MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

HUA DAI

About an inverse eigenvalue problem arising in vibration analysis

Modélisation mathématique et analyse numérique, tome 29, nº 4 (1995), p. 421-434

<http://www.numdam.org/item?id=M2AN_1995__29_4_421_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODELISATION MATHEMATIQUE ET ANALYSE NUMERIQUE

(Vol 29, n° 4, 1995, p 421 à 434)

ABOUT AN INVERSE EIGENVALUE PROBLEM ARISING IN VIBRATION ANALYSIS (*)

by DAI HUA (2)

Communicated by Françoise CHAITIN-CHATELIN

Abstract — *The problem of the best approximation for a given real matrix in the Frobenius and the weighted Frobenius norm by real symmetric positive semidefinite matrices under the given spectral constraints is considered Such a problem is related to the frequently encountered engineering problem of a structural modification on the dynamic behaviour of a structure Necessary and sufficient conditions are obtained for the solubility of the problem A numerical algorithm is suggested Two numerical examples are given*

AMS (MOS) Subject Classifications 65F15, 15A60, 49D10

Key words inverse eigenvalue problem, best approximation, structural modification, modal analysis, vibration test

Résumé — *On considere le problème de la meilleure approximation d'une matrice réelle dans la norme de Frobenius et celle de Frobenius avec poids par des matrices réelles symétriques semi-définies positives, sous des contraintes spectrales données On rencontre fréquemment ce type de problème en ingénierie lorsqu'on étudie la modification structurelle du comportement dynamique d'une structure On obtient des conditions nécessaires et suffisantes pour résoudre le problème On propose un algorithme de résolution et deux exemples numériques sont donnés*

1. INTRODUCTION

Let $\mathbb{R}_r^{n \times m}$ denote the set of all matrices with rank r in $\mathbb{R}^{n \times m}$, $O\mathbb{R}^{n \times n}$ the set of all orthogonal matrices in $\mathbb{R}^{n \times n}$, $SP\mathbb{R}^{n \times n}$ the set of all symmetric positive semidefinite matrices in $\mathbb{R}^{n \times n}$. $A \geqslant 0$ means that A is a real symmetric positive semidefinite matrix. $R(A)$ is the column space of the matrix A. A^+ and $\|A\|_F$ represent the Moore-Penrose generalized inverse matrix and the Frobenius norm of a matrix A, respectively. $\lambda(A)$ and $\lambda(K, M)$ stand for the set of all eigenvalues of the standard eigenvalue problem $Ax = \lambda x$ and the generalized eigenvalue problem $Kx = \lambda Mx$, respectively

(*) Received in September 1993, revised in January 1994

(2) Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, The People's Republic of China

For $0 \leqslant \alpha \leqslant \beta$, let

$$SP\mathbb{R}^{n \times n}_{[\alpha\ \beta]} = \{A \in SP\mathbb{R}^{n \times n} \mid \lambda(A) \subset [\alpha, \beta]\}$$

In this paper, the following two kinds of inverse eigenvalue problems are considered

PROBLEM I *Given a matrix* $\tilde{A} \in \mathbb{R}^{n \times n}$, *a matrix* $X \in \mathbb{R}^{n \times m}$, *a real diagonal matrix* $\Lambda = \mathrm{diag}(\lambda_1, \ \ , \lambda_m) \in \mathbb{R}^{m \times m}$ *and two nonnegative real numbers* α, β $(\alpha < \beta)$ *Let*

$$\mathbb{S}_A = \{A \in SP\mathbb{R}^{n \times n} \mid AX = X\Lambda, \lambda(A) \backslash \lambda(\Lambda) \subset [\alpha, \beta]\} \tag{1 1}$$

find $\hat{A} \in \mathbb{S}_A$ *such that*

$$\|\hat{A} - \tilde{A}\|_F = \inf_{\forall A \in \mathbb{S}_A} \|A - \tilde{A}\|_F \tag{1 2}$$

PROBLEM II *Given a matrix* $\tilde{K} \in \mathbb{R}^{n \times n}$, *an* $n \times n$ *real symmetric positive definite matrix* M, *a matrix* $X \in \mathbb{R}^{n \times m}$, *a real diagonal matrix* $\Lambda = \mathrm{diag}(\lambda_1, \ \ , \lambda_m) \in \mathbb{R}^{m \times m}$ *and two nonnegative real numbers* $\alpha, \beta (\alpha < \beta)$ *Let*

$$\mathbb{S}_K = \{K \in SP\mathbb{R}^{n \times n} \mid KX = MX\Lambda, \lambda(K, M) \backslash \lambda(\Lambda) \subset [\alpha, \beta]\} \tag{1 3}$$

find $\hat{K} \in \mathbb{S}_K$ *such that*

$$\| W(\hat{K} - \tilde{K})\, W\|_F = \inf_{\forall K \in \mathbb{S}_K} \| W(K - \tilde{K})\, W\|_F \tag{1 4}$$

where $W = M^{-\frac{1}{2}}$

Since the sets $\mathbb{S}_A$ and $\mathbb{S}_K$, further the solutions $\hat{A}$ and $\hat{K}$, are determined by the eigenvalues Λ and corresponding eigenvectors X, the above problems are essentially two kinds of inverse eigenvalue problems A special case of Problem I is considered by Zhang [12, 13], Liao [14], Wang and Chang [15] Clearly, Problem II is a generalization of Problem I to a weighted Frobenius norm

Such problems arise in the field of vibration analysis The dynamic behaviour of an engineering structure is determined by the generalized eigenvalue problem $Kx = \lambda Mx$ [1, 21] (where K and M are the stiffness and the mass matrices of the structure, respectively) or the equivalent standard eigenvalue problem $Ax = \lambda x$ [2] In some applications, due to the complexity of the structure no reasonable analytical model of the stiffness matrix can be evaluated, a preliminary estimate of the unknown matrix A or the stiffness matrix K can be obtained by the finite element analysis, the experimental observation and the information of statistical distribution, whereas the mass matrix is known. Additional information on the dynamic behaviour of the structure is available from a vibration test, where the excitation and the response of the structure at many points are measured experimentally. Identification techniques [3] extract a part of the eigenpairs of the structure from the measurements. However, one usually obtains an incomplete set of eigenpairs from the vibration tests [4].

A frequently encountered engineering problem is one in which the designer would like to improve the stiffness matrix of an existing structure such that the improved model predicts the observed dynamic behaviour. Then the improved model may be considered to be a better dynamic representation of the structure. This model may be used with greater confidence for the analysis of the structure under different boundary conditions or with physical structural changes. This structural modification problem is essentially equivalent to the above-mentioned problems.

There are many publications concerning the stiffness matrix improvements [5-11]. In [5-9], Baruch, Wei and Berman used the Lagrange multiplier method to correct the stiffness matrix from vibration tests. The improved stiffness matrix which satisfies the dynamic equation is symmetric, but not necessarily positive semidefinite. However, from a physical and mechanical point of view, the stiffness matrix should be symmetric positive semi-definite. Accordingly, our corrected stiffness matrix will be symmetric positive semidefinite.

The paper is organized as follows. In Section 2 we study the necessary and sufficient condition under which $\mathbb{S}_A$ is nonempty, give the general form of $\mathbb{S}_A$ and propose the expression of solution to Problem I. Problem II is discussed in Section 3. All results on Problem I can be transferred to Problem II. In Section 4 a numerical algorithm and two numerical examples are presented.

For Problem I and II, without loss of generality we may assume that there are t $(t \leqslant m)$ distinct eigenvalues in the given m eigenvalues, the t distinct eigenvalues are still written by $\lambda_1, \dots, \lambda_t$, and their multiplicities are $m_1, \dots, m_t$, respectively. Then $m = \sum_{i=1}^{t} m_i$ and let

$$\Lambda = \operatorname{diag}\left(\lambda_1 I_{m_1}, \dots, \lambda_t I_{m_t}\right). \tag{1.5}$$

The given matrix X can be expressed as

$$X = (X_1, X_2, \dots, X_t) \tag{1.6}$$

where $X_i \in \mathbb{R}^{n \times m_i}$ $(i = 1, \dots, t)$, and every column of X_i is the eigenvector corresponding to the eigenvalue λ_i.

2. THE SOLUTION OF PROBLEM I

We decompose the given matrix $X \in \mathbb{R}^{n \times m}$ by the singular-value decomposition (SVD) (see, e g , Horn and Johnson [16, pp 414-415])

$$X = U_1 \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} V_1^T \qquad (2\ 1)$$

where $U_1 = (U_1^{(1)}, U_2^{(1)}) \in O\mathbb{R}^{n \times n}$, $V_1 = (V_1^{(1)}, V_2^{(1)}) \in O\mathbb{R}^{m \times m}$, $\Sigma_1 = \operatorname{diag}(\sigma_1^{(1)}, \quad, \sigma_r^{(1)})$, $\sigma_i^{(1)} > 0$ $(i = 1, \quad, r)$, $r = \operatorname{rank}(X)$, $U_1^{(1)} \in \mathbb{R}^{n \times r}$, $V_1^{(1)} \in \mathbb{R}^{m \times r}$

LEMMA 2 1 ([13]) *Suppose* $X = (X_1, \quad, X_t) \in \mathbb{R}_r^{n \times m}$, $X_i \in \mathbb{R}^{n \times m}$ $(i = 1, \quad, t)$ *and* $\Lambda = \operatorname{diag}(\lambda_1 I_{m_1}, \quad, \lambda_t I_{m_t})$ $(\lambda_i \neq \lambda_j, i \neq j)$, *then there exists a matrix* $A \in SP\mathbb{R}^{n \times n}$ *such that* $AX = X\Lambda$ *if and only if*

$$\lambda_i \geqslant 0, \quad (\lambda_i - \lambda_j) X_i^T X_j = 0, \quad i, j = 1, \quad, t \qquad (2\ 2)$$

When the above condition is satisfied, the general solution $A \in SP\mathbb{R}^{n \times n}$ of the equation $AX = X\Lambda$ can be expressed as

$$A = X\Lambda X^+ + U_2^{(1)} G U_2^{T(1)} \qquad (2\ 3)$$

where G is an arbitrary $(n - r) \times (n - r)$ symmetric positive semidefinite matrix

Furthermore, if $\operatorname{rank}(X_i) = r_i$ $(i = 1, \quad, t)$ and the equation $AX = X\Lambda$ has a solution $A \in SP\mathbb{R}^{n \times n}$, then $r = \sum_{i=1}^{t} r_i$ and $t \leqslant r$

The following theorem gives a necessary and sufficient condition for $\mathbb{S}_A$ being nonempty

THEOREM 2 1 *Suppose* $X = (X_1, \quad, X_t) \in \mathbb{R}_r^{n \times m}$, $X_i \in \mathbb{R}^{n \times m}$ $(i = 1, \quad, t)$, $\Lambda = \operatorname{diag}(\lambda_1 I_{m_1}, \quad, \lambda_t I_{m_t})$ $(\lambda_i \neq \lambda_j, i \neq j, i, j = 1, \quad, t)$, *then* $\mathbb{S}_A$ *is nonempty if and only if* Λ *and* X *satisfy the condition* (2 2)

Moreover, under the condition, $\mathbb{S}_A$ can be expressed as

$$\mathbb{S}_A = \{ X\Lambda X^+ + U_2^{(1)} G U_2^{T(1)} \mid G \in SP\mathbb{R}_{[\alpha\ \beta]}^{(n-r) \times (n-r)} \} \qquad (2\ 4)$$

Proof It follows by Lemma 2 1 that the general solution $A \in SP\mathbb{R}^{n \times n}$ of the equation $AX = X\Lambda$ may be expressed as (2 3) Let $\mu_1, \quad, \mu_{n-r}$ be the eigenvalues of $G \in SP\mathbb{R}^{(n-r) \times (n-r)}$, $y_1, \quad, y_{n-r}$ the eigenvectors of G corresponding to the eigenvalues $\mu_1, \quad, \mu_{n-r}$

$$Y = (y_1, \quad, y_{n-r}), \quad \mu = \operatorname{diag}(\mu_1, \quad, \mu_{n-r}) \qquad (2\ 5)$$

Then $GY = Y\mu$

From $X^+ U_2^{(1)} = 0$, $U_2^{T(1)} U_2^{(1)} = I$ and (2.3), we have

$$AU_2^{(1)} Y = XAX^+ U_2^{(1)} Y + U_2^{(1)} GU_2^{T(1)} U_2^{(1)} Y = U_2^{(1)} GY = U_2^{(1)} Y\mu .$$

From $AX = X\Lambda$ and $A(U_2^{(1)} Y) = (U_2^{(1)} Y)\mu$, it then follows that $\lambda_1, ..., \lambda_t$ are the eigenvalues of A, their multiplicities are $r_1, ..., r_t$, respectively, and the corresponding eigenvectors belong to $R(X_1), ..., R(X_t)$, respectively, whereas the $n - r$ other eigenvalues of A are exactly the $n - r$ eigenvalues of G, and the corresponding eigenvectors are the corresponding columns of the matrix $U_2^{(1)} Y$.

Let $G \in SP\mathbb{R}_{[\alpha,\beta]}^{(n-r)\times(n-r)}$, i.e., $\alpha \leqslant \mu_i \leqslant \beta$ $(i = 1, ..., n-r)$, then $\lambda(A)\backslash\lambda(\Lambda) \subset [\alpha, \beta]$. Thus $\mathbb{S}_A$ can be expressed as (2.4). ■

It is easy to obtain the following corollary from Theorem 2.1.

COROLLARY 2.1 : *In Problem I, if* $X \in \mathbb{R}^{n\times m}$ *is a column orthogonal matrix and* $\Lambda \geqslant 0$, *then* $\mathbb{S}_A$ *is nonempty.*

THEOREM 2.2 : *If* $\mathbb{S}_A$ *is nonempty, then* $\mathbb{S}_A$ *is a closed convex set.*

Proof: It follows from Theorem 2.1 that $\mathbb{S}_A$ is closed. Let $c_1 \geqslant 0$, $c_2 \geqslant 0$, $c_1 + c_2 = 1$,

$$A_0 = XAX^+ \tag{2.6}$$

$$A_1 = A_0 + U_2^{(1)} G_1 U_2^{T(1)} \in \mathbb{S}_A$$

$$A_2 = A_0 + U_2^{(1)} G_2 U_2^{T(1)} \in \mathbb{S}_A$$

where $G_1, G_2 \in SP\mathbb{R}_{[\alpha,\beta]}^{(n-r)\times(n-r)}$.

Clearly, $c_1 G_1 + c_2 G_2 \in SP\mathbb{R}^{(n-r)\times(n-r)}$. By the bounds for the eigenvalues of the sum of two symmetric matrices (see, e.g., Wilkinson [17, p. 101]) we have

$$c_1 G_1 + c_2 G_2 \in SP\mathbb{R}_{[\alpha,\beta]}^{(n-r)\times(n-r)} .$$

Thus, $c_1 A_1 + c_2 A_2 \in \mathbb{S}_A$. ■

Since any matrix $B \in \mathbb{R}^{n\times n}$ may be uniquely expressed as the sum of the symmetric matrix B_1 and the skew-symmetric matrix B_2, i.e.,

$$\begin{cases} B = B_1 + B_2 \\ B_1 = (B + B^T)/2, \quad B_2 = (B - B^T)/2 . \end{cases} \tag{2.7}$$

We decompose the matrix B_1 using the spectral decomposition

$$B_1 = \sum_{i=1}^{n} \mu_i u_i u_i^T \tag{2.8}$$

where
$\mu_1 \geqslant \cdots \geqslant \mu_k > \beta \geqslant \mu_{k+1} \geqslant \cdots \geqslant \mu_{k+l} \geqslant \alpha > \mu_{k+l+1} \geqslant \cdots \geqslant \mu_n$,
u_i $(i = 1, \ldots, n)$ are the orthonormal eigenvectors of B_1.

Let

$$[B]_{\alpha,\beta} = \sum_{i=1}^{k} \beta u_i u_i^T + \sum_{i=k+1}^{k+l} \mu_i u_i u_i^T + \sum_{i=k+l+1}^{n} \alpha u_i u_i^T . \tag{2.9}$$

It is clear that $[B]_{\alpha,\beta}$ is uniquely determined by B, α and β, and $[B]_{\alpha,\beta} \in SP\mathbb{R}^{n\times n}_{[\alpha,\beta]}$.

It is easy to prove the following lemma from the definition of $[B]_{\alpha,\beta}$.

LEMMA 2.2 : (1) *If* $B \in SP\mathbb{R}^{n\times n}_{[\alpha,\beta]}$, *then* $[B]_{\alpha,\beta} = B$. (2) *If D is a diagonal matrix, then there exists a unique matrix* $\hat{D} \in SP\mathbb{R}^{n\times n}_{[\alpha,\beta]}$ *such that*

$$\|\hat{D} - D\|_F = \inf_{\forall B \in SP\mathbb{R}^{n\times n}_{[\alpha,\beta]}} \|B - D\|_F .$$

Moreover. $\hat{D} = [D]_{\alpha,\beta}$.

LEMMA 2.3 : *Suppose* $\tilde{B} \in \mathbb{R}^{n\times n}$, *then there is a unique matrix* $\hat{B} \in SP\mathbb{R}^{n\times n}_{[\alpha,\beta]}$ *such that*

$$\|\hat{B} - \tilde{B}\|_F = \inf_{\forall B \in SP\mathbb{R}^{n\times n}_{[\alpha,\beta]}} \|B - \tilde{B}\|_F . \tag{2.10}$$

Moreover,

$$\hat{B} = [\tilde{B}]_{\alpha,\beta} . \tag{2.11}$$

Proof: Let $B \in SP\mathbb{R}^{n\times n}_{[\alpha,\beta]}$. From the fact that $\|S + K\|_F^2 = \|S\|_F^2 + \|K\|_F^2$ if $S^T = S$ and $K^T = -K$, we have

$$\|B - \tilde{B}\|_F^2 = \|B - \tilde{B}_1\|_F^2 + \|\tilde{B}_2\|_F^2$$

where $\tilde{B}_1 = (\tilde{B} + \tilde{B}^T)/2$, $\tilde{B}_2 = (\tilde{B} - \tilde{B}^T)/2$, and so the problem reduces to that of approximating $\tilde{B}_1$ in $SP\mathbb{R}^{n\times n}_{[\alpha,\beta]}$. Let $\tilde{B}_1 = Q^T DQ$ $(Q \in O\mathbb{R}^{n\times n}$, $D = \operatorname{diag}(d_1, \ldots, d_n))$ be a spectral decomposition, and let $QBQ^T = F$. Then $F \in SP\mathbb{R}^{n\times n}_{[\alpha,\beta]}$ and $\|B - \tilde{B}_1\|_F = \|F - D\|_F$. From Lemma 2.2, it follows that there is a unique matrix $\hat{F} = [D]_{\alpha,\beta}$ such that $\|\hat{F} - D\|_F = \inf_{\forall F \in SP\mathbb{R}^{n\times n}_{[\alpha,\beta]}} \|F - D\|_F$. By the definition (2.9), $\hat{B} = Q^T \hat{F} Q = Q^T [D]_{\alpha,\beta} Q = [\tilde{B}_1]_{\alpha,\beta} = [\tilde{B}]_{\alpha,\beta}$. ■

If $[\alpha, \beta] = [0, +\infty)$, then $SP\mathbb{R}^{n\times n}_{[\alpha,\beta]} = SP\mathbb{R}^{n\times n}$. By the definition (2.9) and Lemma 2.3, we obtain the following corollary.

COROLLARY 2.2 ([19]) : *Suppose* $\tilde{B} \in \mathbb{R}^{n\times n}$, *then there is a unique matrix* $\hat{B} \in SP\mathbb{R}^{n\times n}$ *such that* $\|\hat{B} - \tilde{B}\|_F = \inf_{\forall B \in SP\mathbb{R}^{n\times n}} \|B - \tilde{B}\|_F$. *Moreover,* $\hat{B} = (\tilde{B}_1 + H)/2$, *where* $\tilde{B}_1 = (\tilde{B} + \tilde{B}^T)/2$, $\tilde{B}_1 = QH$ *is a polar decomposition* $(H \geq 0, Q \in O\mathbb{R}^{n\times n})$.

Now we prove the following theorem.

THEOREM 2.3 : *Suppose* $\tilde{A} \in \mathbb{R}^{n\times n}$, $X = (X_1, \ldots, X_t) \in \mathbb{R}^{n\times m}_r$, $X_i \in \mathbb{R}^{n\times m_i}$ $(i = 1, \ldots, t)$, $\Lambda = \mathrm{diag}\,(\lambda_1 I_{m_1}, \ldots, \lambda_t I_{m_t})$ $(\lambda_i \neq \lambda_j, i \neq j, i, j = 1, \ldots, t)$, *X and* Λ *satisfy the condition* (2.2), *and the* SVD *of X is* (2.1). *Then there is a unique matrix* $\hat{A} \in \mathbb{S}_A$, *satisfying* (1.2), *and* $\hat{A}$ *can be expressed as*

$$\hat{A} = X\Lambda X^+ + U_2^{(1)} [U_2^{T(1)} \tilde{A} U_2^{(1)}]_{\alpha,\beta} U_2^{T(1)}. \tag{2.12}$$

Proof : From Theorem 2.2 and the best approximation theorem (see, e.g., Aubin [18, p. 15]), it follows that there is a unique matrix $\hat{A} \in \mathbb{S}_A$ satisfying (1.2).

By Theorem 2.1, $A \in \mathbb{S}_A$ can be expressed as

$$A = X\Lambda X^+ + U_2^{(1)} G U_2^{T(1)} = X\Lambda X^+ + U_1 \begin{pmatrix} 0 & 0 \\ 0 & G \end{pmatrix} U_1^T \tag{2.13}$$

where $G \in SP\mathbb{R}^{(n-r)\times(n-r)}_{[\alpha,\beta]}$.

From $U_2^{T(1)} X = 0$, $X^+ U_2^{(1)} = 0$ and (2.13), we have

$$\begin{aligned} \|A - \tilde{A}\|_F^2 &= \left\| X\Lambda X^+ + U_1 \begin{pmatrix} 0 & 0 \\ 0 & G \end{pmatrix} U_1^T - \tilde{A} \right\|_F^2 \\ &= \|U_1^{T(1)} X\Lambda X^+ U_1^{(1)} - U_1^{T(1)} \tilde{A} U_1^{(1)}\|_F^2 \\ &\quad + \|U_1^{T(1)} \tilde{A} U_2^{(1)}\|_F^2 \\ &\quad + \|U_2^{T(1)} \tilde{A} U_1^{(1)}\|_F^2 + \|G - U_2^{T(1)} \tilde{A} U_2^{(1)}\|_F^2 . \end{aligned} \tag{2.14}$$

Thus, Problem I is reduced to find $\hat{G} \in SP\mathbb{R}^{(n-r)\times(n-r)}_{[\alpha,\beta]}$ such that

$$\| \hat{G} - U_2^{T(1)} \tilde{A} U_2^{(1)} \|_F = \inf_{\forall G \in SP\mathbb{R}^{(n-r)\times(n-r)}_{[\alpha,\beta]}} \| G - U_2^{T(1)} \tilde{A} U_2^{(1)} \|_F \quad (2.15)$$

It then follows from Lemma 2.3 that

$$\hat{G} = [U_2^{T(1)} \tilde{A} U_2^{(1)}]_{\alpha,\beta} \quad (2.16)$$

Substituting (2.16) in (2.13), we get (2.12) ■

If $[\alpha, \beta] = [0, +\infty)$ in Problem I, the following result is obtained immediately from Theorem 2.3 and Corollary 2.2

COROLLARY 2.3 *Suppose* $\tilde{A} \in \mathbb{R}^{n\times n}$, $X = (X_1, \ldots, X_t) \in \mathbb{R}^{n\times m}_r$, $X_i \in \mathbb{R}^{n\times m_i}$ $(i = 1, \ldots, t)$, $\Lambda = \operatorname{diag}(\lambda_1 I_{m_1}, \ldots, \lambda_t I_{m_t})$ $(\lambda_i \neq \lambda_j, i \neq j, i, j = 1, \ldots, t)$, X and Λ *satisfy the condition* (2.2), *and the* SVD *of* X *is* (2.1) *Then there exists a unique matrix* $\hat{A} \in \mathbb{S}_{A_0} = \{A \in SP\mathbb{R}^{n\times n} \mid AX = X\Lambda\}$ *such that* $\|\hat{A} - \tilde{A}\|_F = \inf_{\forall A \in \mathbb{S}_{A_0}} \|A - \tilde{A}\|_F$, *and* $\hat{A}$ *can be expressed as*

$$\hat{A} = X\Lambda X^+ + \frac{1}{2} U_2^{(1)} (A_1 + H_1) U_2^{T(1)} \quad (2.17)$$

where $A_1 = \frac{1}{2} U_2^{T(1)} (\tilde{A} + \tilde{A}^T) U_2^{(1)}$, $A_1 = Q_1 H_1$ *is a polar decomposition* $(H_1 \geq 0, Q_1 \in O\mathbb{R}^{(n-r)\times(n-r)})$

3. THE SOLUTION OF PROBLEM II

Let $M^{\frac{1}{2}}$ be the unique positive definite square root of the positive definite matrix M (see, e.g., Horn and Johnson [16, pp 405-406]),

$$Z = M^{\frac{1}{2}} X \quad (3.1)$$

$$E = M^{-\frac{1}{2}} K M^{-\frac{1}{2}} \quad (3.2)$$

Then the equation $KX = MX\Lambda$ is equivalent to

$$EZ = Z\Lambda \quad (3.3)$$

and

$$\lambda(K, M) = \lambda(E) \quad (3.4)$$

We decompose the matrix $Z \in \mathbb{R}^{n \times m}$ using SVD

$$Z = U_2 \begin{pmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{pmatrix} V_2^T \tag{3.5}$$

where $U_2 = (U_1^{(2)}, U_2^{(2)}) \in O\mathbb{R}^{n \times n}$, $V_2 = (V_1^{(2)}, V_2^{(2)}) \in O\mathbb{R}^{m \times m}$, $\Sigma_2 = \operatorname{diag}(\sigma_1^{(2)}, \dots, \sigma_r^{(2)})$, $\sigma_i^{(2)} > 0$ $(i = 1, \dots, r)$, $r = \operatorname{rank}(Z) = \operatorname{rank}(X)$, $U_1^{(2)} \in \mathbb{R}^{n \times r}$, $V_1^{(2)} \in \mathbb{R}^{m \times r}$.

From Theorem 2.1, we obtain the following result.

THEOREM 3.1 : *Suppose* $X = (X_1, \dots, X_t) \in \mathbb{R}_r^{n \times m}$, $X_i \in \mathbb{R}^{n \times m_i}$ $(i = 1, \dots, t)$, $\Lambda = \operatorname{diag}(\lambda_1 I_{m_1}, \dots, \lambda_t I_{m_t})$ $(\lambda_i \neq \lambda_j, i \neq j, i, j = 1, \dots, t)$, *then* $\mathbb{S}_K$ *is non-empty if and only if*

$$\lambda_i \geq 0, \quad (\lambda_i - \lambda_j) X_i^T M X_j = 0, \quad i, j = 1, \dots, t. \tag{3.6}$$

Moreover, when the above condition is satisfied, $\mathbb{S}_K$ may be expressed as

$$\mathbb{S}_K = \{ MX\Lambda(M^{\frac{1}{2}} X)^+ M^{\frac{1}{2}} + M^{\frac{1}{2}} U_2^{(2)} G U_2^{T(2)} M^{\frac{1}{2}} | G \in SP\mathbb{R}_{[\alpha, \beta]}^{(n-r) \times (n-r)} \}. \tag{3.7}$$

Similarly to Theorem 2.3, we have the following theorem.

THEOREM 3.2 : *Suppose* $\tilde{K} \in \mathbb{R}^{n \times n}$, $X = (X_1, \dots, X_t) \in \mathbb{R}_r^{n \times m}$, $X_i \in \mathbb{R}^{n \times m_i}$ $(i = 1, \dots, t)$, $\Lambda = \operatorname{diag}(\lambda_1 I_{m_1}, \dots, \lambda_t I_{m_t})$ $(\lambda_i \neq \lambda_j, i \neq j, i, j = 1, \dots, t)$, X, Λ *and* M *satisfy the condition* (3.6), *and the* SVD *of* $M^{\frac{1}{2}} X$ *is* (3.5). *Then there exists a unique matrix* $\hat{K} \in \mathbb{S}_K$ *satisfying* (1.4), *and* $\hat{K}$ *can be expressed as*

$$\hat{K} = MX\Lambda(M^{\frac{1}{2}} X)^+ M^{\frac{1}{2}} + M^{\frac{1}{2}} U_2^{(2)} [U_2^{T(2)} M^{-\frac{1}{2}} \tilde{K} M^{-\frac{1}{2}} U_2^{(2)}]_{\alpha, \beta} U_2^{T(2)} M^{\frac{1}{2}}. \tag{3.8}$$

Proof : By Theorem 3.1, $K \in \mathbb{S}_K$ may be expressed as

$$K = MX\Lambda(M^{\frac{1}{2}} X)^+ M^{\frac{1}{2}} + M^{\frac{1}{2}} U_2 \begin{pmatrix} 0 & 0 \\ 0 & G \end{pmatrix} U_2^T M^{\frac{1}{2}} \tag{3.9}$$

where $G \in SP\mathbb{R}_{[\alpha, \beta]}^{(n-r) \times (n-r)}$.

From $U_2^{T(2)} M^{\frac{1}{2}} X = 0$, $(M^{\frac{1}{2}} X)^+ U_2^{(2)} = 0$ and (3.9), we have

$$\| W(K - \tilde{K}) W\|_F^2 = \|M^{-\frac{1}{2}} K M^{-\frac{1}{2}} - M^{-\frac{1}{2}} \tilde{K} M^{-\frac{1}{2}}\|_F^2 = \| (M^{\frac{1}{2}} X) \Lambda (M^{\frac{1}{2}} X)^+$$

$$+ U_2 \begin{pmatrix} 0 & 0 \\ 0 & G \end{pmatrix} U_2^T - M^{-\frac{1}{2}} \tilde{K} M^{-\frac{1}{2}} \|_F^2$$

$$= \| U_1^{T(2)} (M^{\frac{1}{2}} X \Lambda (M^{\frac{1}{2}} X)^+ - M^{-\frac{1}{2}} \tilde{K} M^{-\frac{1}{2}}) U_1^{(2)} \|_F^2$$

$$+ \| U_1^{T(2)} M^{-\frac{1}{2}} \tilde{K} M^{-\frac{1}{2}} U_2^{(2)} \|_F^2$$

$$+ \| U_2^{T(2)} M^{-\frac{1}{2}} \tilde{K} M^{-\frac{1}{2}} U_1^{(2)} \|_F^2$$

$$+ \| G - U_2^{T(2)} M^{-\frac{1}{2}} \tilde{K} M^{-\frac{1}{2}} U_2^{(2)} \|_F^2 .$$

It then follows from Lemma 2.3 that there is a unique matrix $\hat{G} = [U_2^{T(2)} M^{-\frac{1}{2}} \tilde{K} M^{-\frac{1}{2}} U_2^{(2)}]_{\alpha,\beta}$ minimizing $\| G - U_2^{T(2)} M^{-\frac{1}{2}} \tilde{K} M^{-\frac{1}{2}} U_2^{(2)} \|_F$. Thus there exists a unique matrix $\hat{K} = MX\Lambda (M^{\frac{1}{2}} X)^+ M^{\frac{1}{2}} + M^{\frac{1}{2}} U_2^{(2)} [U_2^{T(2)} M^{-\frac{1}{2}} \tilde{K} M^{-\frac{1}{2}} U_2^{(2)}]_{\alpha,\beta} U_2^{T(2)} M^{\frac{1}{2}}$ minimizing $\| W(K - \tilde{K}) W\|_F$. ∎

If $[\alpha, \beta] = [0, +\infty)$ in Problem II, it is easy to obtain the following corollary from Theorem 3.2 and Corollary 2.2.

COROLLARY 3.1 : *Suppose* $\tilde{K} \in \mathbb{R}^{n\times n}$, $X = (X_1, \dots, X_t) \in \mathbb{R}_r^{n\times m}$, $X_i \in \mathbb{R}^{n\times m_i}$ $(i = 1, \dots, t)$, $\Lambda = \operatorname{diag}(\lambda_1 I_{m_1}, \dots, \lambda_t I_{m_t})$ $(\lambda_i \neq \lambda_j, i \neq j, i, j = 1, \dots, t)$, X, Λ *and* M *satisfy the condition* (3.6), *and the* SVD *of* $M^{\frac{1}{2}} X$ *is* (3.5). *Then there is a unique matrix* $\hat{K} \in \mathbb{S}_{K_0} = \{K \in SP\mathbb{R}^{n\times n} | KX = MX\Lambda\}$ *such that* $\|M^{-\frac{1}{2}} (\hat{K} - \tilde{K}) M^{-\frac{1}{2}}\|_F = \inf_{\forall K \in \mathbb{S}_{K_0}} \|M^{-\frac{1}{2}} (K - \tilde{K}) M^{-\frac{1}{2}}\|_F$, *and* $\hat{K}$ *can be expressed as*

$$\hat{K} = MX\Lambda (M^{\frac{1}{2}} X)^+ M^{\frac{1}{2}} + \frac{1}{2} M^{\frac{1}{2}} U_2^{(2)} (K_2 + H_2) U_2^{T(2)} M^{\frac{1}{2}} \quad (3.10)$$

where $K_2 = \frac{1}{2} U_2^{T(2)} M^{-\frac{1}{2}} (\tilde{K} + \tilde{K}^T) M^{-\frac{1}{2}} U_2^{(2)}$, $K_2 = Q_2 H_2$ *is a polar decomposition* $(H_2 \geqslant 0, Q_2 \in O\mathbb{R}^{(n-r)\times(n-r)})$.

4. NUMERICAL ALGORITHM AND EXAMPLES

According to the conclusions in Section 2 and 3, the numerical algorithm for solving Problem II is presented as follows.

ALGORITHM IEP :

1) Compute the square root $M^{\frac{1}{2}}$ of M and $Z = M^{\frac{1}{2}} X$.
2) Compute the SVD of Z as (3.5).
3) If X, Λ and M satisfy (3.6), go to 4) ; otherwise no solution exists, and stop.
4) Compute $(M^{\frac{1}{2}} X)^+$ and $K_1 = \frac{1}{2} U_2^{T(2)} M^{-\frac{1}{2}} (\bar{K} + \bar{K}^T) M^{-\frac{1}{2}} U_2^{(2)}$.
5) Make the spectral decomposition of K_1.
6) Compute $\hat{K}$ by (3.8).

Remark 4.1 : By letting $M = I$ in the above algorithm, we obtain immediately a numerical algorithm for solving Problem I.

Now we give two numerical examples.

Example 4.1 ([20]). In Problem I $n = 5$, $m = 2$, the accurate matrix A is as follows :

$$A = \begin{pmatrix} 10 & 1 & 2 & 3 & 4 \\ 1 & 9 & -1 & 2 & -3 \\ 2 & -1 & 7 & 3 & -5 \\ 3 & 2 & 3 & 12 & -1 \\ 4 & -3 & -5 & -1 & 15 \end{pmatrix}.$$

Let the two smallest eigenvalues of the matrix A and corresponding eigenvectors be $\Lambda \in \mathbb{R}^{2 \times 2}$ and $X \in \mathbb{R}^{5 \times 2}$, respectively, $\alpha = 9.365555$, $\beta = 19.175421$,

$$\tilde{A} = \begin{pmatrix} 10.5 & 0.9 & 1.8 & 3.2 & 3.9 \\ 1.2 & 8.8 & -1.3 & 1.7 & -3.2 \\ 2.3 & -1.1 & 7.5 & 2.6 & -4.8 \\ 2.9 & 2.4 & 3.5 & 11.6 & -1.2 \\ 4.2 & -2.7 & -5.3 & -1.1 & 15.8 \end{pmatrix}.$$

We obtain the solution $\hat{A}$ of Problem I using the above algorithm on the computer IBM-4341.

$$\hat{A} = \begin{pmatrix} 10.07599775 & 0.98021677 & 2.00383676 & 3.04770252 & 4.08744433 \\ 0.98021677 & 8.98430476 & -1.13808698 & 1.87516977 & -2.82244908 \\ 2.00383676 & -1.13808698 & 7.12765001 & 3.03148332 & -5.07523492 \\ 3.04770252 & 1.87516977 & 3.03148332 & 11.99745382 & -0.90800744 \\ 4.08744433 & -2.82244908 & -5.07523492 & -0.90800744 & 15.07229038 \end{pmatrix}.$$

Let λ_i and $\hat{\lambda}_i$ $(i = 1, \dots, 5)$ be the eigenvalues arranged in increasing order of the matrices A and $\hat{A}$, respectively. We calculate that $\frac{\|\hat{A} - A\|_F}{\|A\|_F} = 1.6687 \times 10^{-2}$, $\frac{|\hat{\lambda}_i - \lambda_i|}{|\lambda_i|} = 0.00000000$ $(i = 1, 2)$, $\frac{|\hat{\lambda}_3 - \lambda_3|}{|\lambda_3|} = 1.6394 \times 10^{-2}$, $\frac{|\hat{\lambda}_4 - \lambda_4|}{|\lambda_4|} = 6.5885 \times 10^{-3}$, $\frac{|\hat{\lambda}_5 - \lambda_5|}{|\lambda_5|} = 5.2150 \times 10^{-8}$.

Example 4.2. In Problem II $n = 5$, $m = 3$, the accurate matrices K and M are as follows :

$$K = \begin{pmatrix} 10 & 2 & 3 & 1 & 1 \\ 2 & 12 & 1 & 2 & 1 \\ 3 & 1 & 11 & 1 & -1 \\ 1 & 2 & 1 & 9 & 1 \\ 1 & 1 & -1 & 1 & 15 \end{pmatrix}, \quad M = \begin{pmatrix} 12 & 1 & -1 & 2 & 1 \\ 1 & 14 & 1 & -1 & 1 \\ -1 & 1 & 16 & -1 & 1 \\ 2 & -1 & -1 & 12 & -1 \\ 1 & 1 & 1 & -1 & 11 \end{pmatrix}.$$

Let the three smallest eigenvalues of $Kx = \lambda Mx$ and corresponding eigenvectors be $\Lambda \in \mathbb{R}^{3 \times 3}$ and $X \in \mathbb{R}^{5 \times 3}$, respectively, $\alpha = 1$, $\beta = 1.5$,

$$\tilde{K} = \begin{pmatrix} 11 & 1 & 3 & 2 & 0 \\ 3 & 11 & 2 & 1 & 1 \\ 3 & 0 & 10 & 1 & -2 \\ 1 & 4 & 0 & 10 & 0 \\ 2 & 1 & -1 & 2 & 16 \end{pmatrix}.$$

Using Algorithm IEP, we get

$$\hat{K} = \begin{pmatrix} 10.08457957 & 2.08707884 & 3.08002574 & 1.05934994 & 0.99774933 \\ 2.08707884 & 12.09319760 & 1.08021530 & 2.06578516 & 1.01308064 \\ 3.08002574 & 1.08021530 & 11.07705150 & 1.05328254 & -1.01157568 \\ 1.05934994 & 2.06578516 & 1.05328254 & 9.04782732 & 1.01875114 \\ 0.99774933 & 1.01308064 & -1.01157568 & 1.01875114 & 15.06692894 \end{pmatrix}.$$

We calculate that $\frac{\|\hat{K} - K\|_F}{\|K\|_F} = 1.1317 \times 10^{-2}$, $\frac{|\hat{\lambda}_i - \lambda_i|}{|\lambda_i|} = 0.00000000$ $(i = 1, 2, 3)$, $\frac{|\hat{\lambda}_4 - \lambda_4|}{|\lambda_4|} = 1.8605 \times 10^{-2}$, $\frac{|\hat{\lambda}_5 - \lambda_5|}{|\lambda_5|} = 5.1240 \times 10^{-3}$, where λ_i and $\hat{\lambda}_i$ $(i = 1, \dots, 5)$ are the eigenvalues arranged in increasing order of $Kx = \lambda Mx$ and $\hat{K}x = \lambda Mx$, respectively.

ACKNOWLEDGEMENTS

I would like to thank Professor Françoise Chaitin-Chatelin for her careful reading of the manuscript and for helpful comments.

REFERENCES

[1] G. SRANG, 1980, Linear algebra and its applications, Academic Press, New York.

[2] K. J. BATHE and E. L. WILSON, 1976, *Numerical methods in finite element analysis,* Prentice-Hall, Englewood Cliffs, New Jersey.

[3] D. H. F. CHU, 1983, *Modal testing and modal refinement,* American Society of Mechanical Engineers, New York.

[4] A. BERMAN and W. G. FLANNELY, 1971, Theory of incomplete models of dynamic structures, *AIAA J.,* **9,** pp. 1481-1487.

[5] M. BARUCH and I. Y. BAR-ITZHACK, 1978, Optimal weighted orthogonalization of measured modes, *AIAA J.,* **16,** pp. 346-351.

[6] M. BARUCH, 1978, Optimization procedure to correct stiffness and flexibility matrices using vibration tests, *AIAA J.,* **16,** pp. 8-10.

[7] F. S. WEI, 1980, Stiffness matrix correction from incomplete test data, *AIAA J.,* **18,** pp. 1274-1275.

[8] M. BARUCH, 1982, Optimal correction of mass and stiffness matrices using measured modes, *AIAA J.,* **20,** pp. 1623-1626.

[9] A. BERMAN and E. J. NAGY, 1983, Improvement of a large analytical model using test data, *AIAA J.,* **21,** pp. 1168-1173.

[10] DAI HUA, 1988, Optimal correction of stiffness, flexibility and mass matrices using vibration tests, *J. of Vibration Engineering,* **1,** pp. 18-27.

[11] DAI HUA, 1994, Stiffness matrix correction using test data, Acta Aeronautica et Astronautica Sinica, **15,** pp. 1091-1094.

[12] ZHANG LEI, 1987, A kind of inverse problem of matrices and its numerical solution, *Mathematica Numerica Sinica,* **9,** pp. 431-437.

[13] ZHANG LEI, 1989, The solvability conditions for the inverse problem of symmetric nonnegative definite matrices, *Mathematica Numerica Sinica,* **11,** pp. 337-343.

[14] LIAO ANPING, 1990, A class of inverse problems of matrix equation $AX = B$ and its numerical solution, *Mathematica Numerica Sinica,* **12,** pp. 108-112.

[15] WANG JIASONG and CHANG XIAOWEN, 1992, The best approximation of symmetric positive semidefinite matrices with spectral constraints, *Numer. Math. — A J. of Chinese Universities,* **14,** pp. 78-86.

[16] R. A. HORN and C. R. JOHNSON, 1985, *Matrix analysis,* Cambridge University Press, New York.

[17] J. H. WILKINSON, 1965, *The algebraic eigenvalue problem,* Clarendon Press, Oxford.

[18] J. P. AUBIN, 1979, Applied functional analysis, John Wiley, New York.

[19] N. J. HIGHAM, 1988, Computing a nearest symmetric positive semi-definite matrix, *Linear Algebra Appl.,* **103,** pp. 103-118.

[20] J. H. WILKINSON and C. REINSCH, 1971, *Handbook for automatic computations,* vol. II, Linear Algebra, Springer-Verlag, New York.

[21] F. CHATELIN, 1993, *Eigenvalues of matrices,* Wiley, Chichester.