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A NEW FORMULATION FOR ARCH STRUCTURES. APPLICATION TO OPTIMIZATION PROBLEMS (*)

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Abstract — *To minimize costs, which depend on the displacement of a loaded arch, studied by following the Budiansky-Sanders's model, we use optimization algorithms The design variable is the shape ϕ of the arch The difficulty is to calculate the descent directions A method, used for example by Habbal and Moriano [6, 8], consists in approximating the exact derivative of the cost Here, the aim is to justify these calculation of the descent direction For that, we introduce a mixed formulation, equivalent to the state equation and the coefficients of which only depend on ϕ and on its first derivative, while the coefficients of the usual state equation depend on the third derivative ϕ''' of the shape of the arch By using this mixed formulation, we can compare these descent directions to the gradient of the approached cost*

Résumé — *Pour minimiser des coûts, qui dépendent régulièrement du déplacement d'une arche chargée, étudiée sous le modèle de Budiansky-Sanders, on utilise des algorithmes de descente La variable de conception est la forme ϕ de l'arche La difficulté ici est de calculer les directions de descente Une méthode, utilisée par Habbal et Moriano [6, 8], consiste à approcher la différentielle exacte du coût Le but ici est de justifier cette démarche L'idée est de comparer cette direction de descente avec le gradient du coût approché, dit gradient discret Pour cela, on introduit une formulation mixte, équivalente à l'équation d'état, et dont les coefficients dépendent seulement de ϕ et de sa dérivée première, alors que les coefficients de l'équation d'état sont fonction de la dérivée troisième de la forme ϕ de l'arche*

INTRODUCTION

We consider an elastic loaded arch, studied by following the Budiansky-Sanders's model. The coefficients of the usual state equation depend on the third derivative ϕ''' of the shape ϕ of the arch. We presently give a mixed formulation, the coefficients of which only depend on ϕ and on its first derivative. This mixed formulation is equivalent to the state equation.

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Then, like Bernadou-Ducatel [2], we approach the arch by beams, linked by rigid hinges. By correctly choosing the discrete mixed spaces, we can prove the equivalence between the discrete equation of Bernadou-Ducatel and the discrete mixed problem, which is conforming. Then, we show the convergence of the discrete displacement, already proved by Bernadou-Ducatel, with another method.

Now, we wish to numerically minimize costs, which regularly depend on the displacement of the arch, the design variable being the shape of the arch. We use descent algorithms. The difficulty is to calculate the descent direction. The idea is to approach the exact differential of the cost, which depends on the displacement and on an adjoint state, by using a finite element code. Thust, we obtain a descent direction, which is called discretized continuous gradient. We can hence use the finite element code as a black box, and avoid calculating the gradient of the rigidity matrix. But, the convergence of descent algorithms has been proved only if the descent direction is equal to the discrete gradient, which is, by definition, equal to the gradient of the approached cost. Numerically, we can observe that the discretized continuous gradient is not equal to the discrete gradient. So, in the general case, if the step h of the finite element method is too large, the optimization algorithm may give wrong results, if the descent direction is choosen equal to the discretized continuous gradient. By using the mixed formulation, we here show that the difference between the discretized continuous gradient and the discrete gradient converges to zero. So, we can here use the discretized continuous gradient in our optimization problem. Numerical results obtained by Habbal are correct.

1. THE CONTINUOUS PROBLEM

1.1. The state equation

The shape of the arch is given by a function ϕ belonging to the space :

$$\Lambda = \{\phi \in W^{3,\infty}(I), \quad \text{such that :} \quad \phi(0) = \phi(1) = 0\},$$

where $I =]0, 1[$. If l denotes the length of the arch, we define the midsurface of the arch by :

$$\omega = \{(x, y, z) \in \mathbb{R}^3, \quad x \in I, \quad z = \phi(x), \quad y \in]0, l[\}$$

and thus the arch Ω is given by :

$$\Omega = \left\{m + x_3 \vec{n}(m), m \in \omega, x_3 \in \left]-\frac{e}{2}, \frac{e}{2}\right[\right\},$$

where $\vec{n}(m)$ denotes the unit vector normal to ω and e denotes the thickness of the arch, which is assumed to be sufficiently small, compared to the curvature of ω, in order to apply the usual approximations of the Budiansky-Sanders's model. The loading of the arch is assumed to be invariant with respect to y, so that the displacement vector belongs to the (x, z) plane. The problem is then two dimensional. The local basis $(\vec{t}(m), \vec{n}(m))$, denoted also by $(\vec{t}(x), \vec{n}(x))$, is given by :

$$\vec{t}(x) = \frac{1}{S(\phi)(x)}\left\{\vec{i} + \phi'(x)\vec{j}\right\} \quad \text{and} \quad \vec{n}(x) = \frac{1}{S(\phi)(x)}\left\{-\phi'(x)\vec{i} + \vec{j}\right\},$$

where $S(\phi) = \sqrt{1 + \phi'^2}$, and $(\vec{i}, \vec{j})$ is the canonical basis of $\mathbb{R}^2$.

According to the Kirchhoff-Love hypothesis, the displacement vector of the arch can be calculated from the displacement field of the points belonging to the midsurface ω. The displacement of a point of ω is given by its pair of tangential and normal components $u(x) = (u_1(x), u_2(x))$ on the local basis $(\vec{t}(x), \vec{n}(x))$. The arch being embedded, the pair $u = (u_1, u_2)$ of components belongs to the space :

$$V = H_0^1(I) \times H_0^2(I).$$

From the virtual work principle, the displacement u^ϕ, which depends on the shape function ϕ, satisfies the *elliptic* state equation ([1], [4]) :

$$u^\phi \in V, \quad a(\phi ; u^\phi, v) = L(\phi ; v), \quad \text{for all} \quad v \in V, \tag{1}$$

where :

- the energy a of the arch is given by :

$$a(\phi ; u, v) = \int_0^1 \{C\,\varepsilon(\phi ; u)\,\varepsilon(\phi ; v) + D\kappa(\phi ; u)\,\kappa(\phi ; v)\}\, S(\phi)\, dx$$

with :

$$C = Ee \quad \text{and} \quad D = E\,\frac{e^3}{12} \quad \text{where } E \text{ is the Young modulus}$$

and the ε membrane energy and the κ bending energy are equal to :

$$\varepsilon(\phi ; v) = \frac{1}{S(\phi)}v_1' + \frac{1}{R(\phi)}v_2 \quad \text{and} \quad \kappa(\phi ; v) = \frac{1}{S(\phi)}\{\theta(\phi ; v)\}'$$

where the curvature $\dfrac{1}{R(\phi)}$ of ω and the rotation of the normal vector $\theta(\phi ; v)$ are defined by :

$$\frac{1}{R(\phi)} = -\frac{\phi''}{S(\phi)^3} \quad \text{and} \quad \theta(\phi ; v) = \frac{v_1}{R(\phi)} - \frac{v_2'}{S(\phi)}$$

• the virtual work of the external load $L(\phi\,;\,.\,)$ is a linear form on the space V, which is here choosen equal to the self weight of the arch :

$$L(\phi\,;v) = -\int_0^1 \rho e(\phi' v_1 + v_2)\,dx$$

where ρ denotes the density of the material.

In the state equation appears ϕ''' (because of the derivative of the curvature). We look for a variational formulation for the arch, with coefficients that depend only on ϕ and its first derivative.

1.2. The continuous mixed formulation

When the arch is approached by beams, we impose the continuity of the displacement *vector* and of the rotation of the normal vector at each node. The idea here is to choose the components (α, β) of the displacement vector on the fixed basis $(\vec{i}, \vec{j})$, and the rotation θ of the normal vector as the new variables. But, to find again a mixed finite element scheme equivalent to the finite element scheme of Bernadou-Ducatel, we have to introduce too the ε membrane energy as a new unknown. Finally, the new unknown is :

$$u_m = (\alpha, \beta, \theta, \varepsilon) \in V_m = H_0^1(I) \times H_0^1(I) \times H_0^1(I) \times L^2(I)\,.$$

The following lemma gives the relations between the four variables $(\alpha, \beta, \theta, \varepsilon)$.

LEMMA 1 : *Let ϕ be a function of the space $W^{3,\infty}(I)$.*

1) *Let $v = (v_1, v_2)$ be an element of the space V, then we have the equalities :*

$$\alpha' = \theta\phi' + \varepsilon \quad \textit{and} \quad \beta' = -\theta + \phi'\varepsilon\,, \tag{2}$$

in the space $\Sigma_m = L^2(I) \times L^2(I)$, where :

$$\alpha\vec{i} + \beta\vec{j} = v_1\,\vec{t}(\phi) + v_2\,\vec{n}(\phi)\,, \quad \theta = \theta(\phi\,;v)\,, \quad \varepsilon = \varepsilon(\phi\,;v)\,. \tag{3}$$

2) *We define the space :*

$$W(\phi) = \{v_m = (\alpha, \beta, \theta, \varepsilon) \in V_m, \textit{ such that} : b(\phi\,;\mu, v_m) = 0, \quad \textit{for all} \quad \mu \in \Sigma_m\}\,,$$

where the continuous bilinear form $b(\phi\,;.\,,.\,) : \Sigma_m \times V_m \to \mathbb{R}$ is given by :

$$b(\phi\,;\mu, v_m) = \int_0^1 \{\mu_1(\alpha' - \theta\phi' - \varepsilon) + \mu_2(\beta' + \theta - \phi'\varepsilon)\}\,dx\,. \tag{4}$$

Then the mapping : $G(\phi): V \to W(\phi): v = (v_1, v_2) \to v_m = (\alpha, \beta, \theta, \varepsilon)$, defined by the relations (3), is an isomorphism.

Proof : 1) By differentiating the equality $\vec{v} = v_1 \vec{t}(\phi) + v_2 \vec{n}(\phi)$, we directly obtain :

$$\begin{cases} \left(\vec{n}(\phi), \dfrac{d\vec{v}}{dx}\right) \mathbb{R}^2 = S(\phi)\left\{-\dfrac{v_1}{R(\phi)} + \dfrac{v_2'}{S(\phi)}\right\} = -S(\phi)\,\theta(\phi\,; v), \\ \left(\vec{t}(\phi), \dfrac{d\vec{v}}{dx}\right) \mathbb{R}^2 = S(\phi)\left\{-\dfrac{v_2}{R(\phi)} + \dfrac{v_1'}{S(\phi)}\right\} = -S(\phi)\,\varepsilon(\phi\,; v). \end{cases}$$

We deduce the relation (2) by using the equality $\vec{v} = \alpha\vec{i} + \beta\vec{j}$, where the basis $(\vec{i}, \vec{j})$ is fixed.

2) Let v_m be an element of the space $W(\phi)$. We define $v = (v_1, v_2)$ by the relation :

$$v_1\,\vec{t}(\phi) + v_2\,\vec{n}(\phi) = \alpha\vec{i} + \beta\vec{j}\,.$$

It is easy to verify the equalities :

$$v \in V, \quad \theta = \theta(\phi\,; v) \quad \text{and} \quad \varepsilon = \varepsilon(\phi\,; v)\,.$$

Then, $v_m = G(\phi)(v)$ and consequently, $G(\phi)$ obviously being an injection, we deduce that it is an isomorphism. □

By using the isomorphism $G(\phi)$, we deduce that the state equation (1) *is equivalent* to find the function $u_m = (\alpha, \beta, \varepsilon, \theta) \in W(\phi)$ such that :

$$c(\phi\,; u_m, v_m) = M(\phi\,; v_m)\,, \quad \text{for all} \quad v_m \in W(\phi) \tag{5}$$

where :

$$c(\phi\,; u_m, v_m) =$$

$$= \int_0^1 \left\{C\varepsilon\underline{\varepsilon} + D\theta'\,\underline{\theta}'\,\frac{1}{S(\phi)^2}\right\} S(\phi)\,dx\,, \quad \text{with} \quad v_m = (\underline{\alpha}, \underline{\beta}, \underline{\varepsilon}, \underline{\theta})$$

and :

$$M(\phi\,; v_m) = -\int_0^1 \rho e S(\phi)\,\underline{\beta}\,dx\,,$$

and we have the relations :

$$u_m = G(\phi)(u^\phi)\,.$$

To find the mixed formulation, we characterize u_m as the solution of the optimization problem :

$$\text{minimize}\,\frac{1}{2}\,c(\phi\,; v_m, v_m) - M(\phi\,; v_m)$$

under the constraint $v_m \in W(\phi)$, which is equivalent to :

$$b(\phi\ ;\ \mu,\ v_m) = 0\ , \quad \text{for all} \quad \mu \in \Sigma_m\ .$$

By writing the Euler's equation of the Lagrangian :

$$\mathcal{L}(\phi\ ;\ \mu,\ v_m) = \frac{1}{2} c(\phi\ ;\ v_m,\ v_m) - M(\phi\ ;\ v_m) + b(\phi\ ;\ \mu,\ v_m)\ ,$$

we obtain the mixed formulation :

$$\text{find } (u_m,\ \lambda) \in V_m \times \Sigma_m \quad \text{such that :}$$

$$\begin{cases} c(\phi\ ;\ u_m,\ v_m) + b(\phi\ ;\ \lambda,\ v_m) = M(\phi\ ;\ v_m), & \text{for all} \quad v_m \in V_m \\ b(\phi\ ;\ \mu,\ u_m) \qquad\qquad\qquad\quad\ = 0 & \text{for all} \quad \mu \in \Sigma_m\ . \end{cases} \tag{6}$$

Let us observe that the shape function ϕ and its first derivative only appears in this formulation.

We now prove that the state equation is equivalent to the mixed problem. As the state equation is equivalent to the equation (5), it is enough to show the equivalence between the equation (5) and the mixed problem. For that, we apply Brezzi's theorem [3]. We have then to verify that :

1) the continuous bilinear form $c(\phi\ ;\ .,\ .)$ is elliptic on the space $W(\phi)$,

2) the continuous bilinear form $b(\phi\ ;\ .\ ;\ .)$ satisfies the L.B.B. condition [3] :

$$\inf_{\substack{\mu \in \Sigma_m \\ \|\mu\|_{\Sigma_m} = 1}} \ \sup_{\substack{v_m \in V_m \\ \|v_m\|_{V_m} = 1}} \ b(\phi\ ;\ \mu,\ v_m) > 0\ .$$

PROPOSITION 1 : *Let ϕ be a function of the space :*

$$\Lambda_m = \{\phi \in W^{1,\infty}(I),\ \textit{such that: } \phi(0) = \phi(1) = 0\}\ .$$

The properties $1 - 2$ are satisfied. Moreover, let μ be an element of the space Σ_m, there exists :

$$v_m = (\alpha,\ \beta,\ \theta,\ \varepsilon) \in V_m$$

such that :

(i) *θ is P_1 on $[0,\ 0.5]$ and on $[0.5,\ 1]$, ε is constant on $[0,\ 1]$,*
(ii) *$\alpha' = \theta\phi' + \varepsilon + \mu_1$, $\beta' = -\theta + \phi'\varepsilon + \mu_2$,*
(iii) *$b(\phi\ ;\ \mu,\ v_m) \geqslant R(\|\phi'\|_{L^\infty})\|\mu\|_{\Sigma_m}\|v_m\|_{V_m}$,*
where R is a strictly positive rational fraction.

Proof : 1) Let $u_m = (\alpha, \beta, \varepsilon, \theta)$ be an element of the subspace $W(\phi)$ of the space V_m. We have to bound below the form :

$$c(\phi \,;\, u_m, u_m) = \int_0^1 \left\{ C\,\varepsilon^2 + D\,\theta'^2 \frac{1}{S(\phi)^2} \right\} S(\phi)\, dx\,.$$

By applying to θ to the Poincaré's inequality, we obtain the existence of a constant $F > 0$, that depends on the function ϕ, such that :

$$c(\phi \,;\, u_m, u_m) \geq F \int_0^1 \{\varepsilon^2 + \theta^2 + \theta'^2\}\, dx\,. \tag{7}$$

From the definition of the space $W(\phi)$ and from Poincaré's inequality, we deduce the existence of a constant F', which depends also on ϕ, such that :

$$\|\alpha\|_{H^1} \leq F' \{\|\theta\|_{L^2} + \|\varepsilon\|_{L^2}\} \quad \text{and} \quad \|\beta\|_{H^1} \leq F' \{\|\theta\|_{L^2} + \|\varepsilon\|_{L^2}\}\,. \tag{8}$$

Then, from inequality (7), we deduce the ellipticity of the form $c(\phi\,;.,.)$.

2) Let μ be an element of the space Σ_m. We define $v_m = (\alpha, \beta, \theta, \varepsilon)$ as follows :

$$\theta = \psi \int_0^1 \mu_2\, dx, \quad \varepsilon = -\int_0^1 \mu_1\, dx - \int_0^1 \theta\,\phi'\, dx\,,$$

$$\alpha(x) = \int_0^x (\theta\,\phi' + \varepsilon + \mu_1)\, dx, \quad \beta(x) = \int_0^x (-\,\theta + \phi'\,\varepsilon + \mu_2)\, dx\,,$$

with :

$$\psi(x) = 4\,x \quad \text{on} \quad [0, 0.5], \quad 4(1-x) \quad \text{on} \quad [0.5\,;1]\,.$$

Thus, the function ψ satisfies :

$$\psi(0) = \psi(1) = 0 \quad \text{and} \quad \int_0^1 \psi\, dx = 1\,.$$

From these definitions, we immediatly deduce (i)-(ii), and, after a brief calculation :

$$v_m \in V_m = H_0^1(I) \times H_0^1(I) \times H_0^1(I) \times L^2(I)\,.$$

Moreover, it follows from the definition of the mapping b and from the relation (ii), that :

$$b(\phi\,;\mu, v_m) = \|\mu\|^2_{L^2(I)}\,,$$

and we can easily verify the existence of a strictly positive polynomial function C, such that :

$$\|v_m\|_{V_m} \leq C\,(\|\phi'\|_{L^\infty})\|\mu\|_{L^2(I)},$$

by applying Poincaré's inequality to the functions α and β. Finally, we obtain the inequality (iii), and so the L.B.B. condition is obvious. □

Remark 1 : If we choose :

$$V = H_0^1(I) \times \{H^2(I) \cap H_0^1(I)\} \text{ and } V_m = H_0^1(I) \times H_0^1(I) \times H^1(I) \times L^2(I),$$

the properties (1) — (2) are still verified [7]. □

Finally, we have proved the following theorem.

THEOREM 1 : *Let $\phi \in \Lambda$. Then the state equation* (1) *is equivalent to the mixed problem* (6), *and we have the following relation between the solution u^ϕ of the state equation and the mixed solution (u_m, λ) :*

$$\alpha\vec{i} + \beta\vec{j} = u_1\,\vec{t}(\phi) + u_2\,\vec{n}(\phi), \quad \theta = \theta(\phi\,;u), \quad \varepsilon = \varepsilon(\phi\,;u), \tag{9}$$

where

$$u_m = (\alpha, \beta, \theta, \varepsilon) \quad and \quad u^\phi = (u_1, u_2).$$

Remark 2 : • For elastic shells, Ph. Destuynder and M. Salaun [5] have obtained a quite complex mixed formulation, which also depends only on the shape of the shell and on its first differential.

• Let us notice that the Lagrange multipliers $\lambda = (\lambda_1, \lambda_2)$ can be calculated from the mixed displacement u_m. In particular, we have the relations :

$$\lambda_1' = 0 \quad \text{and} \quad \lambda_2' = e\rho S(\phi),$$

which are obtained from the first equation of the mixed problem, by choosing test functions $v_m = (\alpha, \beta, \theta, \varepsilon)$ such that $\theta = 0$ and $\varepsilon = 0$. □

We now discretize the state equation.

2. THE FINITE ELEMENT METHOD

2.1. The usual discrete equation

We choose to approximate the displacement by using the finite element scheme of Bernadou-Ducatel [2]. At first, let us introduce, for each step

h, a regular subdivision $(x_i)_{i=0,\ \dots,\ m+1}$ of the closed set $\bar{I} = [0, 1]$, and let us denote :

$$K_i = [x_i, x_{i+1}], \quad \text{for all} \quad i = 0, \dots, m .$$

The principle of this scheme is to approximate the arch by beams. Thus, we define the finite element space $\tilde{\Lambda}_h$ of functions $\tilde{\phi}_h$ such that :

- $\tilde{\phi}_{h|K_i}$ belongs to $P_1(K_i)$, for all $i = 0, \dots, m$
- $\tilde{\phi}_h$ is continuous on the closed set $\bar{I} = [0, 1]$
- $\tilde{\phi}_h(0) = \tilde{\phi}_h(1) = 0.$

Now, we have to define the finite element test space. The arch being approximated by beams, linked with rigid hinges, the finite element test space depends on the geometry. To be precise, with each function $\tilde{\phi}_h$ of the space $\tilde{\Lambda}_h$, we associate the discrete space :

$$\tilde{V}_h(\tilde{\phi}_h) = \left\{\tilde{v}_h \in \mathscr{V}_h,\ \tilde{v}_h \text{ satisfies compatibility conditions}\right\}$$

where the space $\mathscr{V}_h$ is the space of functions $\tilde{v}_h = (\tilde{v}_{h1}, \tilde{v}_{h2})$ such that :

- $\tilde{v}_{h1\,|K_i}$ belongs to $P_1(K_i)$, for all $i = 0, \dots, m,$
- $\tilde{v}_{h1}(0) = \tilde{v}_{h1}(1) = 0,$
- $\tilde{v}_{h2\,|K_i}$ belongs to $P_3(K_i)$, for all $i = 0, \dots, m,$
- $\tilde{v}_{h2}(0) = \tilde{v}_{h2}(1) = \tilde{v}'_{h2}(0) = \tilde{v}'_{h2}(1) = 0.$

The compatibility conditions require the continuity of the displacement *vector* and of the rotation of the normal vector, at each node :

$$\left\{\tilde{v}_{h1}\,\vec{t}_h + \tilde{v}_{h2}\,\vec{n}_h\right\}_{|K_{i-1}}(x_i) = \left\{\tilde{v}_{h1}\,\vec{t}_h + \tilde{v}_{h2}\,\vec{n}_h\right\}_{|K_i}(x_i)\,, \tag{10}$$

and

$$\left\{\frac{1}{S(\phi_h)}\,\tilde{v}'_{h2}\right\}_{|K_{i-1}}(x_i) = \left\{\frac{1}{S(\phi_h)}\,\tilde{v}'_{h_2}\right\}_{|K_i}(x_i)\,, \tag{11}$$

for all $i = 1, \dots, m,$
where $(\vec{t}_h = \vec{t}\,(\tilde{\phi}_h),\ \vec{n}_h = \vec{n}(\tilde{\phi}_h))$ denotes the local basis of the approximating arch.

Since :

$$\tilde{V}_h(\tilde{\phi}_h) \text{ is not included in } V \quad \text{and} \quad \tilde{\Lambda}_h \text{ is not included in } W^{3,\infty}(I)\,,$$

these approximations are non-conforming. So, we have to introduce :

• a new energy, which is equal to the sum of the energies of each beam :

$$a_h(\tilde{\phi}_h ; \tilde{u}_h, \tilde{v}_h) = \sum_{i=0}^{m} \int_{K_i} \left\{ C \frac{1}{S(\tilde{\phi}_h)^2} \tilde{u}'_{h1} \tilde{v}'_{h1} + D \frac{1}{S(\tilde{\phi}_h)^4} \tilde{u}''_{h2} \tilde{v}''_{h2} \right\} S(\tilde{\phi}_h)\, dx \tag{12}$$

• the new external work, which is equal to :

$$L(\tilde{\phi}_h ; \tilde{v}_h) = - \int_0^1 \rho e (\tilde{\phi}'_h \tilde{v}_{h1} + \tilde{v}_{h2})\, dx . \tag{13}$$

Finally, the discrete displacement satisfies the elliptic equation (2) :

$$\tilde{u}_h \in \tilde{V}_h(\tilde{\phi}_h) \quad a_h(\tilde{\phi}_h ; \tilde{u}_h, \tilde{v}_h) = L(\tilde{\phi}_h ; \tilde{v}_h), \quad \text{for all} \quad \tilde{v}_h \in \tilde{V}_h(\tilde{\phi}_h) . \tag{14}$$

To simplify the notations, we here did not mention the dependence of $\tilde{u}_h$ on $\tilde{\phi}_h$.

2.2. The convergence of the finite element scheme

Let us recall the method used in [2]. Let ϕ be a function belonging to an open set Φ of the space Λ, and let $\tilde{\phi}_h$ be its interpolated function on the space $\tilde{\Lambda}_h$. The finite element scheme being non conforming, how can we prove the convergence of the discrete displacement $\tilde{u}_h$? The idea of Bernadou-Ducatel is to define a function u_h of the space V, calculated from the function $\tilde{u}_h$ of the space $\tilde{V}_h(\tilde{\phi}_h)$. The scheme will be convergent because of the estimate : $\|u_h - u^{\phi}\| \to 0$, when $h \to 0$.

To define the function u_h, Bernadou and Ducatel introduce a bijection F_h from the space $\tilde{V}_h(\tilde{\phi}_h)$ into a subspace V_h of the space V. The subspace V_h is the space of the functions $v_h = (v_{h1}, v_{h2})$ such that :

— $v_{h1\,|K_i} \in P_1(K_i)$, for all $i = 0, \ldots, m$,
— $v_{h1}(0) = v_{h1}(1) = 0$,
— v_{h1} is continuous on the set $\bar{I}$,

and

— $v_{h2\,|K_i} \in P_3(K_i)$, for all $i = 0, \ldots, m$,
— $v_{h2}(0) = v_{h2}(1) = v'_{h2}(0) = v'_{h2}(1) = 0$,
— v_{h2} is C^1 on the set $\bar{I}$.

The bijection F_h is given by :

$$F_h(\tilde{v}_h) = v_h$$

where the function $v_h = (v_{h1}, v_{h2})$ is defined from the pair $\tilde{v}_h = (\tilde{v}_{h1}, \tilde{v}_{h2})$ as follows :

$$(v_{h1}\, \vec{t}\,(\phi) + v_{h2}\, \vec{n}(\phi))(x_i) = (\tilde{v}_{h1}\, \vec{t}_h + \tilde{v}_{h2}\, \vec{n}_h)(x_i) \tag{15}$$

$$\theta(\phi\ ;\ v_h)(x_i) = -\left(\frac{\tilde{v}'_{h2}}{S(\tilde{\phi}_h)}\right)(x_i)\,, \tag{16}$$

for all $i = 1, \ldots, m$.

From Bernadou-Ducatel's results [2], we can prove, under the assumptions :

- the function ϕ belongs to the space $W^{4,\infty}(I)$,
- the functions u^{ϕ} is regular, i.e. :

$$u^{\phi} \in H^2(I) \times H^3(I)\,,$$

that :

$$\left\| u_h - u^{\phi} \right\|_V = O(h)\,, \quad \text{when} \quad h \to 0\,,$$

where $u_h = F_h(\tilde{u}_h)$ and $\tilde{u}_h$ is the only solution of the elliptic equation (14).

2.3. The discretized mixed formulation

The aim is again to recover the finite element scheme of Bernadou-Ducatel, by discretizing the mixed problem. So, we don't derive here the « best » finite element method of the mixed problem.

Naturally, we here still approach the arch by beams. But now, observe that the approximation of the geometry is conforming (for the mixed formulation), because the space $\tilde{\Lambda}_h$ is included in the space $W^{1,\infty}(I)$.

The discrete test space is chosen to derive again the scheme of Bernadou-Ducatel. So, we define :

$$V_{mh} = V_{mh1} \times V_{mh1} \times V_{mh2} \times V_{mh3}$$

where :

- V_{mh1} is the space of functions α_h such that :

— $\alpha_{h\,|\,K_i} \in P_3(K_i)$, for all $i = 0, \ldots, m$

— $\alpha_h(0) = \alpha_h(1) = 1$

— α_h is continuous on the set I.

- V_{mh2} is the space of functions θ_h such that :

— $\theta_{h\,|\,K_i} \in P_2(K_i)$, for all $i = 0, \ldots, m$

— $\theta_h(0) = \theta_h(1) = 1$

— θ_h is continuous on the set I.

- V_{mh3} is the space of functions ε_h such that :

— $\varepsilon_{h|K_i} \in P_0(K_i)$, for all $i = 0, ..., m$,

and :

- $\Sigma_{mh} = \Sigma_{mh1} \times \Sigma_{mh1}$ where the space Σ_{mh1} is space of functions μ_h such that :

— $\mu_{h|K_i} \in P_2(K_i)$, for all $i = 0, ..., m$.

Let us recall that we choose these discrete test spaces in order to derive the finite element scheme of Bernadou-Ducatel, but other choices can be more interesting.

Then these approximations are conforming because the spaces V_{mh} and Σ_{mh} are respectively included in the spaces V_m and Σ_m. So, the discrete mixed problem is :

$$\text{find } (u_{mh}, \lambda_h) \in V_{mh} \times \Sigma_{mh} \quad \text{such that :}$$

$$\begin{cases} c(\tilde{\phi}_h ; u_{mh}, v_{mh}) + b(\tilde{\phi}_h ; \lambda_h, v_{mh}) = M(\tilde{\phi}_h ; v_{mh}) & \text{for all } v_{mh} \in V_{mh} \\ b(\tilde{\phi}_h ; \mu_h, u_{mh}) = 0 & \text{for all } \mu_h \in \Sigma_{mh} . \end{cases} \tag{17}$$

To prove the existence of one and only one solution of this discrete mixed problem, and the convergence of the discrete mixed solution to the mixed solution, we apply Brezzi's theorem. From proposition 1 and from the definition of the mapping b, we can easily verify the following lemma.

LEMMA 2. *Let $\tilde{\phi}_h$ be a function of the space $\tilde{\Lambda}_h$.*

1) *The space :*

$$W_h(\tilde{\phi}_h) = \{ v_{mh} \in V_{mh} \text{ such that : } b(\tilde{\phi}_h ; \mu_h, v_{mh}) = 0, \quad \text{for all} \quad \mu_h \in \Sigma_{mh} \} \tag{18}$$

is the space of functions $v_{mh} = (\alpha_h, \beta_h, \theta_h, \varepsilon_h)$ such that :

$$\alpha_h' = \theta_h \tilde{\phi}_h' + \varepsilon_h \quad \text{and} \quad \beta_h' = - \theta_h + \tilde{\phi}_h' \varepsilon_h . \tag{19}$$

Consequently, the space $W_h(\tilde{\phi}_h)$ is included in the space $W(\tilde{\phi}_h)$.

2) *The bilinear form $c(\tilde{\phi}_h ; ., .)$ is uniformly elliptic on the space $W_h(\tilde{\phi}_h)$.*

3) *The bilinear form $b(\tilde{\phi}_h ; ., .)$ satisfies the L.B.B. condition :*

$$\inf_{\substack{\mu_h \in \Sigma_{mh} \\ \|\mu_h\|_{\Sigma_{mh}} = 1}} \sup_{\substack{v_{mh} \in V_{mh} \\ \|v_{mh}\|_{V_{mh}} = 1}} b(\tilde{\phi}_h ; \mu_h, v_{mh}) \geq R\left(\|\tilde{\phi}_h'\|_{L^\infty} \right) .$$

where R is a strictly positive rational fraction.

Finally, from Brezzi's theorem and some calculation, we can now prove the following result.

THEOREM 2 : *Let* ϕ *be a function of the space* $\Lambda_m \cap W^{2,\infty}(I)$ *and let* $\tilde{\phi}_h$ *be its interpolated function on* $\tilde{\Lambda}_h$.

- *The mixed discrete problem* (17) *has one and only one solution* (u_{mh}, λ_h).
- *If we suppose that the mixed solution* (u_m, λ) *of the system* (6) *satisfies :*

$$u_m = (\alpha, \beta, \theta, \varepsilon) = \{H^2(I)\}^3 \times H^1(I), \quad \lambda \in \{H^1(I)\}^2$$

then :

$$\|u_m - u_{mh}\|_{V_m} + \|\lambda - \lambda_h\|_{\Sigma_m} = O(h).$$

- *The discrete mixed problem* (17) *is equivalent to find* $u_{mh} \in W_h(\tilde{\phi}_h)$ *such that :*

$$c(\tilde{\phi}_h ; u_{mh}, v_{mh}) = M(\tilde{\phi}_h ; v_{mh}), \quad \textit{for all} \quad v_{mh} \in W_h(\tilde{\phi}_h). \tag{20}$$

Let us notice that λ belongs to the space $\{H^1(I)\}^2$, from remark 2.

Now, we can prove the equivalence between the discrete mixed problem (17) and the finite element scheme of Bernadou-Ducatel. Thus, from theorem 2, we shall deduce the convergence of the discrete solution $\tilde{u}_h$ of the discrete equation of Bernadou-Ducatel to the solution u^{ϕ}.

2.4. Equivalence between the discrete mixed problem and the finite element scheme of Bernadou-Ducatel

From theorem 2, the discrete mixed problem is equivalent to equation (20). So, we have only to prove the equivalence between this equation and the discrete equation (14).

PROPOSITION 2 : *Let* $\tilde{\phi}_h$ *be an element of the space* $\tilde{\Lambda}_h$. *We define the mapping :*

$$G_h : \tilde{v}_h = (\tilde{v}_{h1}, \tilde{v}_{h2}) \in \tilde{V}_h(\tilde{\phi}_h) \to v_{mh} = (\alpha_h, \beta_h, \theta_h, \varepsilon_h) \in V_{mh}$$

by :

$$\alpha_h \vec{i} + \beta_h \vec{j} = \tilde{v}_{h1} \vec{t}(\tilde{\phi}_h) + \tilde{v}_{h2} \vec{n}(\tilde{\phi}_h), \tag{21}$$

$$\theta_h = -\frac{1}{S(\tilde{\phi}_h)} \tilde{v}'_{h2}, \tag{22}$$

$$\varepsilon_h = \frac{1}{S(\tilde{\phi}_h)} \tilde{v}'_{h1}. \tag{23}$$

The mapping G_h is an isomorphism from the space $\tilde{V}_h(\tilde{\phi}_h)$ onto the space $W_h(\tilde{\phi}_h)$.

Proof : At the first, let us verify that :

$$v_{mh} = G_h(\tilde{v}_h) \in V_{mh}\,, \quad \text{for all} \quad \tilde{v}_h \in \tilde{V}_h(\tilde{\phi}_h)\,.$$

From the definition of the space $\tilde{V}_h(\tilde{\phi}_h)$, the functions $\tilde{v}_{h1}$ and $\tilde{v}_{h2}$ respectively are P_1 and P_3 on each closed set K_ι. So, from the definition of G_h, the functions α_h, β_h are P_3, while θ_h and ε_h respectively are P_2 and P_0, on each closed set K_ι. The boundary conditions being satisfied, we have only to verify that the functions α_h, β_h and θ_h are continuous on the set I. Or, conditions (10)-(11), which are satisfied for all $\tilde{v}_h$ of the space $\tilde{V}_h(\tilde{\phi}_h)$, ensure the continuity of the displacement vector and of the rotation of the normal vector. Then we deduce, from relations (21)-(22), the continuity of the three functions α_h, β_h and θ_h at each node, and then on I.

So, the mapping G_h is well defined. It is obvious that G_h is an injection. Thus, to prove that G_h is an isomorphism from the space $\tilde{V}_h(\tilde{\phi}_h)$ onto the space $W_h(\tilde{\phi}_h)$, we have just to verify the equality :

$$G_h(\tilde{V}_h(\tilde{\phi}_h)) = W_h(\tilde{\phi}_h)\,.$$

At first, we prove that the space $G_h(\tilde{V}_h(\tilde{\phi}_h))$ is included in the space $W_h(\tilde{\phi}_h)$. Let $\tilde{v}_h = (\tilde{v}_{h1}, \tilde{v}_{h2})$ be an element of the space $\tilde{V}_h(\tilde{\phi}_h)$, and let us denote :

$$v_{mh} = (\alpha_h, \beta_h, \theta_h, \varepsilon_h) = G_h(\tilde{v}_h)\,.$$

From lemma 2, to prove that v_{mh} belongs to the space $W_h(\tilde{\phi}_h)$, we have only to verify the equality :

$$\alpha_h' \vec{i} + \beta_h' \vec{j} = S(\tilde{\phi}_h)\left\{\varepsilon_h \vec{t}(\tilde{\phi}_h) - \theta_h \vec{n}(\tilde{\phi}_h)\right\}\,. \tag{24}$$

For that, we differentiate equality (21) and we thus obtain, the function $\tilde{\phi}_h$ being P_1 on each closed set K_ι :

$$\alpha_h' \vec{i} + \beta_h' \vec{j} = \tilde{v}_{h1}' \vec{t}(\tilde{\phi}_h) + \tilde{v}_{h2}' \vec{n}(\tilde{\phi}_h)\,.$$

We deduce equality (24) by using relations (22)-(23). We have thus proved the inclusion :

$$G_h(\tilde{V}_h(\tilde{\phi}_h)) \subset W_h(\tilde{\phi}_h)\,.$$

We now verify the second inclusion :

$$W_h(\tilde{\phi}_h) \subset G_h(\tilde{V}_h(\tilde{\phi}_h)) .$$

Let $v_{mh} = (\alpha_h, \beta_h, \theta_h, \varepsilon_h)$ be an element of the space $W_h(\tilde{\phi}_h)$. We define then $\tilde{v}_h = (\tilde{v}_{h1}, \tilde{v}_{h2})$ by the equality :

$$\alpha_h \vec{i} + \beta_h \vec{j} = \tilde{v}_{h1} \vec{t}(\tilde{\phi}_h) + \tilde{v}_{h2} \vec{n}(\tilde{\phi}_h) . \tag{25}$$

We have then to prove that :

$$\tilde{v}_h \in \tilde{V}_h(\tilde{\phi}_h) ,$$

$$\theta_h = -\frac{1}{S(\tilde{\phi}_h)} \tilde{v}'_{h2} , \tag{26}$$

$$\varepsilon_h = \frac{1}{S(\tilde{\phi}_h)} \tilde{v}'_{h1} , \tag{27}$$

to have the equality :

$$G_h(\tilde{v}_h) = v_{mh} .$$

At first, formulas (26)-(27) are immediately obtained by differentiating the equality :

$$\alpha_h \vec{i} + \beta_h \vec{j} = \tilde{v}_{h1} \vec{t}(\tilde{\phi}_h) + \tilde{v}_{h2} \vec{n}(\tilde{\phi}_h) ,$$

and by using lemma 2, which gives the relation :

$$\alpha'_h \vec{i} + \beta'_h \vec{j} = S(\tilde{\phi}_h)\left\{\varepsilon_h \vec{t}(\tilde{\phi}_h) - \theta_h \vec{n}(\tilde{\phi}_h)\right\} .$$

Let us verify that $\tilde{v}_h$ belongs to the space $\tilde{V}_h(\tilde{\phi}_h)$. From the equalities :

$$\varepsilon_h = \frac{1}{S(\tilde{\phi}_h)} \tilde{v}'_{h1} \quad \text{and} \quad \alpha_h \vec{i} + \beta_h \vec{j} = \tilde{v}_{h1} t(\tilde{\phi}_h) + \tilde{v}_{h2} n(\tilde{\phi}_h) ,$$

we deduce that the function $\tilde{v}_{h1}$ is P_1 on each set K_i and that the function $\tilde{v}_{h2}$ is P_3 on each set K_i. On the other hand the functions α_h, β_h and θ_h being continuous on the set I, we deduce, from equalities (25)-(26), the continuity of the displacement vector and of the rotation of the normal vector at each node. Consequently, the function $\tilde{v}_h$ belongs to the space $\tilde{V}_h(\tilde{\phi}_h)$.

Finally, we have proved the desired inclusion, and so, the equality :

$$G_h(\tilde{V}_h(\tilde{\phi}_h)) = W_h(\tilde{\phi}_h)\,.$$

Thus, the mapping G_h is an isomorphism from the space $\tilde{V}_h(\tilde{\phi}_h)$ onto the space $W_h(\tilde{\phi}_h)$. □

Now, we can prove the equivalence between the finite element scheme of Bernadou-Ducatel and the discrete mixed formulation.

PROPOSITION 3 : *Let $\tilde{\phi}_h$ be a function of the space $\tilde{A}_h$.*

The discrete mixed problem (17) *is equivalent to the discrete equation* (14), *and we have the relation :*

$$u_{mh} = G_h(\tilde{u}_h)\,.$$

Proof : From Brezzi's theorem [3], we already know that the discrete mixed problem is equivalent to equation (20) :

$$u_{mh} \in W_h(\tilde{\phi}_h)\,, \quad c(\tilde{\phi}_h\,;u_{mh}, v_{mh}) = M(\tilde{\phi}_h\,;v_{mh})\,, \quad \text{for all} \quad v_{mh} \in W_h(\tilde{\phi}_h)\,.$$

Then, we have to prove the equivalence between this equation and the discrete equation :

$$\tilde{u}_h \in \tilde{V}_h(\tilde{\phi}_h)\,, \quad a_h(\tilde{\phi}_h\,;\tilde{u}_h, \tilde{v}_h) = L(\tilde{\phi}_h\,;\tilde{v}_h) \quad \text{for all} \quad \tilde{v}_h \in \tilde{V}_h(\tilde{\phi}_h)\,. \tag{14}$$

The mapping G_h being an isomorphism from the space $\tilde{V}_h(\tilde{\phi}_h)$ onto the space $W_h(\tilde{\phi}_h)$, it suffices to verify the equalities :

$$a_h(\tilde{\phi}_h\,;\tilde{v}_h, \tilde{w}_h) = c(\tilde{\phi}_h\,;v_{mh}, w_{mh}) \quad \text{and} \quad L(\tilde{\phi}_h\,;\tilde{v}_h) = M(\tilde{\phi}\,;v_{mh})\,, \tag{28}$$

where :

$$v_{mh} = G_h(\tilde{v}_h) \quad \text{and} \quad w_{mh} = G_h(\tilde{w}_h)\,,$$

for all functions $\tilde{v}_h$ and $\tilde{w}_h$ belonging to the space $\tilde{V}_h(\tilde{\phi}_h)$.

Let $\tilde{v}_h$ and $\tilde{w}_h$ be two functions belonging to the space $\tilde{V}_h(\tilde{\phi}_h)$, and let be $v_{mh} = G_h(\tilde{v}_h)$, $W_h = G_h(\tilde{w}_h)$.

From the definitions of the mapping G_h, we directly obtain the equalities :

$$a_h(\tilde{\phi}_h\,;\tilde{v}_h, \tilde{w}_h) = \sum_{i=0}^{m} \int_{K_i} \left\{ C\,\varepsilon_h\,\underline{\varepsilon}_h + D\,\frac{1}{S(\tilde{\phi}_h)^2}\,\theta_h'\,\underline{\theta}_h' \right\} S\,(\tilde{\phi}_h)\,dx$$

and :

$$L_h(\tilde{\phi}_h\,;\tilde{v}_h) = -\int_0^1 \rho\,e\beta_h\,S(\tilde{\phi}_h)\,dx\,,$$

where we have denoted :

$$v_{mh} = (\alpha_h, \beta_h, \theta_h, \varepsilon_h) \quad \text{and} \quad w_{mh} = (\underline{\alpha}_h, \underline{\beta}_h, \underline{\theta}_h, \underline{\varepsilon}_h).$$

Consequently, we deduce equalities (28). □

From theorem 1 ans proposition 3, which give the equivalence, on the one hand, between the continuous mixed problem (6) and state equation (1), and, on the other hand, between discrete mixed problem (17) and discrete equation (14), we find again the results of convergence of the discrete displacement of Bernadou-Ducatel :

THEOREM 3 : *Let ϕ be a function of the space Λ and let $\tilde{\phi}_h$ be its interpolated function on $\tilde{\Lambda}_h$. We suppose that the solution $u^\phi = (u_1, u_2)$ of state equation* (1) *satisfies :*

$$u^\phi \in H^2(I) \times H^3(I)$$

then the discrete displacement $\tilde{u}_h$, which is the solution of the discrete equation (14), *« converges » when the step $h \to 0$, as follows :*

- $\tilde{u}_{h1}\,\vec{t}\,(\tilde{\phi}_h) + \tilde{u}_{h2}\,\vec{n}(\tilde{\phi}_h) \to u_{h1}\,\vec{t}\,(\tilde{\phi}) + u_2\,\vec{n}(\phi)$ *in the space* $H^1(I)$.

$\theta(\tilde{\phi}_h, \tilde{u}_h) \to \theta(\phi\,;u^\phi)$ *in the space* $H^1(I)$.

- $\varepsilon(\tilde{\phi}_h, \tilde{u}_h) \to \varepsilon(\phi\,;u^\phi)$ *in the space* $L^2(I)$.

We now study an optimization problem.

3. THE OPTIMIZATION PROBLEM

3.1. The continuous optimization problem

The design variable is the shape ϕ of the arch, which belongs to an open set Φ of the space Λ. We want to minimize costs, which regularly depend on the displacement. We write these costs on the following way :

$$j(\phi) = J(\phi\,;u^\phi) \tag{29}$$

where $J : \Phi \times V \to \mathbb{R}$ is a C^1 mapping. We choose, for example, to minimize the energy of the arch, and so, from the state equation, this cost is equal to :

$$j(\phi) = L(\phi, u^\phi).$$

Our purpose is to minimise numerically the cost $j(\phi)$, by using descent algorithms. At each step of the descent algorithm, we have to calculate the descent direction. The descent direction can be derived by two strategies. In one strategy, we first approach the cost, by discretizing the state equation, with a finite element scheme, and then, we calculate the gradient of the approached cost, called discrete gradient. In the other strategy, we calculate an approximation, called discretized continuous gradient, of the exact differential of the cost. The advantage of the second method is to use the finite element code only at the last step, while we have to differentiate the rigidity matrix to calculate the discrete gradient. Otherwise, the discrete gradient seems safer, because the convergence of optimization algorithms is then well known. Before describing the two strategies, we recall the following result, ([1], [9]) :

THEOREM 4 : *We suppose that the mapping : $\phi \to L(\phi\,;.) : \Phi \to V'$ is differentiable, where V' is the dual of the space V. Then the mappings :*

$$\phi \to u^{\phi} : (\Phi \subset W^{3,\infty}(I)) \to V \quad \textit{and} \quad \phi \to j(\phi) : (\Phi \subset W^{3,\infty}(I)) \to \mathbb{R}$$

are differentiable, and :

$$\frac{dj}{d\phi}(\phi)\cdot\psi = -\frac{\partial a}{\partial\phi}(\phi\,;u^{\phi},p^{\phi})\cdot\psi + \frac{\partial L}{\partial\phi}(\phi\,;p^{\phi}).\psi + \frac{\partial J}{\partial\phi}(\phi\,;u^{\phi}).\psi \tag{30}$$

where p^{ϕ}, which belongs to the space V, is the only solution of the adjoint equation :

$$a(\phi\,;p^{\phi},v) = \frac{\partial J}{\partial v}(\phi\,;u^{\phi}).v\,, \quad \textit{for all } v \textit{ of the space } V\,. \tag{31}$$

Let us notice that the adjoint state p^{ϕ} is equal to u^{ϕ}, when we minimize the energy of the arch.

3.2. The approached cost

To use the optimization algorithm, we have to approach the cost $j(\phi)$, which is, for example, equal to the virtual work L of the self weight of the arch. In this case, we naturally approach the function j by the function j_h defined on the space $\tilde{\Lambda}_h$ by :

$$j_h(\tilde{\phi}_h) = L_h(\tilde{\phi}_h\,;\tilde{u}_h)\,. \tag{32}$$

For another cost, we introduce a mapping $J_h : \tilde{\Lambda}_h \times \mathcal{V}_h \to \mathbb{R}$ « approaching » the map J, and we let :

$$j_h(\tilde{\phi}_h) = J_h(\tilde{\phi}_h\,;\tilde{u}_h)\,. \tag{33}$$

Usually, we write the discrete cost with the degrees of freedom of the geometry $\tilde{\phi}_h$ and of the displacement $\tilde{u}_h$. The vector of degrees of freedom of a function $\tilde{\phi}_h$ of the space $\tilde{\Lambda}_h$ is :

$$\Phi_d = (\tilde{\phi}_h(x_1),\ \tilde{\phi}_h(x_2),\ \dots,\ \tilde{\phi}_h(x_m))\,,$$

which belongs to the space $\mathbb{R}^m$.

The function $\tilde{\phi}_h$ being given, the displacement $\tilde{u}_h$, which satisfies the discrete equation (14), belongs to the space $\tilde{V}_h(\tilde{\phi}_h)$. Let us recall that this space is isomorphic to the subspace V_h of the test space V. Consequently, the dimension of the space $\tilde{V}_h(\tilde{\phi}_h)$ is independent of the geometry, and it is equal to $3\,m$. Thus, the discrete equation can be written as follows :

$$K_d(\Phi_d)\,U_d = L_d(\Phi_d)\,, \tag{34}$$

where $K_d(\Phi_d)$ is the rigidity matrix, U_d is the vector of degrees of freedom of the displacement $\tilde{u}_h$, which belongs to $\mathbb{R}^{3\,m}$, and $L_d(\Phi_d)$ is the vector of $\mathbb{R}^{3\,m}$, associated to the linear form $L_h(\tilde{\phi}_h\,;\,.)$. The calculation of the ridigity matrix and of the vector $L_d(\Phi_d)$, which depend on the vector Φ_d, are detailed in [6], [8].

Finally, we write the approached $j_h(\tilde{\phi}_h)$ on the following way :

$$j_h(\tilde{\phi}_h) = j_d(\Phi_d)\,, \tag{35}$$

where j_d is given by :

$$j_d(\Phi_d) = J_d(\Phi_d\,;\,U_d)\,,$$

the mapping J_d being defined by the relation :

$$J_d(\Phi_d\,;\,V_d) = J_h(\tilde{\phi}_h\,;\,\tilde{v}_h)\,,\quad \text{for all}\quad V_d \quad \text{of} \quad \mathbb{R}^{3\,m}\,,$$

where V_d is the vector of degrees of freedom of the displacement $\tilde{v}_h$.

The optimization algorithm, which can be used as a black box, allows to minimize the cost $j_d(\Phi_d)$. For that, at each step, a simulator requires the calculation of the cost and of the descent direction. We now give some details about the calculation of the descent direction.

3.3 The discrete gradient

The discrete gradient is equal to the derivative of the approximated cost $j_d(\Phi_d)$. To calculate it, we work like in the continuous case, (see theorem 4). We then obtain the following result :

PROPOSITION 4 *We suppose that the mapping* $\Phi_d \to L_d(\Phi_d,)$ $\mathbb{R}^m \to \mathbb{R}^{3m}$ *is differentiable and that the mapping* J_d $\mathbb{R}^m \times \mathbb{R}^{3m} \to \mathbb{R}$ *is* C^1

Then the mappings

$$\Phi_d \to U_d \quad \mathbb{R}^m \to \mathbb{R}^{3m} \quad \text{and} \quad \Phi_d \to j_d(\Phi_d) \quad \mathbb{R}^m \to \mathbb{R}$$

are differentiable, and, the components of the discrete gradient, on the canonical basis $(e_i)_{i=1\ \ m}$ *of the space* $\mathbb{R}^m$, *are given by*

$$\frac{dj_d}{d\Phi_d}(\Phi_d)\ e_i = -\left(\left(\frac{\partial K_d}{\partial \Phi_d}(\Phi_d)\cdot e_i\right) U_d, P_d\right)_{\mathbb{R}^{3m}} + \\ + \frac{\partial L_d}{\partial \Phi_d}(\Phi_d, P_d)\cdot e_i + \frac{\partial J_d}{\partial \Phi_d}(\Phi_d, U_d)\cdot e_i$$

where $P_d \in \mathbb{R}^{3m}$ *is the only solution of the adjoint equation*

$$(K_d(\Phi_d)\,P_d, V_d)_{\mathbb{R}^{3m}} = \frac{\partial J_d}{\partial V_d}(\Phi_d, U_d)\cdot V_d\,, \quad \text{for all} \quad V_d \in \mathbb{R}^{3m}, \tag{36}$$

and $(\,.\,,\,.\,)_{\mathbb{R}^{3m}}$ *is the scalar product into the space* $\mathbb{R}^{3m}$

It is not difficult to calculate this discrete gradient, but it requires to differentiate the rigidity matrix, and so, to know very well the finite element code To avoid such heavy calculation, we prefer here to use the discretized continuous gradient

3.4. The discretized continuous gradient

At each iteration, we have to calculate the descent direction at a vector Φ_d of the space $\mathbb{R}^m$, which is the vector of degrees of freedom of a function $\tilde{\phi}_h$ of the space $\tilde{\Lambda}_h$ The method consists in approaching the exact differentiall of the cost j, by using the finite element scheme as a black box

The approximation of the geometry being non conforming, we can't use the formula (30) with the function $\tilde{\phi}_h$ The idea of S Moriano [8] is to construct a function ϕ_h belonging to a subspace Λ_h of the space $W^{3,\infty}(I)$, from the data of Φ_d The subspace Λ_h is chosen equal to the splines of five degrees, satisfying the boundary conditions at the points $x = 0$ and $x = 1$ Thus, it is isomorphic to the spaces $\mathbb{R}^m$ and $\tilde{\Lambda}_h$ Consequently, at each vector Φ_d of $\mathbb{R}^m$, we associate

- a function ϕ_h of the space $\Lambda_h \subset \Lambda$,
- a function $\tilde{\phi}_h$ of the space $\tilde{\Lambda}_h$

The vector of degrees of freedom of these two functions is Φ_d

Finally, we will replace the function ϕ by the function ϕ_h in the formula (30). Let us now to detail the calculations of the direct state and of the adjoint state :

— At first, from the data of vector Φ_d, the finite element code provides the vector U_d. Then, by interpolation, we obtain the direct state $\tilde{u}_h$, which is the only solution of the discrete equation (14) and the function $u_h = F_h(\tilde{u}_h)$.

— Secondly, to obtain the adjoint state, we use the same finite element scheme, which give us the vector P_d of degrees of freedom of the function $\tilde{p}_h$, which is the only solution of the discrete adjoint equation :

$$\tilde{p}_h \in \tilde{V}_h(\tilde{\phi}_h), \quad a_h(\tilde{\phi}_h ; \tilde{p}_h, \tilde{v}_h) = \\ = \frac{\partial J_h}{\partial \tilde{v}_h}(\tilde{\phi}_h ; \tilde{u}_h) \cdot \tilde{v}_h, \quad \text{for all} \quad \tilde{v}_h \in \tilde{V}_h(\tilde{\phi}_h). \tag{37}$$

Then, we calculate the function $p_h = F_h(\tilde{p}_h)$.

All these calculation are detailed in [8].

Finally, the components of the discretized continuous gradient, calculated at the point Φ_d of $\mathbb{R}^m$ are given by :

$$D.C.G.(\Phi_d) \cdot e_\iota = \\ = -\frac{\partial a}{\partial \phi}(\phi_h ; u_h, p_h) \cdot S_{h\iota} + \frac{\partial L}{\partial \phi}(\phi_h ; p_h) \cdot S_{h\iota} + \frac{\partial J}{\partial \phi}(\phi_h ; u_h) \cdot S_{h\iota}, \tag{38}$$

where $S_{h\iota}$ is the spline of the space Λ_h, associated with the vector of degrees of freedom e_ι. We propose now to compare the discrete gradient to the discretized continuous gradient. From definition (38) of the discretized continuous gradient, and from the convergence of the finite element scheme (see theorem 3), it is easy to prove the following result :

PROPOSITION 5 : *Let be ϕ a function of the space Λ. We denote by :*

- *ϕ_h its interpolated spline function on the space Λ_h ;*
- *Φ_d the vector of degrees of freedom, associated with ϕ_h.*

We suppose that :

- *the functions ϕ belongs to the space $W^{4,\infty}(I)$*
- *the functions u^ϕ and p^ϕ are regular, i.e. :*

$$u^\phi, p^\phi \in H^2(I) \times H^3(I).$$

Then it exists two constants $D > 0$ and $h_0 > 0$ such that :

$$\left| D.C.G.(\Phi_d) \cdot e_\iota - \frac{dj}{d\phi}(\phi_h) \cdot S_{h\iota} \right| \leq Dh \, \|S_{h\iota}\|_{W^{3,\infty}(I)}, \ \textit{for all} \ h \in \,]0, h_0[\,.$$

Proof : The function ϕ being regular, we have the estimation :

$$\| \phi - \phi_h \|_{W^{3,\infty}(I)} = O(h) ,$$

hence, because of the expression of the energy $a(\phi ; ., .)$, which is V-elliptic :

$$\| u^{\phi} - u^{\phi_h} \|_V = O(h) ,$$

and, consequently, by using the Bernadou-Ducatel's results ([2]), we have :

$$\| u_h - u^{\phi} \|_V = O(h) \quad \text{and} \quad \| u_h - u^{\phi_h} \|_V = O(h) ,$$

and on the same way, we can then prove that :

$$\| p_h - p^{\phi} \|_V = O(h) \quad \text{and} \quad \| p_h - p^{\phi_h} \|_V = O(h) .$$

Let us bound below the partial derivatives of the energy, which appear in the expression of the difference between the « exact » differential and the discrete gradient, i.e. :

$$A_h = \frac{\partial a}{\partial \phi} (\phi_h ; u^{\phi_h}, p^{\phi_h}) \cdot S_{hi} - \frac{\partial a}{\partial \phi} (\phi_h ; u_h, p_h) \cdot S_{hi} .$$

We write A_h as follows :

$$A_h = \frac{\partial a}{\partial \phi} (\phi_h ; u^{\phi_h}, p^{\phi_h} - p_h) \cdot S_{hi} + \frac{\partial a}{\partial \phi} (\phi_h ; u^{\phi_h} - u_h, p_h) \cdot S_{hi} .$$

From the expression of the energy of the arch, we have the estimations :

- $$\left| \frac{\partial a}{\partial \phi} (\phi_h ; u^{\phi_h}, p^{\phi_h} - p_h) \cdot S_{hi} - \frac{\partial a}{\partial \phi} (\phi ; u^{\phi_h}, p^{\phi_h} - p_h) \cdot S_{hi} \right| \leqslant$$
$$\leqslant C \; \| \phi_h - \phi \|_{W^{3,\infty}(I)} \| u^{\phi_h} \|_V \; \| p^{\phi_h} - p_h \|_V ,$$

- $$\left| \frac{\partial a}{\partial \phi} (\phi_h ; u^{\phi_h} - u_h, p_h) \cdot S_{hi} - \frac{\partial a}{\partial \phi} (\phi ; u^{\phi_h} - u_h, p_h) \cdot S_{hi} \right| \leqslant$$
$$\leqslant C \; \| \phi_h - \phi \|_{W^{3,\infty}(I)} \| p_h \|_V \; \| u^{\phi_h} - u_h \|_V ,$$

- $$\left| \frac{\partial a}{\partial \phi} (\phi ; u^{\phi_h} - p^{\phi_h} - p_h) \cdot S_{hi} \right| \leqslant C \; \| \phi \|_{W^{3,\infty}(I)} \| u^{\phi_h} \|_V \; \| p^{\phi_h} - p_h \|_V ,$$

- $$\left| \frac{\partial a}{\partial \phi} (\phi ; u^{\phi_h} - u_h, p_h) \cdot S_{hi} \right| \leqslant C \; \| \phi \|_{W^{3,\infty}(I)} \| p_h \|_V \; \| u^{\phi_h} - u_h \|_V .$$

Finally, we deduce that :

$$A_h = O(h).$$

On the same way, we can estimate the other terms, which appear in the expression of the difference between the discretized continuous gradient and the « exact » differential. □

The difficulty then is to compare the discrete gradient to the « exact differential ». For that, we use the mixed formulations.

4. COMPARIZON BETWEEN THE DISCRETE GRADIENT AND THE DISCRETIZED CONTINUOUS GRADIENT

To compare the discrete gradient to the « exact gradient », the idea is to write the discrete gradient and the « exact gradient » with the « mixed » variables.

4.1. Another formula of the « exact » gradient

To simplify the notations, we suppose first that we minimize the energy of the arch. So, the cost j is equal to :

$$j(\phi) = L(\phi\ ; u^{\phi}).$$

From the relations (9) between the direct displacement $u^{\phi} = (u_1, u_2)$ and the « mixed » displacement $u_m = (\alpha, \beta\ ; \theta, \varepsilon)$:

$$\alpha \vec{i} + \beta \vec{j} = u_1 \vec{t}(\phi) + u_2 \vec{n}(\phi), \quad \theta = \theta(\phi\ ; u^{\phi}), \quad \varepsilon = \varepsilon(\phi\ ; u^{\phi}),$$

we can write the cost $j(\phi)$ with the mixed variable u_m as follows :

$$j(\phi) = J_m(\phi\ ; u_m),$$

where $(u_m, \lambda) \in V_m \times \Sigma_m$ is the only solution of the mixed problem and J_m here is equal to the linear form M.

The test spaces V_m and Σ_m being independent of the geometry, we can differentiate the mixed problem.

PROPOSITION 6 : *We suppose that the mapping :* $\phi \to J_m(\phi\ ; .) : W^{1,\infty} \times V_m \to \mathbb{R}$ *is* C^1 *and the mapping* $M : W^{1,\infty}(I) \times V_m \to \mathbb{R}$ *is differentiable.*

Then the mapping : $\phi \to j(\phi)$ *is differentiable, and :*

$$\frac{dj}{d\phi}(\phi)\cdot\psi = -\frac{\partial c}{\partial \phi}(\phi\ ; u_m, p_m)\cdot\psi - \frac{\partial b}{\partial \phi}(\phi\ ; \lambda, p_m)\cdot\psi - \\ -\frac{\partial b}{\partial \phi}(\phi\ ; \eta, u_m)\cdot\psi + \frac{\partial M}{\partial \phi}(\phi\ ; p_m)\cdot\psi + \frac{\partial J_m}{\partial \phi}(\phi\ ; u_m)\cdot\psi,$$

for all ϕ, $\psi \in W^{3,\infty}(I)$, *where the adjoint state* (p_m, η) *is the solution of the mixed problem*

$$(p_m, \eta) \in V_m \times \Sigma_m ,$$

$$\begin{cases} c(\phi\,; p_m, v_m) + b(\phi\,; \eta, v_m) = \dfrac{\partial J_m}{\partial v_m}(\phi\,; u_m)\cdot v_m , & \textit{for all} \quad v_m \in V_m \\ \qquad b(\phi\,; \mu, p_m) = 0 & \textit{for all} \quad \mu \in \Sigma_m . \end{cases}$$

Here, the adjoint state is equal to the direct state, but we shall not use this equality.

Proof We write the mixed problem (6) on the equivalent way :
Find

$$x_m = (u_m, \lambda) \in V_m \times \Sigma_m \text{ such that : } k(\phi\,, x_m, z_m) = \\ = n(\phi\,; z_m) \quad \text{for all} \quad z_m \in V_m \times \Sigma_m , \qquad (39)$$

where :

$$\begin{cases} k(\phi\,; y_m, z_m) = c(\phi, v_m, w_m) + b(\phi\,; \eta, w_m) + b(\phi\,; \iota, v_m) \\ \quad n(\phi\,, z_m) = M(\phi\,; w_m) , \end{cases}$$

for all functions $y_m = (v_m, \eta)$ and $z_m = (\omega_m, \iota)$ belonging to the space $V_m \times \Sigma_m$.

Thus, the cost j is equal to .

$$j(\phi) = J_m(\phi\,; u_m) ,$$

where the couple $x_m = (u_m, \lambda) \in V_m \times \Sigma_m$ satisfies the equation (39).

We can then apply the theorem 4, which implies :

$$\frac{dj}{d\phi}(\phi)\cdot\psi = -\frac{\partial k}{\partial\phi}(\phi\,, x_m, y_m)\cdot\psi + \frac{\partial n}{\partial\phi}(\phi\,, y_m)\cdot\psi + \frac{\partial J_m}{\partial\phi}(\phi\,; u_m)\cdot\psi ,$$

where $x_m = (u_m, \lambda)$ and the adjoint state $y_m = (p_m, \eta)$ is the solution of the equation ·

$$y_m \in V_m \times \Sigma,\ k(\phi\,, y_m, z_m) = \frac{\partial J_m}{\partial v_m}(\phi\,; u_m)\cdot v_m , \text{ for all } v_m \in V_m .$$

From the definitions of the forms k and n, we obtain the result. □

Consequently, for all vectors Φ_d of the space $\mathbb{R}^m$, we have obtained the equality :

$$\begin{aligned}\frac{dj}{d\phi}(\phi_h)\cdot S_{hi} = -\frac{\partial c}{\partial \phi}(\phi_h\,;u_m^h, p_m^h)\cdot S_{hi} - \\ -\frac{\partial b}{\partial \phi}(\phi_h\,;\lambda^h, p_m^h)\cdot S_{hi} - \frac{\partial b}{\partial \phi}(\phi_h\,;\eta^h, u_m^h)\cdot S_{hi} \\ +\frac{\partial M}{\partial \phi}(\phi_h\,;p_m^h)\cdot S_{hi} + \frac{\partial J_m}{\partial \phi}(\phi_h\,;u_m^h)\cdot S_{hi}\end{aligned} \tag{40}$$

where :

- ϕ_h is a function on the space Λ_h
- $(u_m^h, \lambda^h) \in V_m \times \Sigma_m$ is the only solution of the mixed problem :

$$\begin{cases} c(\phi_h\,;u_m^h, v_m) + b(\phi_h\,;\lambda^h, v_m) = M(\phi_h\,;u_m^h)\,.\,v_m, & \text{for all } v_m \in V_m \\ b(\phi_h\,;\mu, u_m^h) \qquad\qquad\qquad\qquad = 0 & \text{for all } \mu \in \Sigma_m\,. \end{cases} \tag{41}$$

- $(p_m^h, \eta^h) \in V_m \times \Sigma_m$ is the only solution of the mixed problem :

$$\begin{cases} c(\phi_h\,;p_m^h, v_m) + b(\phi_h\,;\eta^h, v_m) = \dfrac{\partial J_m}{\partial v_m}(\phi_h\,;u_m^h)\cdot v_m, & \text{for all } v_m \in V_m \\ b(\phi_h\,;\mu, p_m^h) \qquad\qquad\qquad\qquad = 0 & \text{for all } \mu \in \Sigma_m\,. \end{cases} \tag{42}$$

4.2. Another formula of the discrete gradient

To simplify the notations, we again minimize the energy of the arch. So, the cost j is equal to :

$$j(\phi) = L(\phi\,;u^\phi)$$

and it is approached by :

$$j_h(\tilde{\phi}_h) = L_h(\tilde{\phi}_h\,;\tilde{u}_h)\,.$$

To calculate the discrete gradient, we have introduced a cost : $j_d : \mathbb{R}^m \to \mathbb{R}$ which is a function of the vector of degrees of freedom. Here, we work like in the continuous case, by using the mixed formulation, in order to compare the discrete gradient to the discretized continuous gradient. Indeed, we have shown that the finite element method of Bernadou-Ducatel is equivalent to the discrete mixed formulation. To be precise, the displacement of the beams $\tilde{u}_h$ is given by :

$$\tilde{u}_h = G_h^{-1}(u_{mh})\,,$$

where u_{mh} is the « discrete mixed » displacement. Thus, the formula (28) :

$$L_h(\tilde{\phi}_h ; \tilde{v}_h) = M(\tilde{\phi} ; G_h(\tilde{v}_h)), \quad \text{for all} \quad \tilde{v}_h \in \tilde{V}_h(\tilde{\phi}_h),$$

implies :

$$j_h(\tilde{\phi}_h) = M(\tilde{\phi} ; u_{mh}),$$

where $(u_{mh}, \lambda_h) \in V_{mh} \times \Sigma_{mh}$ is the only solution of the mixed problem :

$$\begin{cases} c(\tilde{\phi}_h ; u_{mh}, v_{mh}) + b(\tilde{\phi}_h ; \lambda_h, v_{mh}) = M(\tilde{\phi}_h ; v_{mh}), & \text{for all } v_{mh} \in V_{mh} \\ b(\tilde{\phi}_h ; \mu_h, u_{mh}) = 0 & \text{for all } \mu_h \in \Sigma_{mh}. \end{cases}$$

Like in the mixed continuous problem (see proposition 6), and, by differentiating the equality (35) :

$$j_d(\Phi_d) = j_h(\tilde{\phi}_h),$$

we obtain :

$$\begin{aligned} \frac{dj_d}{d\Phi_d}(\Phi_d) \cdot e_i = &- \frac{\partial c}{\partial \phi}(\tilde{\phi}_h ; u_{mh}, p_{mh}) \cdot S_{hi} - \\ &- \frac{\partial b}{\partial \phi}(\tilde{\phi}_h ; \lambda_h, p_{mh}) \cdot S_{hi} - \frac{\partial b}{\partial \phi}(\tilde{\phi}_h ; \eta_h, u_{mh}) \cdot S_{hi} \\ &+ \frac{\partial M}{\partial \phi}(\tilde{\phi}_h ; p_{mh}) \cdot \tilde{S}_{hi} + \frac{\partial J_m}{\partial \phi}(\tilde{\phi}_h ; u_{mh}) \cdot \tilde{S}_{hi}, \end{aligned} \tag{43}$$

where the « discrete mixed » adjoint state $(p_{mh}, \eta_h) \in V_{mh} \times \Sigma_{mh}$ is the solution of the discrete mixed problem :

$$\begin{cases} c(\tilde{\phi}_h ; p_{mh}, v_{mh}) + b(\tilde{\phi}_h ; \eta_h, v_{mh}) = \dfrac{\partial J_m}{\partial v_m}(\tilde{\phi}_h ; u_{mh}) . v_{mh}, & \text{for all } v_{mh} \in V_{mh} \\ b(\tilde{\phi}_h ; \mu_h, p_{mh}) = 0 & \text{for all } \mu_m \in \Sigma_{mh}. \end{cases} \tag{44}$$

Now, from the formulas of the discrete gradient and of the « exact gradient », we can easily compare the discrete gradient to the discretized continuous gradient.

4.3. Comparizon between the discrete gradient and the discretized continuous gradient

By applying the method used to prove proposition 5, we obtain the following result :

PROPOSITION 7 : *Let be ϕ a function of the space A. We denote by :*

- ϕ_h *its interpolated spline function on the space* A_h
- $\tilde{\phi}_h$ *its interpolated spline function on the space* $\tilde{A}_h$
- Φ_d *the vector of degrees of freedom, associated with* ϕ_h *and* $\tilde{\phi}_h$.

We suppose that :

- *the function* ϕ *belongs to the space* $W^{4,\infty}(I)$
- *the functions* u^{ϕ} *and* p^{ϕ} *are regular, i.e. :*

$$u^{\phi}, p^{\phi} \in H^2(I) \times H^3(I) .$$

Then it exists two constants $C > 0$ *and* $h_0 > 0$ *such that :*

$$\left| \frac{dj_d}{d\Phi_d}(\Phi_d) \cdot e_\iota - D.C.G.(\Phi_d) \cdot e_\iota \right| \leqslant Ch \|S_{h\iota}\|_{W^{3,\infty}(I)}, \text{ for all } h \in]0, h_0[.$$

Proof : On the one hand, as the hypothesis of regularity on the function ϕ implies that :

$$\|\phi - \phi_h\|_{W^{3,\infty}(I)} = O(h)$$

we can prove that, because of the expression of the energy $a(\phi ; ., .)$, which is V-elliptic :

$$\|u^{\phi} - u^{\phi_h}\|_V = O(h) ,$$

and, on the same way for the mixed solutions :

$$\|u_m - u_m^h\|_{V_m} + \|\lambda - \lambda^h\|_{\Sigma_m} = O(h) .$$

On the other hand, the assumptions :

$$u^{\phi}, p^{\phi} \in H^2(I) \times H^3(I) ,$$

implies, thanks to the relations (9) :

$$u_m, p_m \in H^2(I) \times H^2(I) \times H^2(I) \times H^1(I) .$$

As, from the remark 2, λ belongs to the space $H^1(I) \times H^1(I)$, and on the same way, η too, we can apply theorem 2, and so :

$$\|u_{mh} - u_m\|_{V_m} + \|\lambda_h - \lambda\|_{\Sigma_m} = O(h) , \quad \text{when} \quad h \to 0 ,$$

and :

$$\|p_{mh} - p_m\|_{V_m} + \|\eta_h - \eta\|_{\Sigma_m} = O(h) , \quad \text{when} \quad h \to 0 .$$

Consequently, from the equalities (40)-(43) and from the estimation :

$$\left\| \phi - \tilde{\phi}_h \right\|_{W^{1,\infty}(I)} = O(h),$$

we deduce the convergence, by applying the way used in proposition 5 :

$$\left| \frac{dj_d}{d\Phi_d} (\Phi_d) \cdot e_i - \frac{dj}{d\phi} (\phi_h) \cdot S_{hi} \right| = O(h), \quad \text{when} \quad h \to 0 .$$

Thus, from the « convergence » of the discretized continuous gradient :

$$\left| D.C.G. (\Phi_d) \cdot e_i - \frac{dj}{d\phi} (\phi_h) \cdot S_{hi} \right| = O(h), \quad \text{when} \quad h \to 0 ,$$

we deduce the theorem. □

5. CONCLUSION

Consequently, we can use here the discretized continuous gradient in the descent algorithms, when the step h is sufficiently small. So, we avoid to differentiate the rigidity matrix, and we can use the finite element code as a black box. The numberical calculation, made by Habbal [6], who uses the discretized continuous gradient, gave satisfying results.

Let us notice that the mixed formulation given here has been used to compare the discrete gradient to the discretized continuous gradient. But, we can also use it to build new finite element schemes, which allow us to approach the displacement of the arch.

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