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SOME ALGORITHMS FOR DIFFERENTIAL GAMES WITH TWO PLAYERS AND ONE TARGET (*)

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Abstract. — *We provide algorithms finding victory domains of a differential game with two players and one target. For doing this we do not need to compute trajectories of the game. We present two kinds of algorithms. The algorithms in continuous case give the victory set as a limit of an suitable decreasing sequence of closed sets. The algorithms in discrete cases give discrete sets approximating the victory domain. In the two cases, we prove convergence results.*

Résumé. — *Il s'agit dans cet article de présenter des algorithmes permettant de déterminer un ensemble de victoire pour un jeu différentiel à deux joueurs et une cible. Ces nouvelles méthodes fournissent l'ensemble de victoire sans « calculer » de trajectoires. Deux types d'études sont exposés : une version continue où l'ensemble de victoire est la limite d'une suite décroissante de fermés et une version discrète où cet ensemble est approché par des ensembles discrets. Dans ces deux cas des résultats de convergences sont prouvés.*

1. INTRODUCTION

We consider the following dynamical system :

$$x'(t) = f(x(t), u(t), v(t)), \quad \text{where} \quad u(t) \in U \quad \text{and} \quad v(t) \in V. \quad (1)$$

Let Ω be an open target. Two players, Ursule and Victor control this dynamical system throught their respective controls u and v. Ursule wants the system to reach the target Ω. Victor wants the system to avoid Ω, namely he wants the state variable x to remain in $K := R^n \setminus \Omega$.

The main topic of this paper is to provide some algorithm computing discriminating victory domains used in differential games. We define

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Victor's Discriminating victory set as the set of initial condition $x_0 \in K$, such that there exists a strategy choosed by Victor such that, for any measurable control $u(\,.\,)$ choosed by Ursule, the associated trajectory stays forever in K.

We know ([7], [1] and [16] for time dependent dynamics) that this victory set, for some classes of causal strategies, is the largest closed subset A of K satisfying the following tangential (3) condition (4) :

$$\forall x \in A,\ \forall u \in U,\ \exists v \in V,\ f(x, u, v) \in T_A(x)\,. \tag{2}$$

We shall call this set the discriminating kernel of K and we denote it by $\mathrm{Disc}_f(K)$. We refer to [7] for the interpretation of $\mathrm{Disc}_f(K)$ in term of Victor's discriminating victory set. These two sets are equal for Krassovski-Subbotin positional strategies defined in [13] and for nonanticipative strategies used in [9] (see also [21] and [8]). The goal of this paper is not to explain this interpretation but, knowing it, the main aim of this paper is to compute the set of initial conditions such that Victor can win by discriminating victory.

We provide in this paper two kinds of algorithms computing victory sets.

The first kind of algorithms is similar to algorithms explained in [10] for viability kernels and in [18] for invariance kernels (5). This is the case when the dynamics only depend on one control.

The main idea of the algorithm is the following :

For a given closed set K and for a point $x \in K$, if the tangential condition is not fulfilled, it is possible to compute some $r(x) > 0$ such that the ball $B(x, r(x))$ does not meet $\mathrm{Disc}_f(K)$. We obtain a set K_1 by substracting to K all such open balls. This $r(x)$ depends only (6) on $\sup_u \inf_v \langle f(x, u, v), \nu \rangle$, on the Lipschitzian constant of f and on an upper bound of f in K. By iterating this construction, we can define recursively a decreasing sequence of closed sets K_n. We prove that this sequence converges to $\mathrm{Disc}_f(K)$.

The second kind of algorithm is related to discretization methods (*cf.* [20] and [19]) to compute the viability and invariance kernels of a closed set K.

(3) Let us recall the definition of the contingent (or Bouligand's) cone at x to A :

$$T_A(x) := \left\{ v \in \mathbb{R}^n \,\middle|\, \liminf_{h \to 0^+} d(x + hv, A)/h = 0 \right\}.$$

(4) We shall write further this tangential condition using proximal normals.

(5) The reader can refer to [3], for the definitions of viability and invariance kernels.

(6) Where ν is a proximal normal to K at x.

The main idea is the following : we associate with (1) a « discretized equation » of the following form :

$$\forall n \in N,\ x_{n+1} = g(x_n, u_n, v_n), \quad \text{where} \quad u_n \in U \quad \text{and} \quad v_n \in V. \qquad (3)$$

Then, we define suitable *discrete discriminating kernel* $\overrightarrow{\mathrm{Disc}}_g(K)$ of a closed set K. It is easy to prove that this victory set for Victor is the largest closed subset of K satisfying the following condition :

$$\forall x \in \overrightarrow{\mathrm{Disc}}_F(K),\ \forall u \in U,\ \exists v \in V \quad \text{such that} \quad g(x, u, v) \in \overrightarrow{\mathrm{Disc}}_F(K). \quad (4)$$

We use this condition to define a decreasing sequence of closed sets converging to the discrete discriminating kernel of a closed set K. An example of similar algorithm, for a discrete dynamic pursuit game, is provided in [14], by using different technics.

In the third part of our paper, we prove that discrete discriminating kernels approximate the discriminating kernels of (1), for suitable choice of g.

We want to underline that these methods differ sharply from classical methods used in differential games theory. Classical methods (*cf.* [4]) consist in constructing a barrier (i.e., the boundary of the discriminating or the leadership kernel) which is assumed to contain optimal solutions ([7]) and after this it is necessary to verify that the set defined is the victory set. In our methods, we do not need to find solutions of (1). The reader can see [14] and [15] for some other methods.

For doing this, we mainly use set-valued analysis and viability theory (*cf.* [2] and [3]).

2. DISCRIMINATING KERNELS FOR DIFFERENTIAL GAMES

This section is devoted to state basic results and assumptions concerning discriminating domains and kernels.

Since our purpose is to establish algorithms, we do not try to have the weakest assumptions, but assumptions such that our two algorithms converge. For more detailed study the reader is refered to [6].

2.1. Assumptions

We study absolutely continuous solutions to the following system

$$\begin{cases} x'(t) = f(x(t), u(t), v(t)), & \text{where} \quad u(t) \in U \quad \text{and} \quad v(t) \in V \\ \text{for almost every} \quad t \geq 0 & \end{cases}$$

([7]) For a rigorous proof of the barrier phenomenon, see ([17]) in the case of the viability kernels, and ([6]) in the case of the discriminating and leadership kernels.

where the state-variable x belongs to $X := \mathbb{R}^n$ and U and V are compact subsets of two finite dimensional metric spaces.

We assume that the function $f : \mathbb{R}^n \times U \times V \to \mathbb{R}^n$ is ℓ-Lipschitzian. Define

$$F(x, u) := \bigcup_{v \in V} f(x, u, v)$$

for all $x \in \mathbb{R}^n$ and for all $u \in U$. We assume here that $F(x, u)$ is convex for all x and all u. We assume furthermore that f is bounded, *i.e.* there exists a constant $M > 0$ such that :

$$\sup_{x \in R^n} \sup_{u \in U} \sup_{v \in V} \|f(x, u, v)\| \leq M . \tag{5}$$

2.2. Discriminating kernels

Let K be a closed subset of $\mathbb{R}^n$.

DEFINITION 2.1 : *Suppose assumptions of section* 2.1 *holds true. A closed set K is a discriminating domain for* (1) *if and only if for any $x \in K$, for any $u \in U$ there exists a solution to*

$$x'(t) \in F(x(t), u) \tag{6}$$

starting from x which remains in K.

Thanks to Viability Theorem, we can characterize discriminating domains through a tangential condition :

PROPOSITION 2.2 : *Suppose assumptions of section* 2.1 *holds true. A closed set K is a discriminating domain if and only if*

$$\forall x \in K,\ \forall \nu \in NP_K(x),\ \sup_u \inf_v \langle f(x, u, v), \nu \rangle \leq 0 , \tag{7}$$

where $NP_K(x)$ is the set of the proximal normals ([8]) *to K at x.*

Notice that $0 \in NP_K(x)$. For any x.

When K is not a discriminating domain we define by $\mathrm{Disc}_f(K)$ the *largest closed discriminating domain contained* in K. When K is the complement of an open target Ω this set is actually *Victor's discriminating victory set* (*cf.* [7] for a detailed proof).

([8]) If A is a closed set, let us recall that

$$NP_A(x) := \{\nu \in \mathbb{R}^n \text{ such that } d_A(x + \nu) = \|\nu\|\} .$$

2.3. Convexity of Discriminating kernels

Following [10], we obtain under suitable convexity assumptions, that the discriminating kernel is a convex set.

PROPOSITION 2.3 : *Suppose that assumptions of section* 2.1, *hold true. If* K *is a convex set and* $F(x, u)$ *is a convex map* (9) *for any* u.
Then $\mathrm{Disc}_f(K)$ *is convex.*

Proof : We know that (*cf.* the construction of discriminating kernels in [6])

$$\mathrm{Disc}_f(K) = \bigcap_{n \geq 0} V_n$$

where sets V_n are defined by

$$\begin{cases} V_0 := K \\ V_n := \bigcap_{u \in U} Viab_{F(.,u)}(V_{n-1}) . \end{cases}$$

According to [10] section 3, the viability of a convex set for a convex set-valued map is convex. Consequently, any V_n is convex and so is $\mathrm{Disc}_f(K)$.

Q.E.D.

3. AN ALGORITHM IN CONTINUOUS CASE

Our goal is to provide an algorithm finding the set $\mathrm{Disc}_f(K)$ which is the largest subset of K satisfying the following tangential condition :

$$\forall x \in \mathrm{Disc}_f(K), \ \forall \nu \in NP_{\mathrm{Disc}_f(K)}(x), \ \sup_{u \in U} \ \inf_{v \in V} \ \langle f(x, u, v), \nu \rangle \leq 0 . \quad (8)$$

This approach is an extension to differential games of results already presented in [10] and in [18].

3.1. Approximation

We want to underline that at each step of the algorithm, we only use the knowledge of K and f because we do not know in advance $\mathrm{Disc}_f(K)$.

Let us denote by $\tilde{K}$ the set of points x of ∂K which do not satisfy (7), *i.e.*,

$$\exists \nu \in NP_K(x) \quad \text{with} \quad \sup_u \ \inf_v \ \langle f(x, u, v), \nu \rangle > 0 .$$

(9) A set valued map is convex if and only if its graph is convex.

LEMMA 3.1 : *Let x belong to $\tilde{K}$. Define the following nonnegative number*

$$a(K, x) := \sup_{\substack{\nu \in NP_K(x) \\ \|\nu\| \leq 1}} \sup_{u \in U} \inf_{v \in V} \langle f(x, u, v), \nu \rangle .$$

Then

$$d(x, \mathrm{Disc}_f(K)) \geq \frac{a(K, x)}{M + \ell} . \tag{9}$$

Proof of the lemma : Fix $x \in \tilde{K}$. Consider $\nu \in NP_K(x)$ such that $a := \sup_{u \in U} \inf_{v \in V} \langle f(x, u, v), \nu \rangle > 0$, and $\|\nu\| \leq 1$. There exists some $\bar{u}$ in U with $\inf_{v \in V} \langle f(x, u, v), \nu \rangle = a$.

Let y belong [10] to $\prod_{\mathrm{Disc}_f(K)} (x + \nu)$. It is easy to nitice that $x + \nu - y \in NP_{\mathrm{Disc}_f(K)}(x)$. Hence because $\mathrm{Disc}_f(K)$ is a discriminating domain, for any $u \in U$, there exists $v \in V$ such

$$0 \geq \langle f(y, u, v), x + \nu - y \rangle = \langle f(y, u, v), \nu \rangle + \langle f(y, u, v), x - y \rangle .$$

Hence because f is bounded by M, for $u = \bar{u}$, there exists some v with

$$0 \geq \langle f(y, \bar{u}, v), \nu \rangle - M \|x - y\| .$$

Since f is ℓ-Lipschitzean and $\|\nu\| \leq 1$,

$$M \|x - y\| \geq \langle f(y, \bar{u}, v), \nu \rangle \geq \langle f(x, \bar{u}, v), \nu \rangle - \ell \|x - y\| .$$

consequently

$$\|x - y\| \geq \frac{\langle f(x, \bar{u}, v), \nu \rangle}{M + \ell} \geq \frac{a}{M + \ell} .$$

Since this results still holds true for any ν which does not satisfy the tangential condition the proof is completed.

Q.E.D.

[10] We denote by $\prod_A (x)$, the set of points $y \in A$ such that $\|x - y\| = d(A, x)$.

3.2. Construction

We define a decreasing sequence of closed subsets of K.

$$\begin{cases} K_0 := L \\ K_1 := K \setminus \bigcup_{x \in \tilde{K}} \mathring{B}\left(x, \dfrac{a(K, x)}{M + \ell}\right) \\ \dots \\ K_{n+1} := K_n \setminus \bigcup_{x \in \tilde{K}_n} \mathring{B}\left(x, \dfrac{a(K_n, x)}{M + \ell}\right) \end{cases} \tag{10}$$

where $\tilde{K}_n$ denotes the set of points x of K_n where the tangential condition (7) is not fulfilled ([11]) with $\mathrm{Disc}_f(K)$ replaced by K_n.

It is easy to notice that $\mathrm{Disc}_f(K) \subset K_n$ and consequently $K_\infty := \bigcap_n K_n$ contains $\mathrm{Disc}_f(K)$.

3.3. Convergence

We prove the convergence of the algorithm.

THEOREM 3.2 : *Suppose assumptions of section* 2.1 *hold true. Then :*

$$K_\infty := \bigcap_n K_n = \mathrm{Disc}_f(K)\,.$$

Proof : It is enough to prove that K_∞ is a discriminating domain. Fix $x \in K_\infty$ and let us prove that for any $\nu \in NP_{K_\infty}(x)$,

$$0 \geq a := \sup_u \inf_v \langle f(x, u, v), \nu \rangle\,.$$

Fix $\bar{\nu} := \lambda \nu$, with $\lambda \in]0, 1[$ such that $\|\bar{\nu}\| < 1$. Let us point out that the projection onto K_∞ of $x + \bar{\nu}$ is equal to x. For n larger enough, $x + \bar{\nu}$ does not belong to K_n, and we set $x_n := \prod_{K_n} (x + \bar{\nu})$. Since $\prod_{K_\infty} (x + \bar{\nu}) = x$, it is obvious that $\lim_n x_n = x$.

Define $\nu_n := x + \bar{\nu} - x_n$. Then ν_n belongs to $NP_{K_n}(x_n)$ and

$$a(x_n, K_n) \geq \sup_u \inf_v \langle f(x_n, u, v), \nu_n \rangle\,.$$

([11]) Namely :

$$\exists \nu \in NP_{K_n}(x) \quad \text{with} \quad \sup_u \inf_v \langle f(x, u, v), \nu \rangle > 0\,.$$

Since $\|x - x_n\| \geq \dfrac{a(x_n, K_n)}{M + \ell}$ from the construction of the sequence (K_n), we obtain $\lim\limits_n a(x_n, K_n) = 0$. In particular, the limit of the sequence $\sup\limits_u \inf\limits_v \langle f(x_n, u, v), x + \bar{\nu} - x_n \rangle$ is non-positive. But this limit is equal to $\sup\limits_u \inf\limits_v \langle f(x, u, v), \bar{\nu} \rangle$, which is non-positive. This completes the proof.

Q.E.D.

4. THE DISCRETE TARGET PROBLEM

We now consider the discrete dynamical system :

$$\forall n \in N, \quad x_{n+1} = g(x_n, u_n, v_n), \quad \text{where} \quad u_n \in U \quad \text{and} \quad v_n \in V . \tag{11}$$

We shall denote by $\vec{x}$ the sequence $(x_n)_n$. Let Ω be an open target of X. Two players, Ursule and Victor control this discrete dynamical system through their respective controls u and v. Ursule wants the system to reach the target. Victor wants the system to avoid it, namely he wants the system to remain in $K = X \backslash \Omega$.

We study Victor's discrete discriminating victory set, i.e., the set of points x of K such that a strategy $\tilde{v}(\,.\,,\,.\,) : X \times U \to V$ exists, such that for any sequence $(u_n)_n$, the solution $(x_n)_n$ of

$$\begin{cases} x_0 = x \\ x_{n+1} = g(x_n, u_n, \tilde{v}(x_n, u_n)), \quad \forall n \geq 0 \end{cases} \tag{12}$$

remains in K. Let us notice that this definition is similar to the one used in [14].

Set :

$$\forall x \in X, \quad \forall u \in U, \quad G(x, u) = \bigcup_{v \in V} g(x, u, v) . \tag{13}$$

We shall say that a closed set S is a *discrete discriminating domain* for G if S enjoys the following property :

$$\forall x \in S, \quad \forall u \in U, \quad G(x, u) \cap S \neq \emptyset . \tag{14}$$

It is easy to deduce the following.

PROPOSITION 4.1 : *A closed set S is a discrete discriminating domain if and only if, for any x_0 of S, a strategy $\tilde{v}(\,.\,,\,.\,) : X \times U \to V$ exists, such that, for* *any sequence* $(u_n)_n$ *of* U, *the solution* $(x_n)_{n \geq 0}$ *of* (12) *starting from* x_0 *ramains in* S, *i.e.*, x_n *belongs to* S *for any* $n \geq 0$.

Proof: Assume that S is a discrete discriminating domain. For any x of S and any u of U, the intersection $G(x, u) \cap S$ is nonempty. Thus we can choose some $\tilde{v}(x, u)$ of V such that $g(x, u, \tilde{v}(x, u))$ belongs to S. For any x_0 of S, for any sequence $(u_n)_n$, the solution of (12) remains in S. So we have defined the desired strategy $\tilde{v}(\ldots)$.

Assume now that, for any x_0 of S, there is a strategy $\tilde{v}(\ldots)$ such that, for any sequence $(u_n)_n$ of U, the solution $(x_n)_{n \geq 0}$ of (12) starting from x_0 remains in S. Let u belong to U, and define the sequence $u_n = u$ for any n. Then the solution x_n of (12) remains in S. Thus $x_1 = g(x_0, u_0, \tilde{v}(x_0, u_0))$ belongs to S and to $G(x_0, u)$. So $G(x_0, u) \cap S$ is non-empty for any x_0 of S and any u of U, and S is a discrete discriminating domain.

Q.E.D.

If K is not a discrete discriminating domain, it is possible to define the a largest discriminating domain contained in K and furthermore the largest one.

PROPOSITION 4.2 : *Let* K *be a closed set and* $G(\cdot, \cdot): X \times U \rightsquigarrow X$ *be a upper semi-continuous set-valued map with compact values. Then, there exists a largest closed discrete discriminating domain contained in* K. *We call this set* discrete discriminating kernel *of* K, *and we denote it* $\overrightarrow{\mathrm{Disc}}_G(K)$.

Proof: Let us consider the decreasing sequence of closed sets K_n defined as follows :

$$\begin{cases} K_0 := K, \\ K_{n+1} := \{x \in K_n \mid \forall u \in U, G(x, u) \cap K_n \neq \emptyset\}, \quad \forall n \geq 0. \end{cases} \tag{15}$$

Since G is upper semi-continuous with compact values, sets K_n are closed. Let us define $K_\infty := \bigcap_{n \geq 0} K_n$, the decreasing limit of the K_n. We claim that K_∞ satisfies the definition of discrete discriminating kernels.

In fact it is clear that if some $D \subset K$ is a discrete discriminating domain, it is contained in every K_n and consequently in K_∞.

Let us prove that K_∞ is a discrete discriminating domain, namely that it satisfies condition like (14).

Let x belong to K_∞, and u belong to U. We have to show that $G(x, u) \cap K_\infty$ is nonempty. Since x belongs to K_n, for all n, there exists $y_n \in K_n \cap G(x, u)$. A subsequence again denoted y_n converges to some y, because G is semi-continuous with compact values. The point y belongs to K_∞, since the sequence K_n decreases to K_∞, and belongs to $G(x, u)$, since and $G(\,.\,, u)$ is upper semicontinuous with compact values. So

$$\forall x \in K_\infty\,, \quad \forall u \in U\,, \quad G(x, u) \cap K_\infty \neq \emptyset\,.$$

This means precisely that K_∞ is a discrete discriminating domain for G.

Q.E.D.

Now, we shall prove that this discrete discriminating domain can be interpretated in term of *Victor's discrete discriminating victory set.*

DEFINITION 4.3 : *Let us posit assumptions of Proposition 4.2. Victor's discrete discriminating victory set denoted by* $\vec{\mathscr{W}}_U^d$ *is the set of point* $x_0 \in K$ *for which a strategy* $\tilde{v}(\,.\,,\,.\,) : X \times U \to V$ *exists, such that, for any sequence* $(u_n)_n$*, the solution* $(x_n)_{n \geq 0}$ *of* (12) *starting from* x_0 *remains in* K*, i.e.* x_n *belongs to* K *for any* $n \geq 0$.

THEOREM 4.4 : *Let* K *be a closed set and* $G(\,.\,,\,.\,) : X \times U \rightsquigarrow X$ *be a upper semi-continuous set-valued map with compact values. Victor's discrete discriminating victory set is the discrete discriminating kernel of* K.

$$\vec{\mathscr{W}}_U^d = \overrightarrow{\mathrm{Disc}}_G(K)\,.$$

Proof : From Proposition 4.1, we obtain $\vec{\mathscr{W}}_U^d \supset \overrightarrow{\mathrm{Disc}}_G(K)$. To prove the opposite inclusion, we shall prove that

$$X \backslash \vec{\mathscr{W}}_U^d \supset X \backslash \overrightarrow{\mathrm{Disc}}_G(K)\,.$$

Fix $x_0 \in K \backslash \overrightarrow{\mathrm{Disc}}_G(K)$ and a strategy $\tilde{v}(\,.\,,\,.\,) : X \times U \to V$. We shall define a sequence $(u_n)_n$ such that the solution of (12) starting from x_0 leaves K in finite time.

Thanks to (15), $\overrightarrow{\mathrm{Disc}}_G(K) = \bigcap_n K_n$. Since x_0 does not belong to $\overrightarrow{\mathrm{Disc}}_G(K)$, there is n_0 such that x_0 belongs to $K_{n_0} \backslash K_{n_0+1}$. Thus there is $u_0 \in U$ with $G(x_0, u_0) \cap K_{n_0} = \emptyset$. In particular, $x_1 := g(x_0, u_0, \tilde{v}(x_0, u_0))$ does not belong to K_{n_0}. So there is $n_1 < n_0$ such that x_1 belongs to $K_{n_1} \backslash K_{n_1+1}$.

In the same way, we can define by induction a decreasing sequence (n_k) and sequences (u_k) and (x_k) such that :

$$\forall k,\ x_{k+1} := g(x_k, u_k, \tilde{v}(x_k, u_k))$$

and x_k belongs to $K_{n_k} \backslash K_{n_k+1}$. Since (n_k) is a decreasing sequence, there is $n_k \leq n_0 + 1$ such that x_k does not belong to $K_0 = K$. So we have defined a sequence (u_k) such that the solution (x_k) of (12) leaves K in finite time. This ends the proof of Theorem 4.4.

Q.E.D.

5. A DISCRETIZATION ALGORITHM

There is a natural way to approximate the discrete discriminating kernel of a closed set K. We already used this idea to prove Theorem 4.4. We provide these algorithms in a more general framework.

The main motivation of studying the discrete target problem is the approximation the discriminating kernel of any closed set. Theorem 5.2 states that the discrete discriminating domain of a closed set K for the set-valued map $x \rightsquigarrow x + \tau f(x, u, V) + \sigma\tau^2 B$ (where B is the closed unit ball and where the constant σ is computed bellow) converge to the discriminating kernel of K for f when $\tau \to 0^+$.

5.1. Approximation of the discrete discriminating kernel

Let K be a closed set and $g : \mathbb{R}^n \times U \times V \to \mathbb{R}^n$ a continuous map, U and V being metric, compact. Set as previously

$$G(x, u) := \bigcup_{v \in V} g(x, u, v).$$

Consider U_n any sequence of subsets of U, such that:

$$\operatorname*{Lim\,sup}_{n \to \infty} U_n = U. \tag{16}$$

We define the following decreasing sequence of closed sets:

$$\begin{cases} K'_0 = K, \\ K'_{n+1} = \{x \in K'_n \mid \forall u \in U_n, \quad G(x, u) \cap K'_n \neq \emptyset\}. \end{cases} \tag{17}$$

In a similar way that in the proof of Theorem 4.4, we can prove the following.

PROPOSITION 5.1 : *Let K and G as previously. The decreasing sequence of closed sets K'_n defined by* (15) *converges to $\overrightarrow{\mathrm{Disc}}_G(K)$ i.e.*

$$\bigcap_{n \in N} K'_n = \overrightarrow{\mathrm{Disc}}_G(K). \tag{18}$$

Proof: From definitions of K_n (see (15)) and of K'_n, it is obvious that for any $n \in \mathbb{N}$, K'_n contains K_n.

So we have just to prove that $K'_\infty := \bigcap_{n \in N} K'_n$ is a discrete discriminating domain for G.

Let x belong to K'_∞ and $u \in U$. From (16), there exists a sequence u_{n_k} of U_{n_k} which converges to u. From (17), there exists some y_k which belongs to the intersection of $G(x, u_{n_k})$ and K'_{n_k}. Since G is upper semicontinuous with compact values, a subsequence, again denoted y_k converges to some y of $G(x, u)$. Moreover, y belongs to K_∞ because y_k belongs to K'_{n_k}. So we proved that for any x of K'_∞, for any u of U, the intersection between $G(x, u)$ and K'_∞ is not empty. In other words, K'_∞ is a discrete discriminating domain of K for G. Since K'_∞ contains the discrete discriminating kernel of K for G, from the definition of the discrete discriminating kernel, both sets have to be equal.

Q.E.D.

5.2. Approximation by discrete discriminating kernels

We introduce the new notations (where B is the unit closed ball) :

$$\begin{cases} F_\tau(x, u) := F(x, u) + \dfrac{Ml_\tau}{2} B \\ G_\tau(x, u) := x + \tau F_\tau(x, u) . \end{cases} \tag{19}$$

The discrete dynamical system for the set-valued map G_τ is a discretization of the dynamical system (1) for f. It is rather natural to ask if the discriminating kernel of a closed set K for f can be approximated by the discrete discriminating kernel of K for G_τ. The answer is positive :

THEOREM 5.2 : *Let f as previously and K be a closed subset of $\mathbb{R}^n$. Then* [12] :

$$\lim_{\tau \to 0^+} \overrightarrow{\mathrm{Disc}}_{G_\tau}(K) = \mathrm{Disc}_f(K) . \tag{20}$$

The proof is the result of the two following Propositions : First Proposition 5.3 states that any upper limit of discrete discriminating domains for G_τ is a discriminating domain for f when $\tau \to 0^+$.

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[12] The lower limit (in the Kuratowski sense) of a set-valued map $\tau \rightsquigarrow A_\tau$ when $\tau \to 0^+$ is the set of points x for which there is $x_\tau \in A_\tau$ which converge to x when $\tau \to 0^+$. The upper limit of A_τ is the set of points x for which there are sequences $\tau_n \to 0^+$ and $x_n \in A_{\tau_n}$ such that $x_n \to x$. If one has :

$$\limsup_{\tau \to 0^+} A_\tau = \liminf_{\tau \to 0^+} A_\tau$$

we say that A_τ has a limit when $\tau \to 0^+$, and we denote it by $\lim_{\tau \to 0^+} A_\tau$. See for instance [2].

PROPOSITION 5.3 : *Let f as in Theorem 5.2. Suppose that for any $\tau > 0$, the closed set K_τ is a discrete discriminating domain for G_τ. Then* $K^\# := \limsup_{\tau \to 0} K_\tau$ *is a discriminating domain for f.*

Second Proposition 5.4 yields that $\mathrm{Disc}_f(K)$ is contained in the discrete discriminating kernel of K for G_τ.

PROPOSITION 5.4 : *Under the previous assumptions,*

$$\mathrm{Disc}_f(K) \subset \overrightarrow{\mathrm{Disc}}_{G_\tau}(K)\,, \quad \forall \tau > 0\,.$$

Proof of Theorem 5.2 : Set

$$D^\# := \limsup_{\tau \to 0^+} \overrightarrow{\mathrm{Disc}}_{G_\tau}(K)\,.$$

Since $\overrightarrow{\mathrm{Disc}}_{G_\tau}(K)$ are discrete discriminating domains, from Proposition 5.3, $D^\#$ is a discriminating domain for f contained in K. So $D^\#$ is contained in $\mathrm{Disc}_f(K)$.

Conversely, Proposition 5.4 states that $\liminf_{\tau \to 0^+} \overrightarrow{\mathrm{Disc}}_{G_\tau}(K)$ contains $\mathrm{Disc}_f(K)$. Hence

$$\limsup_{\tau \to 0^+} \overrightarrow{\mathrm{Disc}}_{G_\tau}(K) \subset \mathrm{Disc}_f(K) \subset \liminf_{\tau \to 0^+} \overrightarrow{\mathrm{Disc}}_{G_\tau}(K)\,.$$

Since the upper limit always contains the lower limit, Theorem 5.2 is proved.

Proof of Proposition 5.3 : Let us consider $x_0 \in K^\#$ and $u \in U$. There exists a subsequence $x_{\tau 0} \in K_\tau$ which converges to x_0. For any τ, there exists $(x_\tau^n)_{n \in \mathbb{N}}$ solutions of :

$$\begin{cases} x_\tau^{n+1} \in G_\tau(x_\tau^n, u) \\ x_\tau^0 = x_{\tau 0} \end{cases} \tag{21}$$

which remain in K_τ, because K_τ is a discrete discriminating domain for G_τ.

From the definition of G_τ, x_τ^{n+1} belongs to $G_\tau(x_\tau^n)$ and then :

$$\forall n > 0\,, \quad \frac{x_\tau^{n+1} - x_\tau^n}{\tau} \in F_\tau(x_\tau^n, u)\,.$$

With this sequence we associate the piecewise linear interpolation $x_\tau(\,.\,)$

which coincides to x_τ^n at nodes $n\tau$:

$$x_\tau(t) = x_\tau^n + \frac{x_\tau^{n+1} - x_\tau^n}{\tau}(t - n\tau), \quad \forall t \in [n\tau, (n+1)\tau), \quad \forall n > 0 .$$

Then

$$x_\tau'(t) \in F_\tau(x_\tau^n, u), \quad \forall t \in [n\tau, (n+1)\tau).$$

We have

$$d((x_\tau(t), x_\tau'(t)), \text{Graph } (F_\tau(\,.\,, u))) \leqslant \|x_\tau(t) - x_\tau^n\| \leqslant \tau \, \|F(x_\tau^n, u)\| .$$

Since $F(\,.\,, u)$ is bounded by M (See (5))

$$\forall \tau, \quad \forall s \leqslant t, \quad d((x_\tau(t), x_\tau'(t)), \text{Graph } (F_\tau(\,.\,, u))) \leqslant M\tau ,$$

and with (19) we have

$$\text{Graph } (F_\tau(\,.\,, u)) \subset \text{Graph } (F(\,.\,, u)) + \frac{Ml\tau}{2} B .$$

Then, for all $t \geqslant 0$, for all $\tau > 0$,

$$(x_\tau(t), x_\tau'(t)) \in \text{Graph } (F(\,.\,, u)) + M\tau\left(1 + \frac{l}{2}\right) B .$$

By the Ascoli and Alaoglu Theorems, we derive that there exists $x(\,.\,) \in W^{1,1}(0, +\infty\,; X\,; e^{-ct}\,dt)$ and a subsequence (again denoted by) x_τ which satisfy :

$$\begin{cases} \text{i)} \;\; x_\tau(\,.\,) \text{ converges uniformly to } x(\,.\,), \\ \text{ii)} \;\; x_\tau'(\,.\,) \text{ converges weakly to } x'(\,.\,) \text{ in } L^1(0, +\infty\,; X\,; e^{-ct}\,dt). \end{cases} \tag{22}$$

This implies ([3] The Convergence Theorem) that $x(\,.\,)$ is a solution to the differential inclusion :

$$\begin{cases} x'(t) \in F(x(t), u), & \text{for almost all} \quad t \geqslant 0, \\ x(0) = x_0 \in K^{\#} . \end{cases}$$

It remains to prove that the limit is a solution which remains in $K^{\#}$. For any $t > 0$, there exists a sequence $n_t = E\left(\frac{t}{\tau}\right)$ such that $n_t\,\tau \to t$ when $\tau \to 0$. Then $x(t) = \lim_{\tau \to 0} x_\tau(n_t\,\tau)$. Since $\forall \tau : x_\tau(n_t\,\tau) = x_\tau^{n_t} \in K_\tau$, $x(t)$ belongs to the upper limit $K^{\#}$ of the subsets K_τ. So we have proved that for any $t \geq 0$, $x(t)$ belongs to $K^{\#}$, so that $K^{\#}$ is a discriminating domain for f.

Proof of Proposition 5.4 : Let $x_0 \in \mathrm{Disc}_f(K)$ and $u \in U$ fixed. Consider any solution $x(\,.\,)$ of the differential inclusion for $F(\,.\,, u)$ starting from x_0. Let $\tau > 0$ given. We have

$$x(t+\tau) - x(t) = \int_t^{t+\tau} x'(s)\, ds\,, \quad \forall t > 0\,.$$

Since $x'(s) \in F(x(s), u)$ and F Lipschitzian,

$$x(t+\tau) - x(t) \in \tau F(x(t)) + l \int_t^{t+\tau} \|x(s) - x(t)\|\, ds\, B\,, \quad \forall t > 0\,.$$

But F is bounded, and $\|x(s) - x(t)\| \leq (s-t) M$. Thus

$$x(t+\tau) - x(t) \in \tau F(x(t), u) + \frac{Ml}{2} \tau^2 B\,. \tag{23}$$

So, we have proved that if x_0 belongs to the discriminating kernel of K for f and if $x(\,.\,)$ is a solution for $F(\,.\,, u)$ starting from x_0, then the following sequence

$$\xi_n = x(n\tau)\,, \quad \forall n \geq 0 \tag{24}$$

is a solution to the discrete dynamical system associated with $G_\tau(\,.\,, u)$:

$$\xi_{n+1} \in G_\tau(\xi_n, u)\,, \quad \forall n \geq 0\,. \tag{25}$$

Assume now that $x(\,.\,)$ is solution of the differential inclusion for $F(\,.\,, u)$ starting from x_0 and which remains in $\mathrm{Disc}_f(K)$. Such a solution exists from the very definition of $\mathrm{Disc}_f(K)$. Then $(\xi_n)_n$ is a solution to (25) which remains in $\mathrm{Disc}_f(K)$. Since for any x_0 of $\mathrm{Disc}_f(K)$ and any u of U, one can find a solution $(\xi_n)_n$ of (25) with $\xi_0 = x_0$, which remains in $\mathrm{Disc}_f(K)$, this means that $\mathrm{Disc}_f(K)$ is a discrete discriminating domain for G_τ. Thus

$$\mathrm{Disc}_f(K) \subset \overrightarrow{\mathrm{Disc}}_{G_\tau}(K)\,, \quad \forall \tau > 0\,.$$

Q.E.D.

6. APPROXIMATION BY FINITE SETVALUED MAPS

With any $h \in \mathbb{R}$ we associate X_h a countable subset of X, which spans X in the sense that

$$\forall x \in X\,, \quad \exists x_h \in X_h \quad \text{such that} \quad \|x - x_h\| \leq \alpha(h) \tag{26}$$

where $\alpha(h)$ decreases to 0 when $h \to 0$:

$$\lim_{h \to 0} \alpha(h) = 0 . \tag{27}$$

Consider a ℓ-Lipschitzian map $f : \mathbb{R}^n \times U \times V \to \mathbb{R}^n$. For any fixed $\tau > 0$:

$$\begin{cases} F_\tau(x, u) = F(x, u) + \dfrac{Ml\tau}{2} B , \\ G_\tau(x, u) = x + \tau F_\tau(x, u) . \end{cases} \tag{28}$$

Set moreover :

$$\begin{cases} G_\tau^r(x, u) = G_\tau(x, u) + rB \\ G_{\tau, h}^r(x, u) = G_\tau^r(x, u) \cap X_h . \end{cases} \tag{29}$$

The set-valued map $G_{\tau, h}^r$ is the discretization of the extend set-valued map G_τ^r. If K is a closed subset of $\mathbb{R}^n$, we set

$$K_h^r = (K + rB) \cap X_h .$$

We are now able to state the last Theorem of this paper :

THEOREM 6.1 : *Assume that $f : \mathbb{R}^n \times U \times V \to \mathbb{R}^n$ is a ℓ-Lipschitzian map and K is a closed subset of $\mathbb{R}^n$. We assume that f is bounded by some constant M (i.e. f satisfies* (5)) *and that $f(\,.\,, u, V)$ has convex values. Then we can approximate the discriminating kernel of K for f by the finite discrete discriminating kernel of $K_h^{\alpha(h)}$ for the discrete set-valued maps $G_{\tau, h}^{(2 + \tau\ell)\alpha(h)}$ in the following way :*

$$\limsup_{\substack{\tau \to 0 \\ \alpha(h) \leq \tau^2}} \overrightarrow{\mathrm{Disc}}_{G_{\tau, h}^{(2 + \tau\ell)\alpha(h)}}(K_h^{\alpha(h)}) = \mathrm{Disc}_f(K) . \tag{30}$$

Remark : Is S is a closed set such that $X_h \cap S$ consists in a finite number of points and $H : S \times U \rightsquigarrow \mathbb{R}^n$ is a set-valued map with closed values, then $\overrightarrow{\mathrm{Disc}}_H(S)$ can be computed in a finite number of steps thanks to the algorithm described in Proposition 5.1.

We first prove the following Lemma :

LEMMA 6.2 : *Let f and K as in Theorem* 6.1. *Then we have :*

$$[\overrightarrow{\mathrm{Disc}}_{G_\tau}(K) + \alpha(h) B] \cap X_h \subset \overrightarrow{\mathrm{Disc}}_{G_{\tau, h}^{(2 + \tau\ell)\alpha(h)}}(K_h^{\alpha(h)}) . \tag{31}$$

Proof of Lemma (6.2) *:* We have just to prove that $D := [\overrightarrow{\mathrm{Disc}}_{G_\tau}(K) + \alpha(h)B] \cap X_h$ is a discriminating domain for $G_{\tau,h}^{(2+\tau\ell)\alpha(h)}$ contained in $K_h^{\alpha(h)}$. Since $\overrightarrow{\mathrm{Disc}}_{G_\tau}(K)$ is contained in K, the set D is clearly contained in $K_h^{\alpha(h)}$.

Let x_h belong to D and $u \in U$ be fixed. From the very definition of D, there exists some x of $\overrightarrow{\mathrm{Disc}}_{G_\tau}(K)$ such that $\|x - x_h\| \leq \alpha(h)$. Since $\overrightarrow{\mathrm{Disc}}_{G_\tau}(K)$ is a discrete discriminating domain, there exists y in the intersection between $G_\tau(x, u)$ and $\overrightarrow{\mathrm{Disc}}_{G_\tau}(K)$. From (26), one can find some y_h in X_h such that $\|y - y_h\| \leq \alpha(h)$. In particular, y_h belongs to D.

Since f is a ℓ-Lipschitzian map, G_τ is $(1 + \tau\ell)$-Lipschitzian set-valued map. So we have :

$$y_h \in [G_\tau(x, u) + \alpha(h)B] \cap X_h \subset [G_\tau(x_h, u) + (2 + \tau\ell)\,\|x - x_h\|\,B] \cap X_h\,.$$

Since $\|x - x_h\| \leq \alpha(h)$, we have finally :

$$y_h \in [G_\tau(x_h, u) + (2 + \tau\ell)\,\alpha(h)B] \cap X_h\,.$$

This means that for any x of D, for any u of U, the intersection between $G_{\tau,h}^{(2+\tau\ell)\alpha(h)}$ and D is not empty, *i.e.*, D is a discriminating domain for $G_{\tau,h}^{(2+\tau\ell)\alpha(h)}$. So Lemma 6.2 is proved.

Proof of Theorem 6.1 *:* First, point out that if a sequence of closed set $A(\tau)$ converges, for the Kuratowski upper limit, to some closed set A when $\tau \to 0^+$, then :

$$\limsup_{\substack{\tau \to 0^+ \\ h \to 0^+}} A(\tau)_h^{\alpha(h)} = A\,.$$

Indeed, consider $x_{\tau_n}^{h_n}$ of $A(\tau_n)_{h_n}^{\alpha(h_n)}$ which converges to some x. There exists $y_{\tau_n}^{h_n}$ of $A(\tau_n)$ with $\left\|x_{\tau_n}^{h_n} - y_{\tau_n}^{h_n}\right\| \leq \alpha(h_n)$. Since $\alpha(h_n)$ converges to 0^+, we have proved that $y_{\tau_n}^{h_n}$ converges to x and x belongs to A. The opposite inclusion is obvious.

So, Lemma 6.2 and Theorem 5.2 yield :

$$\mathrm{Disc}_f(K) = \limsup_{\tau \to 0^+} \overrightarrow{\mathrm{Disc}}_{G_\tau}(K) \subset \limsup_{\substack{\tau \to 0 \\ \alpha(h) \leq \frac{Ml\tau^2}{2}}} \overrightarrow{\mathrm{Disc}}_{G_{\tau,h}^{(2+\tau\ell)\alpha(h)}}(K_h^{\alpha(h)})\,.$$

We have to show the converse. Fix $\bar{h} > 0$. The set $\overrightarrow{\mathrm{Disc}}_{G_{\tau,h}^{(2+\tau\ell)\alpha(h)}}(K_h^{\alpha(h)})$ is contained in $\overrightarrow{\mathrm{Disc}}_{G_\tau^{(2+\tau\ell)\alpha(h)}}(K+\alpha(\bar{h})B)$ for any h lower than $\bar{h}$, because $\overrightarrow{\mathrm{Disc}}_{G_{\tau,h}^{(2+\tau\ell)\alpha(h)}}(K_h^{\alpha(h)})$ is a discriminating domain contained in $K+\alpha(\bar{h})B$. The upper limit of $\overrightarrow{\mathrm{Disc}}_{G_{\tau,h}^{(2+\tau\ell)\alpha(h)}}(K_h^{\alpha(h)})$ is contained in the upper limit of $\overrightarrow{\mathrm{Disc}}_{G_{\tau,h}^{(2+\tau\ell)\alpha(h)}}(K+\alpha(\bar{h})B)$, when h and τ go to 0^+. Recall that

$$G_{\tau,h}^{(2+\tau\ell)\alpha(h)}(x,u) = x + \tau F(x,u) + \left[\frac{M\ell\tau^2}{2} + (2+\tau\ell)\alpha(h)\right]B .$$

Since $\alpha(h)$ is supposed to be lower than τ^2, Theorem 5.2 yields that the upper limit of $\overrightarrow{\mathrm{Disc}}_{G_{\tau,h}^{(2+\tau\ell)\alpha(h)}}(K+\alpha(\bar{h})B)$ is equal to $\mathrm{Disc}_f(K+\alpha(\bar{h})B)$. Let us notice that $\mathrm{Disc}_f(K+\alpha(\bar{h})B)$ contains $\mathrm{Disc}_f(K)$ because $\mathrm{Disc}_f(K)$ is a discriminating domain contained in $K+\alpha(\bar{h})B$. The Stability Theorem (see [3]) states that any upper limit of viability domains is still a viability domain. When $\bar{h}\to 0^+$, the upper limit of $\mathrm{Disc}_f(K+\alpha(\bar{h})B)$ is still a viability domain for $f(\,.\,,u,V)$ for all $u\in U$, so it is a discriminating domain for f contained in K. In particular since this upper limit contains $\mathrm{Disc}_f(K)$, it is equal to $\mathrm{Disc}_f(K)$. So we have proved the opposite inclusion.

7. EXAMPLE

We provide two examples of computation of the discriminating kernel. This game derived from the classical homicidal chauffeur game is suggested by Pierre Bernhard ([13]).

We give two situations corresponding to different values of the parameters. The closed set K is the complement of an open target Ω. In the first case, the discriminating kernel intersects the boundary of the target Ω. In the second case, it does not intersect the boundary of Ω.

Dynamics of the game are described by the following system :

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \end{pmatrix} + \frac{5}{R}\begin{pmatrix} -y(t) \\ x(t) \end{pmatrix} u(t) + w(x(t), y(t))\, v(t)$$

where $U := [-1, 1]$ and V is the closed unit ball of $\mathbb{R}^2$,

$$w_f(x,y) = \max\left(0, \min\left\{W, r - 10 - 10\ln\left(\frac{1+2\,|\sin\theta|}{3}\right) + \ln r\right\}\right)$$

and $$(x,y) = (r\cos\theta, r\sin\theta)\,.$$

We set $W := 5.2$.

([13]) It is our pleasure to thank him for discussions and advices.

The target is defined by

$$\Omega = \{(x, y), r < 1.5\} .$$

For $R = 4$, we obtain figure 1 and for $R = 2$, we obtain figure 2.

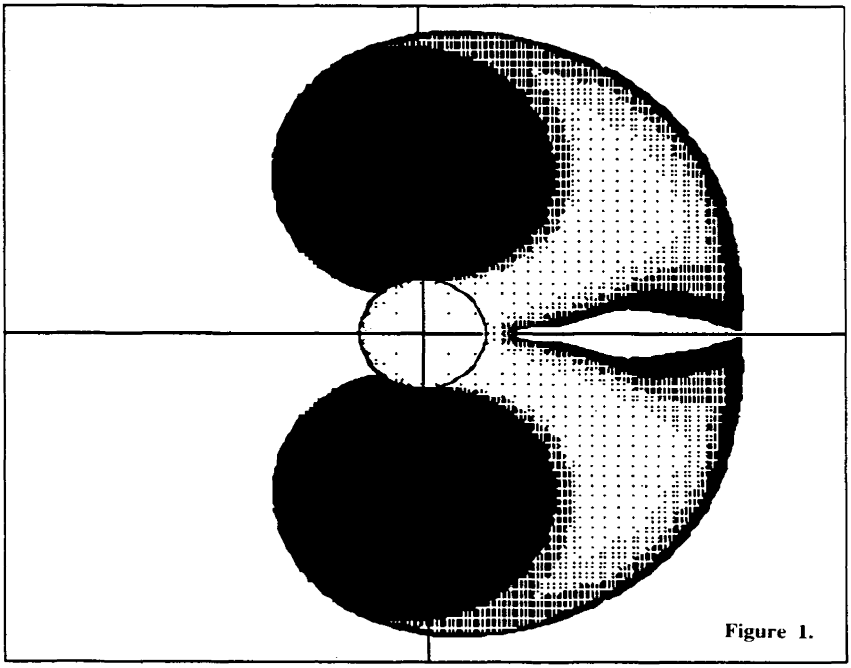

Figure 1.

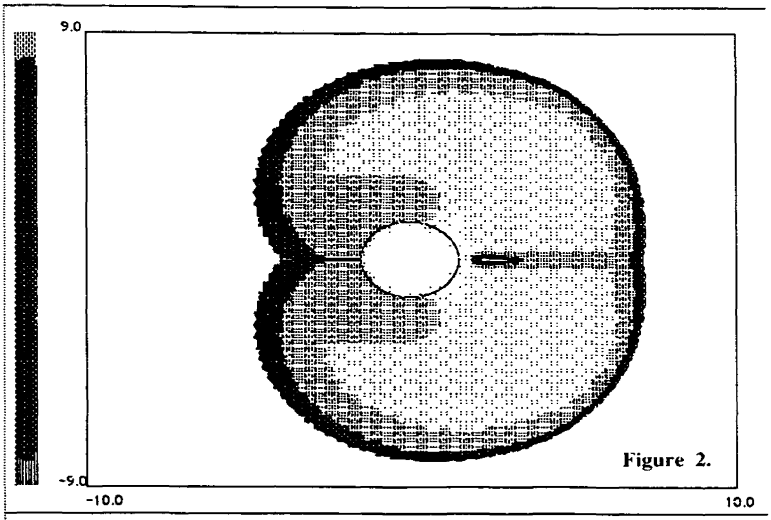

Figure 2.

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