < r \leq 2$, and that the exponent $\gamma^*$ in (2.2) satisfies: $\gamma^* > \gamma_0$, where $\gamma_0$ is given by (2.3). Then there exists a constant $C$, depending only on the data, such that: if $\|f\|_{L^r(\Omega)} \leq C$, with $\frac{1}{\gamma} = \frac{1}{r} + \frac{1}{N}$, then the fixed point iteration is a contraction.

**COROLLARY 2.1:** Under the previous assumptions, Problem $(\mathcal{P})$ has a unique weak solution and the fixed point algorithm $(\mathcal{P}_n)$ is convergent.

### 3. PROOF OF THEOREM 2.1

The proof is based on four propositions:

**PROPOSITION 3.1:** Under the assumptions of theorem 2.1, the fixed point operator $\Phi$ is well defined.

*Proof:* Let us prove existence and uniqueness of a weak solution of $(\mathcal{P}_{n+1})$.

The solution $v^{n+1}$ of the $r$-Stokes problem in $(\mathcal{P}_{n+1})$ exists in $V_r$, is unique owing to the assumptions (1.2)-(1.4); and there exists a corresponding pressure $p^{n+1}$ unique up to a constant, in $L^r_{\gamma}(\Omega)$ (see [12]).

Furthermore, we obtain easily, taking $v^{n+1}$ as a test-function in the first equation of $(\mathcal{P}_{n+1})$, using (1.4) and the Poincaré’s inequality:

$$\|D(v^{n+1})\|_{L^r(\Omega)^2} \leq \left(\frac{C(\Omega)}{K_2}\right)^{-\frac{1}{r-1}} \|f\|_{L^r(\Omega)}^\frac{1}{r} = C(\Omega, r, f).$$  \hspace{1cm} (3.1)

In the second equation in $(\mathcal{P}_{n+1})$, the right-hand side is in $L^1(\Omega)$ since $v$ is in $V_r$ and since $\mu$ satisfies (1.2), (1.3). So we do not have a sufficient regularity for using the classical variational formulation for this problem. Adapting an idea of [3], we decompose this equation in two simpler ones:

Firstly:

$$\begin{cases} 
-k AT_1^{n+1} = \mu(T^n + \tilde{\tau}_0) \left|D(v^{n+1})\right| \left|D(v^{n+1})\right|^2 \text{ in } \Omega . \\
T_1^{n+1} = 0 \text{ on } \Gamma .
\end{cases}$$  \hspace{1cm} (3.2)
Then, we can apply the results on Poisson’s equation with right-hand side in $L^1$, (see for example [5]) and we obtain existence and uniqueness of a solution to (3.2)

$$T_i^{n+1} \in W_0^1 q(\Omega), \quad \forall 1 \leq q < \frac{N}{N-1} = N'.$$

and we have the estimate $\|T_i^{n+1}\|_{W_0^1 q(\Omega)} \leq C(\Omega, N, r, \tau_0), \quad \forall 1 \leq q < N'$

In fact, for $N = 3$, we can use some results from [10] (see Theorem 12.1) to get that the solution of (3.2) lies in $W_0^1 N(\Omega)$

Indeed, using the first equation of $(\mathcal{P}_{n+1})$, we can write formally the right hand side of (3.2) as follows

$$\text{div} \left[ \mu(T^n + \tau_0, |D(v^{n+1})|) D(v^{n+1}) - p^{n+1} I \right] v^{n+1} + f v^{n+1},$$

where

$$[\mu(T^n + \tau_0, |D(v^{n+1})|) D(v^{n+1}) - p^{n+1} I] v^{n+1} \in L^N \quad \text{and} \quad f v^{n+1} \in W^{-1, N}(\Omega)$$

This can be easily seen using Holder’s inequality with exponents $p = \frac{(N-1) r}{N(r-1)}$, $p' = \frac{(N-1) r}{N-r}$, (Note that $p > 1$ for $r < N$). Indeed, we obtain, with (1.2)-(1.3)

$$\int_\Omega |\mu(T^n + \tau_0, |D(v^{n+1})|) D(v^{n+1}) v^{n+1}|^N \leq C \int_\Omega \{|D(v^{n+1})|^{r-1} |v^{n+1}|\}^N \leq \|D(v^{n+1})\|_{L'} \|v^{n+1}\|_{L^r}^N \leq \|D(v^{n+1})\|_{L'}^N, \text{ by Poincaré’s inequality and Sobolev Imbedding Theorem,} \leq C(\Omega, r, f), \text{ by (3.1)}$$

For $f v^{n+1}$, it is easy to see that $\forall \phi \in W_0^1 N(\Omega)$ ($\subset L^p(\Omega), \forall p < \infty$),

$$\int_\Omega f v^{n+1} \phi \leq C \|f\|_{L'} \|v^{n+1}\|_{L^r} \|\phi\|_{L^r} \left(\frac{N}{r}\right)$$

Secondly

$$\left\{ -k \Lambda(T_2^{n+1} + \tau_0) + \rho c_p (T_1^{n+1} + T_2^{n+1} + \tau_0) v^{n+1} \nabla (T_1^{n+1} + T_2^{n+1} + \tau_0) = 0 \text{ in } \Omega \right\}$$

$$T_2^{n+1} = 0 \text{ on } \Gamma \quad (3.3)$$

We have, since $c_p$ is bounded $\forall T \in H^1(\Omega)$,

$$\int_\Omega v^{n+1} c_p (T_1^{n+1} + \tau_0 + T) \nabla (T_1^{n+1} + \tau_0 + T) \leq C \|v^{n+1}\|_{L'} \|T_1^{n+1} + \tau_0 + T\|_{W^{1, r}},$$

$$\leq C \|v^{n+1}\|_{L'} \|T_1^{n+1} + \tau_0 + T\|_{W^{1, r}} \text{ since } (r')' = \frac{Nr}{N + r} < \frac{N}{N-1}, \text{ for } r > N/2.$$
\[ \forall \phi \in H^1_0(\Omega), \quad \left| \int_{\Omega} v^{n+1} T \nabla \phi \right| \leq C \| \phi \|_{H^1_0} \| v^{n+1} \|_{L^2(\Omega)} \| T \|_{L^2(\Omega)}, \]

\[ \leq C \| \phi \|_{H^1_0} \| v^{n+1} \|_{W^{1,q}} \| T \|_{H^1(\Omega)}, \]

this for \( N = 3 \); obtaining a same estimate for \( N = 2 \) being more easy due to Sobolev Imbedding Theorem.

Then, we can apply results of pseudomonotone operators theory, (see [9]), to get existence and uniqueness of a solution \( T^{n+1}_2 \) in \( H^1_0(\Omega) \) to problem (3.3) and that: \( \| T^{n+1}_2 \|_{H^1(\Omega)} \leq C \), where \( C \) depends only on the coefficients of the equation and the data. So, by (3.1), \( C \) depends only on the data.

Note that if \( c_p(T^{n+1} + \tau_0) \) is replaced by \( c_p(T^{n} + \tau_0) \) in the algorithm, then we can deduce existence and uniqueness of a solution of (3.3) in \( H^1_0(\Omega) \) directly from the results of linear elliptic equations with unbounded coefficients (see [8]) since the coefficient \( v^{n+1} \) satisfies: \( \| v^{n+1} \|_{L^\infty(\Omega)} \leq C < +\infty \), with \( p = 2r > N \).

Finally, taking: \( T^{n+1} = T^{n+1}_1 + T^{n+1}_2 \), we obtain a unique weak solution of \( (\mathcal{P}_{n+1}) \), which satisfies:

\[ \| T^{n+1} \|_{W^{1,p}} \leq C(\Omega, N, r, \tau_0). \]  

We conclude that the mapping \( \Phi \) is well defined.

**PROPOSITION 3.2:** If the iterative method converges to \( (v_0, T_0) \), then \( (v_0, T_0 + \tau_0) \) is a weak solution of \( (\mathcal{P}) \).

**Proof:** From the estimates (3.1) and (3.4), we deduce that there exists a subsequence, still denoted by the same symbol, such that:

— firstly: \( v^n \to v_0 \) in \( V_r \) weak. So, by Rellich’s Compactness Theorem,

\[ v^n \to v_0 \text{ in } L^p(\Omega) \text{ strong, for } 1 \leq p < r^* = \frac{Nr}{N-r} \quad \text{if } r < N, \]

for all \( p < \infty \) if \( r = N \)

— secondly: \( T^n \to T_0 \) in \( W^{1,q}_0(\Omega) \) weak, \( 1 \leq q < N' \). Then:

\[ T^n \to T_0 \text{ in } L^{m}(\Omega) \text{ strong for } 1 \leq m < (N')^* = \frac{N}{N-2}, \text{ if } N = 3, \]

for all \( m < \infty \) if \( N = 2 \), and \( T^n \to T_0 \) a.e. in \( \Omega \).

Let us now show that: \( (v^n) \xrightarrow{n \to \infty} v_0 \) in \( V_r \) strong:

We have by (1.8):

\[ \int_{\Omega} \mu(T^n + \tau_0, |D(v^{n+1})|) D(v^{n+1}) : D(\psi) = \int_{\Omega} f\psi, \quad \forall \psi \in V_r, \]

and taking: \( \psi = \phi - v^{n+1} \):

\[ \int_{\Omega} \mu(T^n + \tau_0, |D(v^{n+1})|) D(v^{n+1}) : D(\phi - v^{n+1}) = \int_{\Omega} f(\phi - v^{n+1}). \]
But (1.4) gives:
\[
\int_{\Omega} \left[ \mu(T^n + \bar{T}_0, |D(\varphi)|) D(\varphi) - \mu(T^n + \bar{T}_0, |D(v^n + 1)|) D(v^n + 1) \right] : D(\varphi - v^n + 1) \geq 0.
\]

Then:
\[
\int_{\Omega} \mu(T^n + \bar{T}_0, |D(\varphi)|) D(\varphi) : D(\varphi - v^n + 1) \geq \int_{\Omega} f(\varphi - v^n + 1).
\]

(3.9)

Then, passing to the limit in this inequality, using the continuity of \(\mu\), the a.e. convergence of \(T^n\) to \(T_0\) and the weak convergence of \(v^n\) to \(v_0\), we get:
\[
\int_{\Omega} \mu(T_0 + \bar{T}_0, |D(\varphi)|) D(\varphi) : D(\varphi - v_0) \geq \int_{\Omega} f(\varphi - v_0).
\]

(3.10)

Now, by a usual procedure from Minty's lemma (taking first \(\varphi = v_0 + \alpha \psi\), with \(\alpha > 0\), in (3.10), then letting \(\alpha \to 0\), and taking \(\psi = - \varphi\), we obtain:
\[
\int_{\Omega} \mu(T_0 + \bar{T}_0, |D(v_0)|) D(v_0) : D(\varphi) = \int_{\Omega} f\varphi, \quad \forall \varphi \in V_r.
\]

(3.12)

So in particular:
\[
\int_{\Omega} \mu(T_0 + \bar{T}_0, |D(v_0)|) |D(v_0)|^2 = \int_{\Omega} f|v_0|; \quad \text{and, with (3.7):}
\]
\[
\int_{\Omega} \mu(T^n + \bar{T}_0, |D(v^n + 1)|) |D(v^n + 1)|^2 - \int_{\Omega} \mu(T_0 + \bar{T}_0, |D(v_0)|) |D(v_0)|^2
\]
\[
= \left| \int_{\Omega} f(v^n + 1 - v_0) \right|_{n \to \infty} 0.
\]

(3.13)

Furthermore, we have:
\[
\int_{\Omega} \left[ \mu(T^n + \bar{T}_0, |D(v_0)|) D(v_0) - \mu(T^n + \bar{T}_0, |D(v^n + 1)|) D(v^n + 1) \right] : D(v_0 - v^n + 1)
\]
\[
= \int_{\Omega} \left[ \mu(T^n + \bar{T}_0, |D(v_0)|) - \mu(T_0 + \bar{T}_0, |D(v_0)|) \right] D(v_0) : D(v_0 - v^n + 1)
\]
\[
+ \int_{\Omega} \mu(T_0 + \bar{T}_0, |D(v_0)|) D(v_0) : D(v_0 - v^n + 1)
\]
\[
- \int_{\Omega} \mu(T^n + \bar{T}_0, |D(v^n + 1)|) D(v^n + 1) : D(v_0 - v^n + 1)
\]
\[
= \int_{\Omega} \left[ \mu(T^n + \bar{T}_0, |D(v_0)|) - \mu(T_0 + \bar{T}_0, |D(v_0)|) \right] D(v_0) : D(v_0 - v^n + 1),
\]
by (3.7) and (3.12). This, with condition (1.4), gives:

\[
K_2 \int_{\Omega} |D(v_0 - v^{n+1})|^2 \left( |D(v_0)| + |D(v^{n+1})| \right)^{r-2}
\]

\[
\leq \left| \int_{\Omega} \left[ \mu(T^n + \tilde{\tau}_0, |D(v_0)|) - \mu(T_0 + \tilde{\tau}_0, |D(v_0)|) \right] D(v_0) : D(v_0 - v^{n+1}) \right|
\]

\[
\leq K_1 \int_{\Omega} \beta(|T^n - T_0|) |D(v_0)|^{r-1} |D(v_0 - v^{n+1})|, \text{ by (1.2)}
\]

\[
\leq C \|D(v_0 - v^{n+1})\|_{(L^1(\Omega))^2} \left[ \int_{\Omega} \beta(|T^n - T_0|)^r |D(v_0)|^r \right]^\frac{1}{r},
\]

(3.14)

by Holder’s inequality (with \(\frac{1}{r'} + \frac{1}{r} = 1\)). But, we have (for \(r < 2\)):

\[
\int_{\Omega} |D(v_0 - v^{n+1})|^r
\]

\[
= \int_{\Omega} |D(v_0 - v^{n+1})|^r \left( |D(v_0)| + |D(v^{n+1})| \right)^{r-2} \left[ |D(v_0)| + |D(v^{n+1})| \right]^\frac{2-r}{2},
\]

\[
\leq \left[ \int_{\Omega} |D(v_0 - v^{n+1})|^2 |D(v_0)| + |D(v^{n+1})| \right]^{\frac{r}{2}} \left[ \int_{\Omega} |D(v_0)| + |D(v^{n+1})| \right]^{\frac{2-r}{2}}.
\]

and since:

\[
\int_{\Omega} \left( |D(v_0)| + |D(v^{n+1})| \right)^r \leq 2^{r-1}(\|D(v_0)\|_{L^r} + \|D(v^{n+1})\|_{L^r}) \leq C,
\]

by (3.1), then we get (for \(r \leq 2\)):

\[
\|D(v_0 - v^{n+1})\|_{(L^1(\Omega))^2} \leq C \left[ \int_{\Omega} |D(v_0 - v^{n+1})|^2 \left( |D(v_0)| + |D(v^{n+1})| \right)^{r-2} \right]^\frac{1}{2}.
\]

This gives:

\[
\|D(v_0 - v^{n+1})\|_{(L^1(\Omega))^2}^2
\]

\[
\leq C \int_{\Omega} |D(v_0 - v^{n+1})|^2 \left( |D(v_0)| + |D(v^{n+1})| \right)^{r-2}, \quad (3.15)
\]

\[
\leq C \|D(v_0 - v^{n+1})\|_{(L^1(\Omega))^2} \left[ \int_{\Omega} \beta(|T^n - T_0|)^r |D(v_0)|^r \right]^\frac{1}{r}, \text{ by (3.14)}.
\]
Therefore:
\[ \| D(v_0 - v_n) \|_{L^2(\Omega)} \leq C \left( \int_\Omega \beta(|T^n - T_0|) |D(v_0)| \right)^{\frac{1}{2}}. \] (3.16)

Since \( \beta \) is bounded, then we have:
\[ \forall n, \quad |\beta(|T^n - T_0|)| |D(v_0)| \leq C |D(v_0)| = g \text{ a.e. in } \Omega, \text{ with } g \in L^1(\Omega). \]

Then, using Lebesgue's Dominated Convergence Theorem and the continuity of \( \beta \) (we have: \( T^n \rightarrow T_0 \) a.e.), we deduce from (3.16): \( \| D(v_0 - v_n) \|_{L^2(\Omega)} \xrightarrow{n \rightarrow \infty} 0 \). Consequently,
\[ (v^n) \xrightarrow{n \rightarrow \infty} v_0 \quad \text{in} \quad V_r \text{ strong.} \] (3.17)

For \( (T^n) \), we have, by (3.13): \( \forall \xi \in W_0^{1,\infty}(\Omega) \),
\[ \int_\Omega \mu(T^n + \bar{\tau}_0, |D(v_0)|) |D(v_0)|^2 \xi \xrightarrow{n \rightarrow \infty} \int_\Omega \mu(T_0 + \bar{\tau}_0, |D(v_0)|) |D(v_0)|^2 \xi, \]
and by (3.17) and (3.6):
\[ \int_\Omega v_0 + 1 C_p(T^n + \bar{\tau}_0) V \xi \xrightarrow{n \rightarrow \infty} \int_\Omega v_0 C_p(T_0 + \bar{\tau}_0) V \xi. \]

Indeed:
\[ \rho \int_\Omega \{v_0 + 1 C_p(T^n + \bar{\tau}_0) - v_0 C_p(T_0 + \bar{\tau}_0)\} V \xi \]
\[ \leq C \left\{ \int_\Omega |v_0 + 1 v_0| |C_p(T^n + \bar{\tau}_0)| + \int_\Omega |v_0| |C_p(T^n + \bar{\tau}_0) - C_p(T_0 + \bar{\tau}_0)| \right\} \]
\[ \leq C \left\{ \|v_0 + 1 v_0\|_{L^\infty} \|T^n + \bar{\tau}_0\|_{L^1} + \|v_0\|_{L^1} \|T^n + \bar{\tau}_0 - T_0\|_{L^1} \right\} \xrightarrow{n \rightarrow \infty} 0, \]
by (3.6) and (3.17), since we have: \( r' = \frac{r}{r - 1} < \frac{N}{N - 2} \) for \( r > \frac{N}{2} \).

Furthermore, by (3.6) and the Sobolev Imbedding: \( W_0^{1,\infty}(\Omega) \subset W_0^{1,q}(\Omega), \forall q < \frac{N}{N - 1}, \) we have:
\( \forall \xi \in W_0^{1,\infty}(\Omega), \)
\[ k \int_\Omega \nabla(T^n + \bar{\tau}_0) V \xi \xrightarrow{n \rightarrow \infty} k \int_\Omega \nabla(T_0 + \bar{\tau}_0) V \xi. \]

So, by uniqueness of the limit, we obtain:
\[ k \int_\Omega \nabla(T_0 + \bar{\tau}_0) V \xi - \rho \int_\Omega v_0 C_p(T_0 + \bar{\tau}_0) V \xi \]
\[ = \int_\Omega \mu(T_0 + \bar{\tau}_0, |D(v_0)|) |D(v_0)|^2 \xi; \quad \forall \xi \in W_0^{1,\infty}(\Omega). \] (3.18)
Furthermore, the assumption on $\tau_0$ implies that the limit $T_0 + \tilde{\tau}_0 \geq C_0 > 0$ a.e. in $\Omega$, (see [3], [6]).

This, (3.12) and (3.18) imply that $(v_0^c, \tau_0^c)$ is a weak solution of $(\mathcal{P})$.

There exists a corresponding pressure $p_0$ in $L^r(\Omega)$, convergence of $(v^n)$ giving that of $(p^n)$ in $W^{-1,r}(\Omega)$.

In the sequel, for simplicity, we will take $c_0(T) = 1$, this function being of secondary importance in the obtaining of the following estimates, since it is bounded.

**Proposition 3.3:** Under the assumptions of theorem 2.1, the velocities satisfy the following estimate:

$$\|D(v_1 - v_2)\|_{L^r} \leq C\|f\|_{L^r(\Omega)}^r \|T_{u_1} - T_{u_2}\|_{W^r},$$

where: $(v_1, T_{u_1}) = \phi(u_1, T_{u_1})$ and $(v_2, T_{u_2}) = \phi(u_2, T_{u_2})$, $C$ depending only on the data: $\Omega, N, r, \tau_0, f$.

**Proof:** We easily get from the definition of $\phi$:

$$\int_\Omega [\mu(T_{u_1} + \tilde{\tau}_0, |D(v_1)|)] D(v_1) - \mu(T_{u_1} + \tilde{\tau}_0, |D(v_2)|) D(v_2) : D(v_1 - v_2)$$

$$= -\int_\Omega [\mu(T_{u_1} + \tilde{\tau}_0, |D(v_2)|) - \mu(T_{u_2} + \tilde{\tau}_0, |D(v_2)|)] D(v_2) : D(v_1 - v_2).$$

(3.19)

Therefore, by (1.4) and (1.2):

$$K_2 \int_\Omega |D(v_1 - v_2)|^2 [\|D(v_1)\| + \|D(v_2)\|]^{r-2}$$

$$\leq K_1 \int_\Omega \beta(|T_{u_1} - T_{u_2}|) |D(v_2)|^{r-1} |D(v_1 - v_2)|,$$

$$\leq K_1 \|D(v_1 - v_2)\|_{L^r} \left(\int_\Omega \beta(|T_{u_1} - T_{u_2}|) |D(v_2)|^{r} \right)^{\frac{1}{r}}.$$

Then, similarly as in estimate (3.16), we obtain:

$$\|D(v_1 - v_2)\|_{L^r(\Omega)^{y^2}} \leq C\left(\int_\Omega |T_{u_1} - T_{u_2}|^{y^2} |D(v_2)|^{r} \right)^{\frac{1}{r}}.$$  (3.20)

And, by the Meyers’s regularity property of the $r$-Stokes problem, using Hölder’s inequality, we obtain:

$$\|D(v_1 - v_2)\|_{L^r(\Omega)^{y^2}} \leq C\|D(v_2)\|_{L^r(\Omega)^{y^2}} \|T_{u_1} - T_{u_2}\|_{L^{r} - y'}(\Omega).$$

Hence, by (2.2):

$$\|D(v_1 - v_2)\|_{L^r(\Omega)^{y^2}} \leq C\|f\|_{L^r(\Omega)}^r \|T_{u_1} - T_{u_2}\|_{L^{r} - y'}(\Omega).$$  (3.21)
Then, in order to have an estimate of $\| T_{u_1} - T_{u_2} \|_{L^r(\Omega)}$ with $r > \frac{N}{2}$, we need to add, for $N = 3$, the following regularity assumption: $\gamma^* > \gamma_0$, where $\gamma_0 = \frac{N(r - 1)}{2 r - N} r$, which is a necessary and sufficient condition to have: $\frac{r \gamma^*}{\gamma^* - r} < \frac{N}{N-2}$. This, with (3.21) gives Proposition 3.3.

**Remark 3.1:** The method used in the previous step does not allow us to prove Proposition 3.3 in the case $r > 2$, under a natural assumption on $\mu$, that is:

$$[\mu(s, |\xi|) \xi - \mu(s, |\eta|) \eta] : (\xi - \eta) \geq K_4 |\xi - \eta|^\gamma.$$ 

Indeed, (3.19) and (1.2) would give:

$$K_4 \int_\Omega |D(v_1 - v_2)|^\gamma \leq K_1 \int_\Omega \beta(|T_{u_1} - T_{u_2}|) |D(v_2)|^\gamma - 1 |D(v_1 - v_2)|,$$

$$\leq C \|D(v_1 - v_2)\|_L^{\gamma - 1} \left( \int_\Omega |T_{u_1} - T_{u_2}|^\gamma |D(v_2)|^\gamma \right)^{\frac{1}{\gamma - 1}}.$$

So:

$$\|D(v_1 - v_2)\|_{L^\gamma(\Omega)} \leq C \left( \int_\Omega |T_{u_1} - T_{u_2}|^\gamma |D(v_2)|^\gamma \right)^{\frac{1}{\gamma - 1}}.$$ 

Finally, we would get, by Hölder’s inequality and for $\gamma^* > \gamma_0$,

$$\|D(v_1 - v_2)\|_{L^\gamma(\Omega)} \leq C \|f\|_{L^\gamma(\Omega)} \|T_{u_1} - T_{u_2}\|_{W_0^{1, \gamma}}^{\frac{1}{\gamma - 1}}.$$

Because of the exponent $\frac{1}{\gamma - 1} < 1$, for $r > 2$, we can not deduce from this estimate that $\Phi$ is a contracting mapping in that case.

**Proposition 3.4:** Under the assumptions of theorem 2.1, the temperatures satisfy the following estimate:

$$\| T_{v_1} - T_{v_2} \|_{W_r} \leq C \{ \|f\|_{L^\gamma(\Omega)}^{2 \gamma} + \|f\|_{L^\gamma(\Omega)}^\gamma \} \| T_{u_1} - T_{u_2} \|_{W_r}$$

$$+ C \|f\|_{L^\gamma(\Omega)} \| T_{v_1} - T_{v_2} \|_{W_r},$$

where the constant $C$ depends only of the data: $\Omega, N, r, \tau_0, f$.

**Proof:** $(T_{v_1} - T_{v_2})$ is a solution of the equation:

$$- k_d (T_{v_1} - T_{v_2}) = \{ \mu(T_{u_1} + \tau_0, |D(v_1)|) |D(v_1)|^2 - \mu(T_{u_2} + \tau_0, |D(v_2)|) |D(v_2)|^2 \}$$

$$- \rho \{ v_1 \nabla(T_{v_1} + \tau_0) - v_2 (\nabla T_{v_2} + \tau_0) \}.$$

(3.22)
We get, from the definition of $\phi$:

$$
\int_{\Omega} \left\{ \mu(T_{u_1} + \bar{v}_0, |D(v_1)|) |D(v_1)|^2 - \mu(T_{u_2} + \bar{v}_0, |D(v_2)|) |D(v_2)|^2 \right\}
= \int_{\Omega} \mu(T_{u_1} + \bar{v}_0, |D(v_2)|) D(v_2) : D(v_1 - v_2).
$$

Then:

$$
\int_{\Omega} \left\{ \mu(T_{u_1} + \bar{v}_0, |D(v_1)|) |D(v_1)|^2 - \mu(T_{u_2} + \bar{v}_0, |D(v_2)|) |D(v_2)|^2 \right\}
\leq C \int_{\Omega} |D(v_2)|^r - 1 |D(v_1 - v_2)|, \text{ by (1.2) - (1.3)},
$$

$$
\leq C \|D(v_2)\|_{L^r(\Omega)}^r \|D(v_1 - v_2)\|_{L^r(\Omega)}^r,
$$

$$
\leq \|f\|_{L^r(\Omega)}^r \|T_{u_1} - T_{u_2}\|_{W^r}, \text{ by (2.2) and Proposition 3.3}.
$$

Furthermore,

$$
\rho \left| \int_{\Omega} v_1 \nabla(T_{v_1} + \bar{v}_0) - v_2 \nabla(T_{v_2} + \bar{v}_0) \right|
\leq \rho \int_{\Omega} |(v_1 - v_2) \nabla(T_{v_1} + \bar{v}_0)| + \rho \int_{\Omega} |v_2(\nabla T_{v_1} - \nabla T_{v_2})|
\leq C \|v_1 - v_2\|_{L^{Nr/N - r}} \|\nabla(T_{v_1} + \bar{v}_0)\|_{L^{Nr/N - N + r}}
+ C \|v_2\|_{L^{Nr/N - r}} \|\nabla T_{v_1} - \nabla T_{v_2}\|_{L^{Nr/N - N + r}} \text{ (for } r < N) ;
\leq C \|D(v_1 - v_2)\|_{L^r} \|T_{v_1} + \bar{v}_0\|_{W^r} + C \|D(v_2)\|_{L^r} \|T_{v_1} - T_{v_2}\|_{W^r},
$$

by Poincaré’s inequality and Sobolev imbedding theorem (Recall that: $\frac{Nr}{Nr - N + r} < \frac{N}{N - 1}$, for $r > \frac{N}{2}$).

Then, by Proposition 3.3, estimates (2.2) and (3.4), we obtain:

$$
\rho \left| \int_{\Omega} \{v_1 \nabla(T_{v_1} + \bar{v}_0) - v_2 \nabla(T_{v_2} + \bar{v}_0)\} \right|
\leq C \|f\|_{L^r(\Omega)}^r \|T_{u_1} - T_{u_2}\|_{W^r} + C \|f\|_{L^r(\Omega)}^r \|T_{v_1} - T_{v_2}\|_{W^r}
$$

(3.24)
Then, (3.22)-(3.24) imply that: $T_{v_1} - T_{v_2}$ is a solution of the equation: $-A(T_{v_1} - T_{v_2}) = F$, where $F \in L^1(\Omega)$ and consequently the following estimate holds (see [5]):

$$\|T_{v_1} - T_{v_2}\|_{W^{1,q}(\Omega)} \leq C \|F\|_{L^1(\Omega)}, \quad \forall q < \frac{N}{N-1}.$$ 

This, with estimates (3.23) and (3.24) gives Proposition 3.4.

End of proof of Theorem 2.1: We can now deduce that there exists a closed ball $B_R$ nonempty in $V_r \times W_N$ such that: $\phi(B_R) \in B_R$; and $\phi$ is a contracting mapping on $B_R$, for $r > \frac{N}{2}$ and $\|f\|_{L^r}$ sufficiently small:

By the definition of $v^0$ and $T^0$, we can easily choose $R > 0$ such that: $\|D(v^0)\|_{L^r(\Omega)} + \|\tau_0\|_{L^r(\Gamma)} \leq R$, and consequently $(v^0, T^0) \in B_R$.

Our aim is to prove that there exists $\delta$, $0 < \delta < 1$, such that:

$$\|(v_1, T_{v_1}) - (v_2, T_{v_2})\|_{V_r \times W_N} \leq \delta \|(u_1, T_{u_1}) - (u_2, T_{u_2})\|_{V_r \times W_N}.$$ 

Using Proposition 3.4, we obtain that if $\|f\|_{L^r(\Omega)}$ is sufficiently small, that is: 

$$C \max \{ \|f\|_{L^r(\Omega)}, \|f\|_{L^{r'}(\Omega)}, \|f\|_{L^{r'}(\Omega')} \} < \frac{1}{2},$$ 

then:

$$(1 - \bar{\delta}) \|T_{v_1} - T_{v_2}\|_{W^r_N} \leq \bar{\delta} \|T_{u_1} - T_{u_2}\|_{W^r_N}.$$ 

Finally, taking: $\bar{\delta} = \frac{\bar{\delta}}{1 - \bar{\delta}}$, we get:

$$\|T_{v_1} - T_{v_2}\|_{W^r_N} \leq \delta \|T_{u_1} - T_{u_2}\|_{W^r_N} \quad \text{with } 0 < \delta < 1.$$ 

(3.25)

Analogously, in proposition 3.3, if $f$ is sufficiently small, then:

$$\|D(v_1 - v_2)\|_{L^r} \leq \delta \|T_{u_1} - T_{u_2}\|_{W^r_N}.$$ 

(3.26)

Finally, (3.25) and (3.26) imply that $\phi$ is a contraction mapping, for $r > \frac{N}{2}$, $f$ sufficiently small and, for $N = 3$, $v$ sufficiently regular: $D(v) \in L^r; \gamma^* > \gamma^*_w$. This gives Theorem 2.1.

Then, under the above assumptions, we can apply the Banach fixed-point theorem to get that $\phi$ admits a unique fixed point $(v_0, T_0)$ in $B_R$. Furthermore, there exists a corresponding pressure $p$ unique up to a constant. Then, the algorithm $(\mathcal{P}_n)$ converges to this solution. Since, a solution of $(\mathcal{P})$ corresponds to a fixed point of $\phi$, then, using Proposition 3.2, we obtain that $(v_0, T_0 + \tau_0)$ is the unique weak solution of problem $(\mathcal{P})$. Therefore, Corollary 2.1 is proved.

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