M2AN - Modélisation mathématique et analyse numérique

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M2AN - Modélisation mathématique et analyse numérique, tome 30, nº 1 (1996), p. 103-121
<http://www.numdam.org/item?id=M2AN_1996__30_1_103_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 30, n° 1, 1996, p. 103 à 120)

DIRICHLET PROBLEM ASSOCIATED WITH A RANDOM QUASILINEAR OPERATOR IN A RANDOM DOMAIN (*)

by Y. ABDDAIMI and G. MICHAILLE [1]

Abstract. — *We study the rate of convergence of solutions relative to Dirichlet problems associated with a random quasilinear operator in randomly perforated domains of* $\mathbf{R}^d$ *with holes whose size tends to 0. Our direct method allows to extend the results already obtained by an epi-convergence method in the case of symetric operator with deterministic and constant coefficients in a random domain.*

Key words : Homogenization, Dirichlet problem, ergodic theory.
AMS subject classifications : 35B27, 35J25, 60FXX, 60G10, 73B27.

Résumé. — *On étudie le comportement asymptotique des solutions des problèmes de Dirichlet associés à un opérateur quasilinéaire aléatoire dans des domaines aléatoirement perforés lorsque la taille des perforations tend vers zéro. La méthode directe utilisée permet d'étendre les résultats déjà obtenus par une méthode d'épi-convergence dans le cas d'opérateurs symétriques à coefficients constants et déterministes.*

1. INTRODUCTION

Let Ω be an open bounded subset in $\mathbf{R}^d$, $K(\omega)$ an union of randomly distributed « holes » in $\mathbf{R}^d$ and ε a positive rescaling parameter. We consider the following random Dirichlet problem in the perforated domain $\Omega \backslash K_\varepsilon(\omega)$, $K_\varepsilon(\omega) := \varepsilon K(\omega)$:

$$\begin{cases} - \operatorname{div}\left(A\left(\omega, \frac{x}{\varepsilon}\right) Du_\varepsilon(\omega, .\,) + \beta(\omega, u_\varepsilon(\omega, .\,))\right) + a(\omega, u_\varepsilon(\omega, .\,)) \\ = f \text{ in } \Omega \backslash K_\varepsilon(\omega)) \\ u_\varepsilon(\omega, .\,) = 0 \text{ in } \partial K_\varepsilon(\omega) \cup \partial\Omega \end{cases} \tag{1.1}$$

where $A(\omega, x)$ is a random elliptic matrix $(a_{i,j}(\omega, x))_{i,j}$, not necessarily symetric, β is a vector valued function from $\Sigma \times \mathbf{R}$ into $\mathbf{R}^d$, a is a scalar function from $\Sigma \times \mathbf{R}$ into $\mathbf{R}$ and where f is a given function in $L^2(\Omega)$.

(*) Manuscript received May 16, 1995.
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Under some hypothesis of the distribution of « holes » and on the random operator, one can show that the solution $u_\varepsilon(\omega, .)$ of (1.1), more precisely its extension by zero in $\Omega \cap \varepsilon K(\omega)$, strongly converges to zero in $H_0^1(\Omega)$ when ε tends to 0. A natural question to ask is : *at what rate does $u_\varepsilon(\omega, .)$ converge to zero ?* The aim of this paper is to answer the question. Among the physical motivations of this problem, we mention the applications to heat conduction or to electrostatic problems in wildly perturbed domains with a rapidly oscillating nonlinear conductivity or nonlinear dielectric field in the neighbourhood of random « holes ».

Let us give a more precise description of our probabilistic setting. $(\Sigma, \mathcal{T}, P)$ is the product probability space defined as follows :

- T is a finite family of compact sets included in $Y :=]0, 1[^d$,
- $\Sigma = \{\omega = (\omega_z)_{z \in \mathbf{Z}^d} : \omega_z - z \in T\}$ where ω_z are compact sets included in $Y + z$,
- $\mathcal{T}$ is the σ-field generated by the cylinders

$$E_{K,z} := \{\omega \in \Sigma : \omega_z - z = K\}, z \in \mathbf{Z}^d, K \in T,$$

- P is the probability product, construct from the probability presence of every element of T.

In these conditions, we prove that, almost surely, $u_\varepsilon(\omega, .)/\varepsilon^2$ weakly converges to λf in $L^2(\Omega)$ where λ is defined by $\lambda := \sup_{k \in \mathbf{N}^*} E(\lambda_k(\,.\,))$. $E(\lambda_k(\,.\,))$ denotes the probability average of

$$\lambda_k(\omega) := 1/k^d \int_{kY} w^{kY}(\omega, x)\, dx \quad \text{and} \quad w^{kY}(\omega, .)$$

is the solution of the following random Dirichlet problem in the cell kY and relative to the adjoint operator :

$$\begin{cases} -\operatorname{div}(A^*(\omega, x)\, Dw^{kY}(\omega, .)) = 1 & \text{in } kY \setminus K(\omega) \\ w^{kY}(\omega, .) = 0 & \text{in } K(\omega) \cap kY \\ w^{kY}(\omega, .) \in H_0^1(kY)\,. \end{cases} \tag{1.2}$$

This result generalizes those of H. Attouch [2], A. Brillard [8] or J. L. Lions [13] in the periodic case and the paper E. Chabi-G. Michaille [11] in the stochastic case. In this last paper A is the identity, $\beta = a = 0$ and an *epi-convergence* method — also known as *Γ-convergence* — was used.

Because of the random distribution of holes, the classical method where an oscillating test function is constructed from the only cell problem ($k = 1$) with periodicity condition, breaks down. Nevertheless, in [11], we showed how to recover λ in the periodic case from above expression (Corollary 5.2).

In the situation described in this paper, because of the non symmetry of A and the presence of β, we cannot use an epiconvergence process. Our method is then based on a technic of oscillating random test functions. As in the previous paper, the crucial point in our proof is to remark that the set function

$$I \mapsto \lambda_I(\,.\,) := \int_I w^I(\,.\,, x)\,dx$$

where $w^I(\omega, .)$ denotes the solution of (1.2) in a cell I, is a discrete superadditive process from the set of intervals $]a, b[$, a, b in $\mathbf{Z}^d$ into $L^1(\Sigma, \mathcal{T}, P)$. This property, when A is symmetric, is a direct consequence of the energy formulation and of the subadditivity property of the minimum (see G. Dal Maso-L. Modica [15], E. Chabi-G. Michaille [11]). In our case, this is a consequence of monotonicity properties of the solutions of problems (1.2).

We emphasize that our problem is a probabilistic version of a Dirichlet problem in a perforated domain of the form $\Omega \backslash r(\varepsilon)\,K$ where $r(\varepsilon) \sim \varepsilon$. For the case $r(\varepsilon) \ll \varepsilon$ and more precisely $r(\varepsilon) \sim \varepsilon^3$ and with a Bernoulli distribution of holes, we refer the reader to the thesis of E. Chabi [10]. The mathematical treatment of this last modelling is very similar to that of the deterministic version and the same kind of problem in the only cell Y occurs because of the boundary condition $u = 1$ on ∂Y. This homogenization problem leads to a scalar version of the *Brinkman's law*. For general results concerning Dirichlet problems in perforated domain, we refer the reader to G. Dal Maso-A. Garroni [14] and their references. For other results related to Dirichlet problems in random sets, see M. Balzano [4] and, for other aspects of theory of stochastic homogenization, we refer the reader to A. Bensoussan [5], A. Bourgeat-S. M. Koslov-S. Wright [7], G. Dal Maso-L. Modica [15], S. M. Koslov [16], G. C. Papanicolaou [18], K. Sab [19], and their bibliography.

The paper is organized as follows. The next section contains some notations and a brief summary of some results related to *Ergodic Theory* and to the measurability of *set-valued map*. In section 3, we shall be concerned with the almost sure convergence of the sequence $\{\lambda_k(\omega)\,;\, k \to +\infty\}$ toward λ. Using monotonicity properties with respect to the data satisfied by the solutions of problems of the form (1.2), we prove that λ_I is a discrete superadditive process. According to Ackoglu-Krengel's ergodic theorem, we derive the almost sure convergence of $\lambda_k(\omega)$ towards λ.

Section 4 is devoted to our main result. Assuming Ω to be a cube, the basic idea is to take as a test function in (1.1) the following oscillating random function

$$\phi \sum_{i \in I(\eta)} \varphi_i\, w^{kY + z_i}\!\left(\omega, \frac{\cdot}{\varepsilon}\right)$$

where $(\varphi_i)_{i \in I(\eta)}$ is a partition of unity associated to the partition $(Q_{i,\eta})_{i \in I(\eta)}$ of Ω made of cubes with size η and where ϕ belongs to $\mathscr{D}(\Omega)$. Every cube $Q_{i,\eta}$ is contained in a ε-homothetic of $(kY + z_i)$ where $k = k(\varepsilon) := \left[\frac{\eta}{\varepsilon}\right] + 1$ and $z_i \in \mathbf{Z}^d$. At the limit when ε tends to 0, φ_i tends to 1 and η tends to 0, we obtain the require information on the behaviour of $u_\varepsilon(\omega, .)/\varepsilon^2$. The signification of this choice is the following : omitting the functions φ_i which localize the test function in the ε-homothetic of $(kY + z_i)$ where an information (1.2) can be used, it is easily seen, using the mean value theorem and the convergence result of section 3, that almost surely (for a subsequence) the measure $\sum_{i \in I(\eta)} w^{kY + z_i}(\omega, .)\, dx$ weakly converges whenever ε tends to 0 towards $\lambda \sum_{i \in I(\eta)} \text{meas}(Q_{i,\eta})\, \delta_{a_i}$, $a_i \in \overline{Q}_{i,\eta}$. Going finally to the limit on η, this last measure weakly converges towards λl_Q where l_Q denotes the Lebesgue measure restricted to Q.

The Bernoulli probability space which describes the distribution of the random « holes » in $\mathbf{R}^d$ with some periodic structure, might be replaced by any complete probability space $(\Sigma, \mathcal{T}, P)$ fitted with an *Ergodic group* $(\tau_z)_{z \in \mathbf{Z}^d}$ *of P preserving transformations* such that the following conditions hold for P almost every ω in Σ :

(i) $K(\omega) - z = K(\tau_z(\omega))$, $A(\omega, x + z) = A(\tau_z \omega, x)\, x$ a.e. (see the proofs of Lemmas 3.1, 3.2),

(ii) meas $(K(\omega) \cap Y) > 0$ and Poincaré's constant for $\{u \in H^1(Y) : u = 0 \text{ on } K(\omega)\}$ is bounded by a constant that does not depend on ω (see the proof of lemmas 3.1, 3.3),

(iii) For every interval $I =]a, b[$ where a and b belong to $\mathbf{Z}^d$, the set-valued maps $\omega \mapsto C(\omega, I) := \{u \in H_0^1(I) : u = 0 \text{ on } K(\omega) \cap I\}$ is measurable (see Definition 2.2 and the proof of Lemma 3.2).

For the relevant definitions, see the next section. The validity of our method would not be affected by this more general setting.

2. PRELIMINARIES

In this section $(\Sigma, \mathcal{T}, P)$ is any probability space and $(\tau_z)_{z \in \mathbf{Z}^d}$ a group of P-preserving transformations on $(\Sigma, \mathcal{T})$, that is

(i) τ_z is $\mathcal{T}$-measurable,

(ii) $P \circ \tau_z(E) = P(E)$, for every E in $\mathcal{T}$ and every z in $\mathbf{Z}^d$,

(iii) $\tau_z \circ \tau_t = \tau_{z+t}$, $\tau_{-z} = \tau_z^{-1}$, for every z and t in $\mathbf{Z}^d$.

In addition, if every set E in $\mathcal{T}$ such that $\tau_z(E) = E$ for all $z \in \mathbf{Z}$ has a probability $P(E) = 0$ or 1, $(\tau_z)_{z \in \mathbf{Z}^d}$ is said to be Ergodic.

A sufficient condition to ensure the ergodicity property of $(\tau_z)_{z \in \mathbf{Z}^d}$ is the following *mixing* condition : for every E and F in $\mathcal{T}$

$$\lim_{|z| \to +\infty} P(\tau_z E \cap F) = P(E)\, P(F)\,.$$

which expresses an asymptotic independance.

When $(\Sigma, \mathcal{T}, P)$ is the probability space described in the introduction $(\tau_z)_{z \in \mathbf{Z}^d}$ will be the ergodic group of *P-preserving transformations* defined by $\tau_z(\omega) = (\omega_{t+z})_{t \in \mathbf{Z}^d}$ for all $z \in \mathbf{Z}^d$ and for all ω in Σ, that is the *shift group.*

We denote by $\mathcal{I}$ the set of intervals $[a, b[$ where a and b belong to $\mathbf{Z}^d$ and consider a set function $\mathcal{S}$ from $\mathcal{I}$ into $L^1(\Sigma, \mathcal{T}, P)$ satisfying the three following conditions :

(i) $\mathcal{S}$ is superadditive, that is, for every $I \in \mathcal{I}$ such that there exists a finite family $(I_j)_{j \in J}$ of disjoint sets in $\mathcal{I}$ with $I = \cup\, I_{j \in J}$,

$$\mathcal{S}_I(\,.\,) \geq \sum_{j \in J} \mathcal{S}_{I_j}(\,.\,,)$$

(ii) $\mathcal{S}$ is covariant, that is, for every $I \in \mathcal{I}$, every $z \in \mathbf{Z}^d$,

$$\mathcal{S}_{I+z} = \mathcal{S}_I \circ \tau_z\,,$$

(iii) $\sup \left\{ 1 / \text{meas}\,(I) \int_\Sigma \mathcal{S}_I\, dP, I \in \mathcal{I}, \text{meas}\,(I) \neq 0 \right\} < +\infty.$

Following M. A. Ackoglu-U. Krengel [1], $\mathcal{S}$ is called a discrete superadditive process and the following useful almost sure convergence result holds (see M. A. Ackoglu-U. Krengel [1] Theorem (2.4), Lemma (3.4) and U. Krengel [16] Remark, p. 59) :

THEOREM 2.1 : *When n tends to $+\infty$, $\frac{1}{n^d} \mathcal{S}_{[0,n[^d}$ converges almost surely. Moreover, if $(\tau_z)_{z \in k\mathbf{Z}^d}$ is Ergodic, then, almost surely :*

$$\lim_{n \to +\infty} \frac{1}{n^d} \mathcal{S}_{[0,n[^d}(\omega) = \sup_{n \in \mathbf{N}^*} \frac{1}{n^d} E(\mathcal{S}_{[0,n[^d}(\,.\,))$$

where $E(\,.\,)$ denotes the probability average operator.

To generalize our problem to the more general probability setting invoked at the end of the introduction, we shall use the following important definitions and result about the measurability of set-valued maps (see, for instance, C. Castaing-M. Valadier [9] or J. P. Aubin-H. Frankowska [3]).

DEFINITION 2.2 : *Consider a measurable space* $(\Sigma, \mathcal{T})$, *a complete separable metric space X and a set-valued map* $F : \Sigma \to X$ *with closed images.*

The map F is called measurable if, for every open subset $\mathcal{O}$ *of X, we have*

$$F^{-1}(\mathcal{O}) := \{\omega \in \Sigma ; F(\omega) \cap \mathcal{O} \neq \Phi\} \in \mathcal{T} .$$

A measurable map $s : \Sigma \to X$ *satisfying*

$$\forall \omega \in \Sigma , \quad s(\omega) \in F(\omega)$$

is called a measurable selection of F.

THEOREM 2.3 : *Let* $(\Sigma, \mathcal{T}, P)$ *be a complete probability space and F a set-valued map with closed images like in Definition 2.2. Then the two following properties are equivalent :*

(i) *F is measurable*

(ii) *There exists a sequence of measurable selections* $(s_n)_{n \geq 1}$ *of F such that*

$$\forall \omega \in \Sigma , \quad F(\omega) = \overline{\cup_{n \geq 1 \mathcal{S}_n(\omega)}} .$$

We adopt the following standard notations. For every bounded open set $\mathcal{O}$ in $\mathbf{R}^d$, $H^1(\mathcal{O})$ is the Sobolev space of all functions u in $L^2(\mathcal{O})$ such that the gradient distribution Du belongs to $L^2(\mathcal{O},\mathbf{R}^d)$, endowed with the norm :

$$\left(\int_{\mathcal{O}} |u(x)|^2 \, dx\right)^{1/2} + \left(\int_{\mathcal{O}} |Du(x)|^2 \, dx\right)^{1/2},$$

where we do not distinguish the notations of the norms in $\mathbf{R}$ and $\mathbf{R}^d$. $H_0^1(\mathcal{O})$ will denote the closure in $H^1(\mathcal{O})$ of the set $\mathscr{C}_0^\infty(\mathcal{O})$ of all $\mathscr{C}^\infty$-functions with compact supports in $\mathcal{O}$. For every topologic space E $\mathscr{B}(E)$ will denote its Borel field.

We shall denote by A^* the adjoint matrix $(a_{j,i})$ of the matrix $A = (a_{i,j})$ where $a_{i,j} : \Sigma \times \mathbf{R}^d \to \mathbf{R}$ and we make following hypothesis : almost surely

(2.1) $a_{i,j}$ is $\mathcal{T} \otimes \mathscr{B}(\mathbf{R}^d) - \mathscr{B}(\mathbf{R})$ measurable and there exists a $\mathcal{T} - \mathscr{B}(\mathbf{R})$ measurable function M from Σ into $\mathbf{R}^+$, such that $|a_{i,j}(\omega, x)| \leq M(\omega)$.

Moreover $\forall z \in \mathbf{Z}^d$, $A(\omega, x + z) = A(\tau_z \omega, x) x$ a.e.

(2.2) $a(\omega, .)$ and $\beta(\omega, .)$ are continuous and there exists C_1, $C_2 : \Sigma \to \mathbf{R}^+$ such that $\forall s \in \mathbf{R}$

$$|a(\omega, s)| \leq C_1(\omega)(|s| + 1), \quad |\beta(\omega, s)| \leq C_2(\omega)(|s| + 1) .$$

Moreover $a(\omega, 0) = \beta(\omega, 0) = 0$ and for $t > 0$ large enough

$$t|\beta(\omega, s)| \leq |\beta(\omega, ts)|, \quad \forall s \in \mathbf{R}.$$

(2.3) There exists a function $\alpha > 0$ such that $\frac{1}{\alpha} \in L^1(\Sigma, \mathcal{T}, P)$ and

$$\sum_{i,j=1}^{d} a_{i,j}(\omega, x)\, \xi_i \xi_j \geq \alpha(\omega)\, |\xi|^2, \quad \forall \xi \in \mathbf{R}^d, \quad x \text{ a.e.}$$

Remark 2.4 : For a proof of existence of a solution of problem (1.1), we refer the reader to L. Boccardo-F. Murat-J. P. Puel [7] or, for another method and for unicity, to M. Chipot-G. Michaille [12].

Note that, if A does not depend on x, by (2.1) and ergodicity property of $(\tau_z)_{z \in \mathbf{Z}^d}$, A is almost surely constant.

In the sequel, $(\Sigma, \mathcal{T}, P)$ will be the probability space described in the introduction and $(\tau_z)_{z \in \mathbf{Z}^d}$ will be *the shift group* which obviously satisfies $K(\omega) - z = K(\tau_z \omega)$. In this case, the proofs of the various measurability properties are evident. Nethertheless, we shall always give the proofs in the more general setting invoked at the end of the introduction.

3. DEFINITION AND PROPERTIES OF THE COEFFICIENT λ

Let us denote by $\mathcal{I}$ the set of intervals $]a, b[$, $a, b \in \mathbf{Z}^d$ and, for every $I \in \mathcal{I}$, consider the adjoint problems

$$\begin{cases} -\operatorname{div}(A^*(\omega, x)\, Dw^I(\omega, .)) = 1 \text{ in } I \setminus K(\omega) \\ w^I(\omega, .) = 0 \text{ in } K(\omega) \cap I \\ w^I(\omega, .) \in H_0^1(I). \end{cases} \tag{3.1}$$

We define the following linear subspace of $H_0^1(I)$

$$C(\omega, I) := \{ v \in H_0^1(I) : v = 0 \text{ in } I \cap K(\omega) \}$$

so that (3.1) is equivalent to

$$\begin{cases} w^I(\omega, .) \in C(\omega, I) \\ \int_I A^*(\omega, x)\, Dw^I(\omega, x) \,.\, Dv\, dx = \int_I v dx, \quad \forall v \in C(\omega, I). \end{cases}$$

LEMMA 3.1 : *The solution of* (3.1) *fulfils the following a priori estimates*

$$\frac{1}{meas(I)} \int_I w^I(\omega, x)\, dx \leqslant \frac{C_p}{\alpha(\omega)}, \frac{1}{meas(I)} \cdot \int_I |Dw^I(\omega, x)|^2\, dx \leqslant \frac{C_p}{\alpha^2(\omega)},$$

$$\frac{1}{meas(I)} \int_I |w^I(\omega, x)|^2\, dx \leqslant \left(\frac{C_p}{\alpha(\omega)}\right)^2 .$$

where C_p is a positive constant depending only on Poincaré's constant for the space $\{u \in H^1(Y) : u = 0 \text{ in } K(\omega) \cap Y\}$.

Proof: By Hölder's inequality

$$\int_I w^I(\omega, x)\, dx \leqslant \left(\int_I |w^I(\omega, x)|^2\, dx\right)^{\frac{1}{2}} (meas(I))^{\frac{1}{2}} .$$

According to $meas(I \setminus \cup_{z \in I \cap Z^d}(Y + z)) = 0$ and to Poincaré's inequality in the space $\{u \in H^1(Y) : u = 0 \text{ in } \omega_z - z\}$, there exists a constant $C(\omega_z - z)$, depending only on $\omega_z - z$, such that

$$\begin{aligned} \int_I |w^I(\omega, x)|^2\, dx &= \sum_{z \in I \cap Z^d} \int_Y |w^I(\omega, x + z)|^2\, dx \\ &\leqslant \sum_{z \in I \cap Z^d} C(\omega_z - z) \cdot \int_Y |Dw^I(\omega, x + z)|^2\, dx \\ &\leqslant C_p \sum_{z \in I \cap Z^d} \int_Y |Dw^I(\omega, x + z)|^2\, dx \\ &= C_p \int_I |Dw^I(\omega, x)|^2\, dx \\ &\leqslant \frac{C_p}{\alpha(\omega)} \int_I w^I(\omega, x)\, dx , \end{aligned}$$

where $C_p = \text{Max}\{C(\omega_z - z) : \omega_z - z \in T\}$ and where we have used the *ellipticity* assumption (2.3). It follows that

$$\int_I \omega^I(\omega, x)\, dx \leq \left(\frac{C_p}{\alpha(\omega)}\right)^{\frac{1}{2}} \left(\int_I w^I(\omega, x)\, dx\right)^{\frac{1}{2}} \text{meas}\, I^{\frac{1}{2}}.$$

This finishes the proof. ◇

LEMMA 3.2 : *Let $H_0^1(I)$ be equipped with its Borel field. Then the map $\omega \mapsto w^I(\omega, .)$ from Σ into $H_0^1(I)$ is measurable.*

Proof : It is easily seen that the inverse image of any Borel set of $H_0^1(I)$ is a cylinder and so belongs to $\mathcal{T}$.

In the more general case where $(\Sigma, \mathcal{T}, P)$ is a P-complete probability space, we assume that $\omega \mapsto C(\omega, I)$ is a measurable closed set-valued map from Σ into $H_0^1(I)$ (see (iii) in the introduction and definition 2.2). The basic idea is to give a « measurable constructive » proof of existence of $w^I(\omega, .)$.

It is well known that $w^I(\omega, .)$ is the unique fixed point of the strict contraction $S(\omega)$ of the Hilbert space $H_0^1(I)$ defined by

$$S(\omega)\, v := \pi(\omega)\, (\rho(\omega)\, (\tilde{1} - A^*(\omega)\, v) + v)\ \forall v \in H_0^1(I)$$

where

- $\rho(\omega) = \dfrac{\alpha(\omega)}{M^2(\omega)}$,
- $\pi(\omega)$ is the projection from $H_0^1(I)$ onto its closed linear subspace $C(\omega, I)$,
- $A^*(\omega)$ is the linear operator of $H_0^1(I)$ defined by

$$((A^*(\omega)\, u, v)) := \int_I A^*(\omega, x)\, Du \,.\, Dv\, dx, \forall u, v \text{ in } H_0^1(I)$$

$((\ ,\ ))$ denoting the standard inner product of $H_0^1(I)$,

- $\tilde{1}$ is the element of $H_0^1(I)$ defined by $((\tilde{1}, v)) := \int_I v\, dx$ for all v in $H_0^1(I)$.

Let v_0 be an arbitrary element of $H_0^1(I)$ and consider the sequence $\{v_n(\omega, .) : n \to +\infty\}$ defined by

$$v_{n+1}(\omega, .) = S(\omega)\, v_n(\omega, .) \quad \text{for} \quad n \geq 0\,.$$

Therefore $w^I(\omega, .) = \lim_{n \to +\infty} v_n(\omega, .)$ strongly in $H_0^1(I)$ and it remains to prove the measurability of $\omega \mapsto S(\omega)\, v_0$. For this, let us consider the two maps

$$\Xi : \Sigma \to \mathscr{L}(H_0^1(I)) \times H_0^1(I) \qquad \Theta : \mathscr{L}(H_0^1(I)) \times H_0^1(I) \to H_0^1(I)$$

$$\omega \mapsto (\pi(\omega), \rho(\omega)\,(\bar{1} - A^*(\omega)\, v_0) + v_0) \qquad (L, u) \mapsto Lu$$

where $\mathscr{L}(H_0^1(I))$ denotes the Banach space of continuous linear operator of $H_0^1(I)$ and where $\mathscr{L}(H_0^1(I)) \times H_0^1(I)$ and $H_0^1(I)$ are equipped with their Borel field. According to the diagram

$$\omega \mapsto (\pi(\omega), \rho(\omega)\,(\bar{1} - A^*(\omega)\, v_0) + v_0) \mapsto \pi(\omega)\,(\rho(\omega) \times (\bar{1} - A^*(\omega)\, v_0) + v_0)$$

and to the continuity of Θ, it suffices to establish the measurability of Θ. The measurability of $\rho(\omega)\,(\bar{1} - A^*(\omega)\, v_0)$ derives from the measurability of $\rho(\omega)$, from the separability of $H_0^1(I)$ and the measurability of $\omega \mapsto ((A^*(\omega)\, v_0, v))$ for every $v \in H_0^1(I)$, which is a direct consequence of hypothesis (2.1).

Let us prove the measurability of $\omega \mapsto \pi(\omega)$. It is equivalent to prove that for every u, v in $H_0^1(I)$, $\omega \mapsto ((\pi(\omega)\, u, v))$ is measurable. According to Theorem 2.3, there exists a family $(s_n(\omega))_{n \geq 1}$ of measurable selections of the set-valued map $\omega \mapsto C(\omega, I)$, such that $C(\omega, I) = \overline{\cup_{n \geq 1} s_n(\omega)}$. Therefore, there exists a sequence $\{s_n^u(\omega) : n \to +\infty\}$ of elements of $(s_n(\omega))_{n \geq 1}$ with $\pi(\omega)\, u = \lim_{n \to +\infty} s_n^u(\omega)$ strongly in $H_0^1(I)$. We thus get $((\pi(\omega)\, u, v)) = \lim_{n \to +\infty} ((s_n^u(\omega), v))$ with $\omega \mapsto ((s_n^u(\omega), v))$ measurable. This completes the proof of Lemma 3.1.

For every $I \in \mathscr{I}$ and $\omega \in \Sigma$, we now define

$$\lambda_I(\omega) := \int_I w^I(\omega, x)\, dx\,.$$

The following lemma shows that λ is a discrete superadditive process, more precisely.

LEMMA 3.3 : *For all I in $\mathscr{I}$*

(i) *The map* $\omega \mapsto \lambda_I(\omega)$ *belongs to* $L^1(\Sigma, \mathcal{T}, P)$ *and there exists a constant* $C_p > 0$ *such that*

$$\| \lambda_I \|_{L^1(\Sigma, \mathcal{T}, P)} \leq meas\,(I)\; C_p \int_\Sigma \frac{1}{\alpha(\omega)}\, dP(\omega)\,,$$

(ii) *the map* $I \mapsto \lambda_I$ *from* $\mathcal{I}$ *into* $L^1(\Sigma, \mathcal{T}, P)$ *is superadditive,*

(iii) λ_I *is covariant, that is, for every* $z \in \mathbf{Z}^d$,

$$\lambda_{z+I}(\,.\,) = \lambda_I(\,.\,)\circ\tau_z\,.$$

Proof of (i) : By Lemma 3.2 and the continuity of $u \mapsto \int_I u\, dx$ from $H^1_0(I)$ into $\mathbf{R}$, the map $\omega \mapsto \lambda_I(\omega)$ is measurable. On the other hand, Lemma 3.1 and assumption (2.3) gives the desired inequality.

Proof of (ii) : Let us consider the two Dirichlet problems

$$\begin{cases} -\ \operatorname{div}\,(A^*(\omega, x)\, Du_l(\omega, .\,)) = g_l \text{ in } I\backslash K(\omega) \\ u_l(\omega, .\,) = 0 \text{ in } K(\omega) \cap I \\ u_l(\omega, .\,) = \varphi_l \text{ in } \partial I \quad l = 1, 2 \end{cases}$$

where g_l, $l = 1, 2$ are two given functions in $L^2(I)$ and where φ_l, $l = 1, 2$ are two given functions in $L^2(\partial I)$.

Classicaly, the solutions $u_l(\omega, .\,)$ fulfil the following monotonicity properties with respect to the data g_l, φ_l :

$$g_1 \leq g_2 \text{ a.e. in } I \text{ and } \varphi_1 \leq \varphi_2 \text{ a.e. on } \partial I \Rightarrow u_1(\omega, .\,) \leq u_2(\omega, .\,) \text{ a.e. in } I\,.$$

Therefore, taking $g_1 = 0$, $\varphi_1 = 0$ and $g_2 = 1$, $\varphi_2 = 0$, we obtain $0 \leq w^I(\omega, .\,)$ a.e. in I.

On the other hand, let I, I_1, I_2 be three sets of $\mathcal{I}$ such that meas $(I\backslash I_1 \cup I_2) = 0$. By the previous result, the restrictions $w^I(\omega, .\,)\llcorner I_1$ and $w^I(\omega, .\,), \llcorner I_2$ of $w^I(\omega, .\,)$ to I_1 and I_2 satisfy, in the trace sense

$$w^I(\omega, .\,)\llcorner I_1 \geq \text{ a.e. on } \partial I_1,$$

$$w^I(\omega, .\,)\llcorner I_2 \geq \text{ a.e. on } \partial I_2$$

so that, using again the monotonicity properties with $g_l = 1$, $l = 1, 2$

$$w^{I_1}(\omega, .) \leqslant w^I(\omega, .) \llcorner I_1 \text{ a.e. in } I_1 ,$$

$$w^{I_2}(\omega, .) \leqslant w^I(\omega, .) \llcorner I_2 \text{ a.e. in } I_2 .$$

It follows

$$\lambda_I(\omega) = \int_{I_1} w^I(\omega, x)\, dx + \int_{I_2} w^I(\omega, x)\, dx$$

$$\geqslant \lambda_{I_1}(\omega) + \lambda_{I_2}(\omega)$$

which completes the proof of (ii).

Proof of (iii) : By (2.1), the property $K(\tau_z \omega) = K(\omega) + z$ (see also hypothesis (i) in the introduction) and the unicity of the solutions of problem 3.1, it is straightforward to check that

$$w^I(\tau_z \omega, .) = w^{z+I}(\omega, . + z) \text{ a.e. in } I .$$

Thus

$$\begin{aligned} \lambda(\,.\,) \circ \tau_z(\omega) &= \int_I w^I(\tau_z \omega, x)\, dx \\ &= \int_I w^{z+I}(\omega, x + z)\, dx \\ &= \int_{z+I} w^{z+I}(\omega, x)\, dx \\ &= \lambda_{z+I}(\omega) . \end{aligned}$$

which ends the proof of Lemma 3.2. ◇

We are now in a position to state the main theorem of this section.

THEOREM 3.4 : *There exists* $\Sigma' \in \mathcal{T}$, $P(\Sigma') = 1$, *such that for every* $\omega \in \Sigma'$

$$\lim_{k \to +\infty} \frac{\lambda_{kY}(\omega)}{\text{meas}\,(kY)} = \sup_{k \in N^*} \frac{E(\lambda_{kY}(.))}{\text{meas}\,(kY)} . \tag{3.2}$$

We shall denote this last limit by λ.

Proof : It is a direct consequence of Lemma 3.3 and Theorem 2.1.

Remark 3.5 : Note that, thanks to the covariance property, for every fixed z in $\mathbf{Z}^d$, we also have, almost surely

$$\lambda = \lim_{k \to \infty} \frac{1}{k^d} \int_{kY+z} w^{kY+z}(\omega, x)\, dx .$$

4. LIMIT BEHAVIOUR OF $\dfrac{u_\varepsilon(\omega, .)}{\varepsilon^2}$

In the sequel, the extension by 0 in $\Omega \cap \varepsilon K(\omega)$ of the solution of problem (1.1) will be still denoted by $u_\varepsilon(\omega, .)$.

We begin with a proposition which states that the sequence $u_\varepsilon(\omega, .)/\varepsilon^2$ is bounded in $L^2(\Omega)$. More precisely

PROPOSITION 4.1 : *$u_\varepsilon(\omega, .)$ strongly converges to 0 in $H_0^1(\Omega)$ and there exists a positive constant C_p such that*

$$\| Du_\varepsilon(\omega) \|_{L^2(\Omega)} \leqslant \frac{\sqrt{C_p}}{\alpha(\omega)}\, \varepsilon\, \|f\|_{L^2(\Omega)}$$

$$\| u_\varepsilon(\omega) \|_{L^2(\Omega)} \leqslant \frac{C_p}{\alpha(\omega)}\, \varepsilon^2\, \|f\|_{L^2(\Omega)} .$$

Proof: To shorten notation, we ignore the dependance on ω. Extending u_ε by 0 in $\mathbf{R}^d \backslash \Omega$, we obtain

$$\int_{\Omega \backslash \varepsilon K(\omega)} u_\varepsilon^2\, dy = \sum_{\{z \in \mathbf{Z}^d\ ;\ \Omega \cap \varepsilon(Y+z) \neq \emptyset\}} \int_{\varepsilon(Y+z)} u_\varepsilon^2\, dx ,$$

and a change of scale, $x = \varepsilon(y+z)$, yields

$$\int_{\Omega \backslash \varepsilon K(\omega)} u_\varepsilon^2\, dy = \sum_{\{z \in \mathbf{Z}^d\ ;\ \Omega \cap \varepsilon(Y+z) \neq \emptyset\}} \varepsilon^d \int_{Y \backslash K(\tau_z \omega)} \tilde{u}_\varepsilon^2(y)\, dy$$

where $\tilde{u}_\varepsilon(y) = u_\varepsilon(\varepsilon(y+z))$. Applying, for every z in $\mathbf{Z}^d$, the Poincaré inequality in the space $\{u \in H^1(Y) : u = 0 \text{ in } K(\tau_z \omega)\}$, like in the proof of Lemma 3.1, we obtain the existence of a constant $C_p > 0$ such that

$$\int_{Y \setminus K(\tau_z \omega)} \tilde{u}_\varepsilon^2(y)\, dy \leqslant C_p \int_{Y \setminus K(\tau_z \omega)} |D\tilde{u}_\varepsilon(y)|^2\, dy .$$

Noticing that $D\tilde{u}_\varepsilon(y) = \varepsilon(Du_\varepsilon)(\varepsilon(y+z))$, we get, after summing over z and rescaling,

$$\int_{\Omega \setminus \varepsilon K(\omega)} u_\varepsilon^2\, dy \leqslant \varepsilon^2 C_p \int_{\Omega \setminus \varepsilon K(\omega)} |Du_\varepsilon|^2\, dx .$$

On the other hand, according to ellipticity assumption (2.3)

$$\int_{\Omega \setminus K_\varepsilon(\omega)} |Du_\varepsilon|^2\, dx \leqslant \frac{1}{\alpha(\omega)} \int_{\Omega \setminus K_\varepsilon(\omega)} f u_\varepsilon\, dx$$

$$\leqslant \frac{1}{\alpha(\omega)} \left(\int_\Omega f^2\, dx \right)^{\frac{1}{2}} \left(\int_\Omega u_\varepsilon^2\, dx \right)^{\frac{1}{2}},$$

it follows

$$\| Du_\varepsilon(\omega) \|_{L^2(\Omega)} \leqslant \frac{\sqrt{C_p}}{\alpha(\omega)}\, \varepsilon \|f\|_{L^2(\Omega)} ,$$

and

$$\| u_\varepsilon(\omega) \|_{L^2(\Omega)} \leqslant \frac{C_p}{\alpha(\omega)}\, \varepsilon^2 \|f\|_{L^2(\Omega)} ,$$

which completes the proof of Proposition 4.1.

We construct now the random test function invoked in introduction. There is no loss of generality in assuming Ω to be a cube Q and we consider the partition $(Q_{i,\eta})_{i \in I(\eta)}$ of Q made of cubes with size η (actually η is a sequence converging to 0). Every $Q_{i,\eta}$ is contained in a cube $\varepsilon(kY + z_i)$ where $k = k(\varepsilon) := \left[\frac{\eta}{\varepsilon}\right] + 1$ and $z_i \in \mathbf{Z}^d$. Finally, we consider the family

$(\varphi_i^\delta)_{i \in I(\eta)}$ of functions φ_i^δ in $C_0^\infty(\mathbf{R}^d)$, $0 \leq \varphi_i^\delta \leq 1$ with support in $Q_{i,\eta}$ and where δ is a positive parametter such that $\lim_{\delta \to 0} \varphi_i^\delta = 1$ strongly in $L^2(Q_{i,\eta})$. With the notations of section 3, we define the random measure

$$v_{\eta,\delta,\epsilon}(\omega) := \sum_{i \in I(\eta)} \varphi_i^\delta \, w^{kY+z_i}\left(\omega, \frac{\cdot}{\varepsilon}\right) dx .$$

The following lemma is the key point of the main result of this paper.

LEMMA 4.2 : *For every ω in the set Σ' of Theorem 3.4, we have the following $\sigma(C_0'(Q), C_0(Q)$ convergence, for a subsequence on ε and δ*

$$\lim_{\eta \to 0, \delta \to 0, \varepsilon \to 0} v_{\eta,\delta,\varepsilon}(\omega) = \lambda l_{\lfloor Q} ,$$

where $C_0(Q)$ denotes the set of all continuous functions with compact support in Q and $l_{\lfloor Q}$ the Lebesgue measure restricted to Q.

Proof: Let $\phi \in C_0(Q)$,

$$\begin{aligned}
\langle v_{\eta,\delta,\varepsilon}(\omega), \phi \rangle &= \sum_{i \in I(\eta)} \int_{Q_{i,\eta}} \phi \varphi_i^\delta \, w^{kY+z_i}\left(\omega, \frac{x}{\varepsilon}\right) dx \\
&= \sum_{i \in I(\eta)} \phi(a_{i,\delta,\varepsilon}) \int_{Q_{i,\eta}} \varphi_i^\delta \, w^{kY+z_i}\left(\omega, \frac{x}{\varepsilon}\right) dx \qquad (4.1) \\
&= \sum_{i \in I(\eta)} \phi(a_{i,\delta,\varepsilon}) (\varepsilon K)^d \frac{1}{k^d} \int_{kY+z_i} \varphi_i^\delta(\varepsilon x) \, w^{kY+z_i}(\omega, x) \, dx
\end{aligned}$$

where we have used the mean value theorem, a change of scale and where $a_{i,\delta,\varepsilon} \in \overline{Q}_{i,\eta}$. On the other hand, according to Theorem 3.4, Remark 3.5 and Lemma 3.1, it is easily seen that, for every ω in the set Σ' of Theorem 3.4

$$\lim_{\delta \to 0, \varepsilon \to 0} \frac{1}{k^d} \int_{kY+z_i} \varphi_i^\delta(\varepsilon x) \, w^{kY+z_i}(\omega, x) \, dx = \lambda . \qquad (4.2)$$

By (4.1) and (4.2), for every ω fixed in Σ', there exist a subsequence on ε and δ independant on η and $a_i \in \overline{Q}_{i,\eta}$ such that

$$\lim_{\delta \to 0, \varepsilon \to 0} v_{\eta,\delta,\varepsilon}(\omega) = \lambda \sum_{i \in I(\eta)} \text{meas}\,(Q_{i,\eta}) \, \delta_{a_i}$$

$\sigma(C_0'(Q), C_0(Q))$ (note that $(k\varepsilon)^d \sim \text{meas}(Q_{i,\eta})$ when ε tends to 0). We obtain the desired result by making η tend to 0.

We are now in a position to prove our main theorem.

THEOREM 4.3 : *Let $u_\varepsilon(\omega, .)$ be the solution of problem (1.1). There exists Σ' in $\mathcal{T}$ with $P(\Sigma') = 1$, such that, for every $\omega \in \Sigma'$, $u_\varepsilon(\omega, .)/\varepsilon^2$ weakly converges to λf in $L^2(\Omega)$ where λ is defined by*

$$\lambda := \sup_{k \in \mathbf{N}^*} E(\lambda_k(\,.\,))$$

$$\lambda_k(\omega) := 1/k^d \int_{kY} w^{kY}(\omega, x)\, dx\,,$$

and where, for every $k \in \mathbf{N}^*$, $w^{kY}(\omega, .)$ is the solution of the random Dirichlet problem (1.2).

Proof : One may assume f to be continuous in Ω. An easy density argument allows to extend the result in the general case. From now on, ω is a fixed element in the subset Σ' of Theorem 3.4.

Let us take, as a test function in (1.1)

$$v_{\eta,\delta,\varepsilon} := \Phi \sum_{i \in I(\eta)} \varphi_i^\delta\, w^{kY+z_i}\left(\omega, \frac{\cdot}{\varepsilon}\right)$$

where Φ is any element of $C_0^\infty(\Omega)$. We get (to shorten notation, we ignore again the dependance on ω)

$$\int_\Omega A\left(\frac{x}{\varepsilon}\right) Du_\varepsilon \,.\, Dv_{\eta,\delta,\varepsilon}\, dx + \int_\Omega a(\omega, u_\varepsilon)\, v_{\eta,\delta,\varepsilon}\, dx + \int_\Omega \beta(u_\varepsilon) \,.\, Dv_{\eta,\delta,\varepsilon}\, dx = \int_\Omega f v_{\eta,\delta,\varepsilon}\, dx \,. \tag{4.3}$$

By lemma 4.2, the right hand side converges to $\lambda \int_\Omega \Phi f\, dx$ whenever ε tends to 0, δ tends to 0 and η tends to 0. We estimate now the left hand side.

$$\int_\Omega A\left(\frac{x}{\varepsilon}\right) Du_\varepsilon . Dv_{\eta,\delta,\varepsilon}\, dx + \int_\Omega a(u_\varepsilon)\, v_{\eta,\delta,\varepsilon}\, dx + \int_\Omega \beta(u_\varepsilon) . Dv_{\eta,\delta,\varepsilon}\, dx$$

$$= \sum_{i \in I(\eta)} \int_{Q_{i,\eta}} \frac{1}{\varepsilon} A\left(\frac{x}{\varepsilon}\right) Du_\varepsilon . (Dw^{kY+z_i})\left(\frac{x}{\varepsilon}\right) \Phi\varphi_i^\delta\, dx$$

$$+ \int_\Omega a(u_\varepsilon)\, v_{\eta,\delta,\varepsilon}\, dx + \int_\Omega \beta(u_\varepsilon) . Dv_{\eta,\delta,\varepsilon}\, dx + O(\varepsilon)$$

$$= \sum_{i \in I(\eta)} \int_{Q_{i,\eta}} \frac{1}{\varepsilon} Du_\varepsilon . A^*\left(\frac{x}{\varepsilon}\right) (Dw^{kY+z_i})\left(\frac{x}{\varepsilon}\right) \Phi\varphi_i^\delta\, dx$$

$$+ \int_\Omega a(u_\varepsilon)\, v_{\eta,\delta,\varepsilon}\, dx + \int_\Omega \beta(u_\varepsilon) . Dv_{\eta,\delta,\varepsilon}\, dx + O(\varepsilon) \tag{4.4}$$

$$= \sum_{i \in I(\eta)} \int_{Q_{i,\eta}} \frac{u_\varepsilon(x)}{\varepsilon^2} \left(- \operatorname{div}\,(A^*(.)\, Dw^{kY+z_i}(.))\left(\frac{x}{\varepsilon}\right)\right) \Phi\varphi_i^\delta\, dx$$

$$+ \int_\Omega a(u_\varepsilon)\, v_{\eta,\delta,\varepsilon}\, dx + \int_\Omega \beta(u_\varepsilon) . Dv_{\eta,\delta,\varepsilon}\, dx + O(\varepsilon)$$

$$= \sum_{i \in I(\eta)} \int_{Q_{i,\eta}} \frac{u_\varepsilon(x)}{\varepsilon^2} \Phi\varphi_i^\delta\, dx + \int_\Omega a(u_\varepsilon)\, v_{\eta,\delta,\varepsilon}\, dx + \int_\Omega \beta(u_\varepsilon) . Dv_{\eta,\delta,\varepsilon}\, dx + O(\varepsilon)$$

where $O(\varepsilon)$ denotes various expressions which tend to 0 whenever ε tends to 0. Indeed, this is a straightforward calculation, using (2.1), Hölder's inequality, Lemma 3.1 and Proposition 4.1.

Going to the limit in ε, δ (for a further subsequence), by proposition 4.1, we get the existence of $u \in L^2(\Omega)$ such that

$$\lim_{\delta \to 0, \varepsilon \to 0} \sum_{i \in I(\eta)} \int_{Q_{i,\eta}} \frac{u_\varepsilon(x)}{\varepsilon^2} \Phi\varphi_i^\delta\, dx = \int_\Omega u\Phi\, dx\,. \tag{4.5}$$

Let us estimate the two last integrals of (4.4). By Lemma 3.1 and a straightforward calculation, it is easily seen that $v_{\eta,\delta,\varepsilon}$ is bounded in $L^2(\Omega)$ by a constant C independant on ε. On the other hand, by (2.2), $a(\omega, .)$ is a strongly continuous operator from $L^2(\Omega)$ into itself. Therefore, according to Proposition 4.1, $a(u_\varepsilon)$ strongly converges to 0 in $L^2(\Omega)$ and

$$\int_\Omega a(u_\varepsilon)\, v_{\eta,\delta,\varepsilon}\, dx = O(\varepsilon)\,.$$

For the last integral, with the same arguments, we have

$$\int_\Omega \beta(u_\varepsilon)\,.\,Dv_{\eta,\delta,\varepsilon}\, dx = \sum_{i \in I(\eta)} \int_{Q_{i,\eta}} \frac{1}{\varepsilon}\beta(u_\varepsilon)\,.\,(Dw^{kY+z_i})\left(\frac{x}{\varepsilon}\right)\Phi\varphi_i^\delta\, dx + O(\varepsilon)\,.$$

But, like above, by Lemma 3.1

$$\Phi \sum_{i \in I(\eta)} \varphi_i^\delta (Dw^{kY+z_i})\left(\omega, \frac{\cdot}{\varepsilon}\right)$$

is bounded in $L^2(\Omega)$ by a constant C so that, according to (2.2) for ε large enough

$$\left|\int_\Omega \beta(u_\varepsilon)\,.\,Dv_{\eta,\delta,\varepsilon}\, dx\right| \leqslant C\left(\int_\Omega \left|\beta\left(\frac{1}{\varepsilon}u_\varepsilon\right)\right|^2 dx\right)^{\frac{1}{2}} + O(\varepsilon)\,.$$

Finally, by (2.2) and Proposition 4.1, $\beta\left(\frac{1}{\varepsilon}u_\varepsilon\right)$ strongly converges to 0 in $L^2(\Omega, \mathbf{R}^d)$ and

$$\int_\Omega \beta(u_\varepsilon)\,.\,Dv_{\eta,\delta,\varepsilon}\, dx = O(\varepsilon)\,.$$

Recalling (4.3), (4.4), (4.5) and previous estimates, going to the limit on ε, δ and η, we obtain

$$\int_\Omega u\Phi\, dx = \lambda \int_\Omega \Phi f\, dx$$

which ends the proof. ◇

Remark 4.4 : If $a(\omega, .\,0) \neq 0$, with the same computations, we obtain that almost surely $u_\varepsilon(\omega, .\,)/\varepsilon^2$ weakly converges to $\lambda(f - a(\omega, 0))$ in $L^2(\Omega)$.

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