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SEPARATION OF VARIABLES IN THE STOKES PROBLEM APPLICATION TO ITS FINITE ELEMENT MULTISCALE APPROXIMATION (*)

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Abstract. — *The aim of the paper is to describe a method for the multiscale approximation of the Stokes problem. We first use a transformation of variables to substitute for this problem two unconstrained optimization problems. We then describe a finite element multiscale approximation of these problems. It turns out that this approximation allows us to also approximate the solutions of the Stokes problem. We conclude by describing an algorithm based on this method.*

Résumé. — *Nous proposons ici une méthode d'approximation multi-échelles du problème de Stokes. Dans un premier temps nous prouvons que celui-ci se découple en deux problèmes elliptiques sans contraintes ; ce résultat est obtenu à l'aide d'un simple changement de variables. Dès lors nous décrivons une approximation multi-échelles à l'aide d'espaces d'éléments finis, adaptée à ces nouveaux problèmes. En retour cette approximation permet d'approcher la solution du problème initial. Un algorithme de calcul est alors présenté.*

1. INTRODUCTION

This paper is the first part of a work concerned with finite element multiscale approximations for Navier-Stokes equations, in the framework of the *nonlinear Galerkin methods*. This article is devoted to the stationary Stokes problem.

Let us first give an overview of the nonlinear Galerkin methods, that were introduced by Marion and Temam (see [16]). The aim of these methods is the

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large time approximation of the solutions of dissipative evolution equations. The set that describes the large time behavior of solutions is an attractor that can have a complex structure and even be a fractal ; moreover the convergence of the orbits towards the attractor can be arbitrarily slow.

A first attempt to overcome these difficulties was the introduction of inertial manifolds (see [11], [20]). They are finite dimensional invariant manifolds that contain the attractor, and then allow us to reduce the dynamics of the initial system to the dynamics on these manifolds.

Next came the approximate inertial manifolds (AIM) (see [10]), that are finite dimensional manifolds that contain the attractor into a thin neighborhood. On the one hand sequences of approximate inertial manifolds that approximate the attractor with higher and higher order, have been derived for a broad class of evolution equations (see [7], [9], [19], [21]). On the other hand, since their equations are rather simple, AIMs make easier the implementation of numerical algorithms ; as the classical Galerkin method is related to the simplest of these AIMs, that is the finite dimensional linear space spanned by the first m functions of the Galerkin basis, the nonlinear Galerkin methods feature inertial nonlinear algorithms that correspond to AIMs providing better orders of accuracy.

The theory first developed in the spectral case extends now beyond : for instance see [22], [5] for works about finite differences or [13] about wavelets. In this paper we are interested in finite elements. Hence we return to the framework of [17]. Let V_h be a finite element space corresponding to a triangulation whose mesh size is h. Instead of computing an approximation y_h of a solution u of a dissipative evolution equation as the solution of the approximated problem on V_h, we are looking for a nonlinear approximation $y_h + \phi(y_h)$, where ϕ maps V_h into a suitable supplementary W_h of V_h into $V_{h/2}$. A question we would like to address in this paper is the choice of a pair (V_h, W_h) that is convenient to approximate the solution u of the stationary Stokes problem. In other words, we would like to choose a pair (V_h, W_h), that allows us to obtain a robust decomposition of u as a sum

$$u = y_h + \sum_{k=0}^{+\infty} z_{h_k}, \tag{1.1}$$

where y_h is as above and where the incremental variables z_{h_k} are obtained by successive mesh refinements $h_{k+1} = \frac{h_k}{2}$, such that $z_{h_k} \in W_{h_k}$.

On the one hand, the utilization of multiscale approximations has been advocated and studied for standard elliptic linear problem (see [1], [4], [25]...). On the other hand, it is difficult to use these methods for saddle point problems, even using mixed finite element methods. The difficulty is to find a conforming space V_h that enjoys $V_{2h} \subset V_h$. However we would like to refer to the numerical work in [18].

In this paper, we overcome this difficulty by introducing a separation of variables in the Stokes problem that replaces the saddle point problem by two unconstrained decoupled elliptic problems ; these problems can be each solved by standard multiscale process. Moreover this method provides approximation of the solution u of the Stokes problem that has the same order than the one using the usual mixed finite element methods (*).

This paper is organized as follows. In the first section we introduce the separation of variables. A first subsection is devoted to recalling some classical results for the Stokes problem on a two-dimensional domain whose boundary is a polygon. Then in the next subsection we introduce a separation of variables according to duality arguments. In the second section, we apply these results to theorical finite element multiscale approximations for the Stokes problem. Having addressed such a problem in a first subsection, we then describe expansions of the new (and old) variables into series whose terms are actually incremental variables. Error estimates conclude this section. In the third and last section, we describe a three-steps algorithm to approximate the solution of the Stokes problem ; error estimates are then derived.

2. SEPARATION OF THE VARIABLES

2.1. The Stokes problem

Let us first introduce some notations. Let Ω be a bounded open set of $\mathbb{R}^2$, whose boundary is a convex polygon. We shall consider the following two-dimensional Stokes problem.

For f in $\mathbb{L}^2(\Omega) = L^2(\Omega) \times L^2(\Omega)$ (or $\mathbb{H}^{-1}(\Omega)$), we seek a vector function $u = (u_1, u_2)$ and a scalar function p, which are defined in Ω and which satisfy the following equations and boundary conditions

$$- \Delta u + \nabla p = f \quad \text{in} \quad \Omega , \tag{2.1}$$
$$\operatorname{div} u = 0 \quad \text{in} \quad \Omega , \tag{2.2}$$
$$u = 0 \quad \text{on} \quad \partial\Omega . \tag{2.3}$$

For the sake of convenience, we rewrite (2.1), (2.2) and (2.3) in the following abstract form

$$Lu + {}^tBp = f , \tag{2.4}$$
$$Bu = 0 , \tag{2.5}$$

where we denote by L the Laplacian operator which maps $\mathbb{H}^2(\Omega) \cap \mathbb{H}_0^1(\Omega)$ onto $\mathbb{L}^2(\Omega)$ (and $\mathbb{H}_0^1(\Omega)$ onto its dual space $\mathbb{H}^{-1}(\Omega)$).

(*) These results were announced in [14].

B is defined as follows

$$B : \mathbb{H}_0^1(\Omega) \to \dot{L}^2(\Omega)$$
$$u \mapsto - \operatorname{div} u . \tag{2.6}$$

Hence ${}^tB$ is the Gradient operator and maps $\dot{L}^2(\Omega)$ into $\mathbb{H}^{-1}(\Omega)$. We also set

$$M = \dot{L}^2(\Omega), \quad W = \mathbb{H}_0^1(\Omega), \quad V = \operatorname{Ker} B .$$

We also define H as the closure of V in $\mathbb{L}^2(\Omega)$; we recall the following result (see [23])

$$H = \{u \in \mathbb{L}^2(\Omega) ; \operatorname{div} u = 0 \text{ in } \Omega, \ (u . n) = 0 \text{ on } \partial\Omega\} , \tag{2.7}$$

where n is the unit outward normal. Moreover we denote by A the Stokes operator defined by $D(A) = \mathbb{H}^2(\Omega) \cap V$ and by

$$Au = f, \text{ for } f \text{ in } H \text{ and } u \text{ in } D(A) ,$$

if and only if there exists p such that (u, p) satisfies (2.4)-(2.5).

We also set, for any pair w, w' in W and any p, p' in M

$$b(w, p) = \langle w, {}^tBp \rangle_{W, W^*} , \tag{2.8}$$

$$((w, w')) = \int_\Omega \nabla w . \nabla w' \, dx , \tag{2.9}$$

$$(w, w') = \int_\Omega w . w' \, dx , \tag{2.10}$$

$$(p, p')_M = \int_\Omega pp' \, dx . \tag{2.11}$$

Moreover we denote by $\| . \|$, $| . |$, $| . |_M$ the norms corresponding respectively to (2.9), (2.10) and (2.11).

Let us recall now the following theorem that is crucial for the study of the Stokes problem (we refer to [23] for a proof ; see also [12]).

THEOREM 1 : *The three following assertions, that are equivalent, hold*

$$* \ B : W/V \to M \text{ is an isomorphism} , \tag{2.12}$$

$$* \ {}^tB : M \to (W/V)^* \text{ is an isomorphism} , \tag{2.13}$$

$$* \text{ There exist } \alpha, \ \|b\| > 0 \text{ such that for any } p \text{ in } M$$

$$\alpha |p|_M \leq \sup_{\|x\| = 1} b(x, p) \leq \|b\| \, |p|_M . \tag{2.14}$$

We conclude this section by recalling the following regularity result (see [23], [15]).

PROPOSITION 1 : *There exists* $C > 0$ *such that if the data* f *in* (2.4), (2.5) *belongs to* $\mathbb{L}^2(\Omega)$, *then the solution* (u, p) *satisfies the following*

$$|Lu| + |{}^tBp| \leqslant C\,|f|\,. \tag{2.15}$$

2.2. Separation of variables

In this section we first prove that, if p is solution of (2.4)-(2.5), then p is solution of an unconstrained optimization problem. Next we introduce a simple change of variables that allows us to replace the problem (2.4)-(2.5) by two unconstrained optimization problems. Throughout the section we shall use without proofs some duality results. We refer to [8] for the proofs.

Let us introduce the Lagrangian

$$\mathcal{L}(u, p) = \frac{1}{2}\,\|u\|^2 + b(u, p) - (f, u)\,. \tag{2.16}$$

Then, (u, p) is solution of (2.4)-(2.5) if and only if

$$(u, p) \text{ is solution of } \inf_{u \in W}\ \sup_{p \in M}\ \mathcal{L}(u, p)\,. \tag{2.17}$$

Moreover problem (2.17) is equivalent to its dual problem

$$\sup_{p \in M}\ \inf_{u \in W}\ \mathcal{L}(u, p)\,. \tag{2.18}$$

On the other hand, let q belong to M. Let u_q be solution of the following

$$\inf_{u \in W}\ \mathcal{L}(u, q)\,. \tag{2.19}$$

Hence u_q satisfies

$$((u_q, w)) + b(w, q) = (f, w), \quad \forall w \in W\,. \tag{2.20}$$

Conversely, if u_q satisfies (2.20), then u_q is solution of (2.19). Then (2.16) and (2.20) yield

$$\mathcal{L}(u_q, q) = -\frac{1}{2}\,\|u_q\|^2\,. \tag{2.21}$$

Therefore, due to (2.21), we observe that p is solution of (2.18) if and only if p is solution of problem (2.22) hereafter :

$$\inf_{q \in M}\ \left(\frac{1}{2}\,\|u_q\|^2\right)\,. \tag{2.22}$$

We shall prove now that (2.22) defines on M a well-posed optimization problem. We have the following proposition.

PROPOSITION 2 : *There exists a scalar product* $[.,.]$ *on* M *such that*

* *the corresponding norm* $[.]$ *is equivalent to the usual norm* $|.|_M$.
* *Problem* (2.22) *is the same as problem* (2.23) *hereafter*

$$\inf_{q \in M} \left(\frac{1}{2} [q]^2 - [q, \eta] \right), \tag{2.23}$$

where η *is an element of* M *that will be defined below.*

Proof: We rewrite (2.20) as

$$u_q = L^{-1} f - L^{-1}\, {}^t B q . \tag{2.24}$$

Hence we have the following

$$\frac{1}{2} \|u_q\|^2 = \frac{1}{2} |L^{-1/2}\, {}^t B q|^2 - (L^{-1/2}\, {}^t B q, L^{-1/2} f) + \frac{1}{2} |L^{-1/2} f|^2 . \tag{2.25}$$

Setting for $\tilde{p}$, $\tilde{q}$ in M

$$[\tilde{p}, \tilde{q}] = (L^{-1/2}\, {}^t B \tilde{p}, L^{-1/2}\, {}^t B \tilde{q}), \tag{2.26}$$

we have a scalar product such that (2.23) holds ; actually $\eta = p$ in (2.23), since (2.18) and (2.23) are in fact equivalent.

To complete the proof of Proposition 2, it remains to check that $[.,.]$ defines an equivalent norm to $|.|_M$ on M. For any w in W, due to (2.14), we obtain

$$\left| \langle w, {}^t B q \rangle_{W, W^*} \right| \leq \|b\| \, \|w\| \, |q|_M . \tag{2.27}$$

We take $w = L^{-1}\, {}^t B q$ in (2.27). This yields

$$[q]^2 \leq \|b\| \, [q] \, |q|_M , \tag{2.28}$$

observing that

$$\|L^{-1}\, {}^t B q\| = [q] . \tag{2.29}$$

It remains to prove the reverse inequality in (2.28), i.e.

$$|q|_M \leq c\,[q] . \tag{2.30}$$

We know that any w in W can be written as

$$w = v + L^{-1}\,{}^tB\tilde{q}\,, \tag{2.31}$$

where $(v, \tilde{q})$ belongs to $V \times M$. Hence, for q in M, we have

$$b(w, q) = b(v, q) + [q, \tilde{q}]\,, \tag{2.32}$$

observing that

$$b(L^{-1}\,{}^tB\tilde{q}, q) = [q, \tilde{q}]\,. \tag{2.33}$$

Since v belongs to V, $b(v, q)$ vanishes. On the other hand, we easily derive from (2.31)

$$\|w\|^2 = \|v\|^2 + [\tilde{q}]^2\,, \tag{2.34}$$

observing that $b(v, \tilde{q})$ vanishes as well. Therefore, (2.32) and (2.34) yield

$$\frac{b(w, q)}{\|w\|} \leqslant \frac{[q, \tilde{q}]}{[\tilde{q}]} \leqslant [q]\,. \tag{2.35}$$

Hence (2.14) and (2.35) yield (2.30). ■

Now we introduce in (2.16) the following change of variables

$$u^* = u + L^{-1}\,{}^tBp\,, \tag{2.36}$$

while p is kept unchanged, i.e.

$$p^* = p\,. \tag{2.37}$$

Then (2.16) becomes

$$\mathscr{L}(u, p) = \frac{1}{2}\,\|u^*\|^2 + \frac{1}{2}\,\|L^{-1}\,{}^tBp\|^2 - ((u^*, L^{-1}\,{}^tBp)) + b(u^*, p) - b(L^{-1}\,{}^tBp, p) - (f, u^*) + (f, L^{-1}\,{}^tBp)\,. \tag{2.38}$$

Using (2.8), (2.29) and (2.33), we transform (2.38) into

$$\mathscr{L}(u, p) = \frac{1}{2}\,\|u^*\|^2 + \frac{1}{2}\,[p]^2 - b(u^*, p) + b(u^*, p) - [p]^2 - (f, u^*) + (f, L^{-1}\,{}^tBp)\,. \tag{2.39}$$

In other words, we have the following

$$\mathscr{L}(u, p) = J(u^*) - I(p)\,, \tag{2.40}$$

where

$$J(u^*) = \frac{1}{2} \|u^*\|^2 - (f, u^*), \tag{2.41}$$

$$I(p) = \frac{1}{2} [p]^2 - (f, L^{-1}\,{}^tBp). \tag{2.42}$$

We summarize the results of this section by the following proposition.

PROPOSITION 3 : *Let* (u, p) *be the solution of saddle-point problem* (2.17). *Let* u^* *and* p^* *be respectively the solution of*

$$\inf_{u^* \in W} J(u^*), \tag{2.43}$$

and of

$$\inf_{p \in M} I(p). \tag{2.44}$$

Then u, p, u^*, p^* *are related by*

$$p^* = p, \tag{2.45}$$

$$u^* = u + L^{-1}\,{}^tBp. \tag{2.46}$$

Moreover, the following equality holds

$$\sup_{p \in M} \inf_{u \in W} \mathscr{L}(u, p) = \inf_{u^* \in W} J(u^*) - \inf_{p \in M} I(p). \tag{2.47}$$

Proof : u^* and p are characterized by respectively,

$$((u^*, w)) = (f, w), \tag{2.48}$$

for any w in W, an by

$$[p, q] = (f, L^{-1}\,{}^tBq), \tag{2.49}$$

for any q in M. Hence, taking $w = L^{-1}\,{}^tBq$ in (2.48), we obtain

$$(Bu^*, q)_M = (f, L^{-1}\,{}^tBq). \tag{2.50}$$

We then easily derive from (2.33), (2.46), (2.49) and (2.50) that u satisfies

$$(Bu, q)_M = 0, \quad \forall q \in M. \tag{2.51}$$

On the other hand, we take the scalar product in W of (2.46) with

w to get the following

$$((u^*, w)) = ((u, w)) + (L^{-1}\, {}^tBp, Lw)\,. \tag{2.52}$$

Therefore (2.48) and (2.52) lead to

$$((u, w)) + b(w, p) = (f, w)\,. \tag{2.53}$$

To conclude the proof of Proposition 3, it remains to check (2.47) ; this can be merely derived from (2.40).

3. APPLICATION TO THE FINITE ELEMENTS MULTISCALE ANALYSIS OF THE STOKES PROBLEM

3.1. Multiscale analysis of the Stokes problem

Let us define a finite elements multiscale approximation of a Hilbert space $\tilde{V}$, as a sequence of finite element spaces $\{V_{h_j}\}_{j \in \mathbb{N}}$ that satisfies the two following properties,

* embedding condition

$$V_{h_j} \subset V_{h_{j+1}}\,. \tag{3.1}$$

* density condition

$$\overline{\bigcup_{j \in \mathbb{N}} V_{h_j}} = \tilde{V}\,. \tag{3.2}$$

Here h_j is a nonnegative parameter that represents the mesh size of the triangulation corresponding to V_{h_j}. Actually we set

$$h_j = \frac{h_0}{2^j}\,; \tag{3.3}$$

then h_0 is the mesh size of the coarsest triangulation and j is the number of refinement levels.

In this paper we are interested in the construction of finite elements multiscale approximation to solve the Stokes problem. The natural Hilbert space related to the Stokes problem is $V = \operatorname{Ker} B$. The main difficulty is to construct approximations of V which satisfy (3.1) and (3.2) ; even if we are looking at the Stokes problem in its saddle point formulation, namely (2.4)-(2.5), it is far from easy to construct finite element multiscale approximations of $W \times M$ that fit with the incompressibility condition (2.5).

To overcome this difficulty, we rather solve the two problems (2.43) and (2.44). These problems are convenient for multiscale approximations, since they feature no constraint conditions. Let us then conclude § 3.1 by describing a suitable finite elements multiscale approximation for (2.43) and (2.44).

Let $\{W_{h_j} \times M_{h_j}\}_{j \in \mathbb{N}}$ be a collection of finite elements spaces such that we have the following

$$* \quad W_{h_j} \subset W_{h_{j+1}} \tag{3.4}$$

$$M_{h_j} \subset M_{h_{j+1}} \tag{3.5}$$

$$* \quad \overline{\bigcup_{j \geq 0} W_{h_j}} = W \tag{3.6}$$

$$* \quad \overline{\bigcup_{j \geq 0} M_{h_j}} = M \,. \tag{3.7}$$

We also assume that the following properties involving the finite elements spaces above hold :

* Approximation result for W_{h_j}.

There exists $C > 0$, such that for any u^* in $D(L) = \mathbb{H}^2(\Omega) \cap \mathbb{H}_0^1(\Omega)$, the following inequality holds

$$\inf_{\tilde{y} \in W_{h_j}} \|u^* - \tilde{y}\| \leq C \,.\, h_j \,.\, |Lu^*| \,. \tag{3.8}$$

* Approximation result for M_{h_j}.

There exists $C > 0$, such that for any p in $\dot{H}^1(\Omega)$, the following inequality holds

$$\inf_{\tilde{p} \in M_{h_j}} |p - \tilde{p}|_M \leq C \,.\, h_j \,.\, |{}^tBp| \,. \tag{3.9}$$

Remark : C denotes a constant that is independent of h_j.

There is a broad class of finite element spaces which enjoy (3.4)-(3.9). For instance, let us mention the P_1 iso P_2/P_1 element of Bercovier and Pironneau, see [2], [3] or [12]. For other examples of suitable finite elements we refer to [6].

3.2. The incremental variables

This section has two aims. On the one hand we introduce an expansion of u^* and p, the solutions of (2.43) and (2.44), in series whose terms are actually *incremental variables ;* here we are refering to the framework of the nonlinear Galerkin finite elements methods (see [17]). On the other hand, we establish some results, that also relate to the nonlinear Galerkin framework, and that will be useful to obtain the error estimates of the next section.

Let W_{h_j} be as in § 3.1. For the sake of convenience, and when no confusion is possible, we drop the subscript j to write $h_j = h$. A natural approximation for u^*, the solution of (2.43), is u_h^* defined as follows

$$J(u_h^*) = \inf_{v_h \in W_h} J(v_h) . \tag{3.10}$$

Due to standard results, u_h^* is well-defined and is characterized by the following property

$$((u_h^*, w_h)) = (f, w_h), \quad \text{for any } w_h \text{ in } W_h . \tag{3.11}$$

We then define the incremental variables corresponding to u^* as, for each h,

$$z_{2h} = u_h^* - u_{2h}^* . \tag{3.12}$$

Remark : We observe that the incremental variable z_{2h} is the solution of the following problem

$$\inf_{w_h \in W_h} J(u_h^* + w_h) . \tag{3.13}$$

Moreover z_{2h} is the projection of u_h^* onto the orthogonal complement W_{2h}^0 of W_{2h} in W_h. In other words, when computing the approximations u_{2h}^*, u_h^*, ... of u^*, z_{2h} is along the direction of the steepest descent from u_{2h}^* to u^* for the $((.\,,\,.))$ norm.

We define by induction on j

$$z_{h_j} = u_{h_{j+1}}^* - u_{h_j}^* , \tag{3.14}$$

that leads to the following expansion of u^* as a series

$$u^* = u_{h_0}^* + \sum_{j=0}^{+\infty} z_{h_j} ; \tag{3.15}$$

we observe that (3.15) holds for u^* in W, since

$$\|u^*\|^2 = \|u_{h_0}^*\|^2 + \sum_{j=0}^{+\infty} \|z_{h_j}\|^2 , \tag{3.16}$$

that implies the convergence of the series in W. We shall briefly address other questions related to the convergence of the series involved in (3.15) (and in (3.20) below) in subsequent remarks in Section 3.3.

On the other hand, let us apply the same process to p that is solution of (2.44). For $h = h_j$ as above, we define p_h as

$$I(p_h) = \inf_{\tilde{p}_h \in M_h} I(\tilde{p}_h) . \tag{3.17}$$

We then define the incremental variable corresponding to p as

$$\psi_{2h} = p_h - p_{2h} . \tag{3.18}$$

We observe that ψ_{2h} satisfies

$$[\psi_{2h}, q_{2h}] = 0 , \quad \forall q_{2h} \in M_{2h} . \tag{3.19}$$

This leads to the expansion of p as

$$p = p_{h_0} + \sum_{j=0}^{+\infty} \psi_{h_j} . \tag{3.20}$$

We observe that due to (3.19), for p in M the following equality holds

$$[p]^2 = [p_{h_0}]^2 + \sum_{j=0}^{+\infty} [\psi_{h_j}]^2 . \tag{3.21}$$

Hence the series involved in the right hand side of equality (3.20) converges in M, since $[\,.\,]$ and $|\,.\,|_M$ define equivalent norms.

We then prove some lemmata that relate to the nonlinear Galerkin methods. Let W_{2h} be the orthogonal complement in W of W_{2h} in W_h ; W^0_{2h} is the natural space that contains z_{2h} accordingly to (3.13). Let M^0_{2h} be the orthogonal complement, in M endowed with scalar product $[\,.\,,\,.\,]$ of M_{2h} in M_h ; M^0_{2h} is the natural space that contains ψ_{2h} accordingly to (3.19). We then have the following strenghtened Poincaré inequalities.

LEMMA 1 : *There exists C that is independent of h, such that for any ψ in M^0_h, for any z in W^0_h, we have the following*

- $|\psi|_M \leq C\,.\,h\,.\,|{}^tB\psi|$, (3.22)
- $|L^{-1}\,{}^tB\psi| \leq C\,.\,h\,.\,|\psi|_M$, (3.23)
- $|z| \leq C\,.\,h\,.\,\|z\|$. (3.24)

Remark : C denotes a constant that is independent of h.

Proof : Let us first prove (3.22). Recall $|\,.\,|_M$ and $[\,.\,]$ define equivalent norms on M (see (2.28) and (2.30)). Therefore, the following inequality holds

$$|\psi|_M \leq C\,[\psi] . \tag{3.25}$$

Since ψ belongs to M^0_h, for any q in M_h we have

$$[\psi - q]^2 = [\psi]^2 + [q]^2 . \tag{3.26}$$

Then (3.25) and (3.26) yield

$$|\psi|_M \leqslant C\,[\psi - q]\,; \tag{3.27}$$

(3.27) holds for any q in M_h. We use again the norm equivalence between $|\,.\,|_M$ and $[\,.\,]$ to obtain

$$|\psi|_M \leqslant C\,|\psi - q|_M\,, \tag{3.28}$$

holding for any q in M_h. We then use (3.9) to derive (3.22) from (3.28).

We prove below (3.23) using an Aubin-Nitsche duality argument. Let f be in H. Let (v, p) be in $V \times M$ such that

$$f = Lv + {}^tBp \tag{3.29}$$

holds. Let ψ be in M_h^0. Using $Bv = 0$ and (2.26), it is easy to check that

$$(L^{-1}\,{}^tB\psi, f) = [\psi, p]\,. \tag{3.30}$$

Since ψ belongs to M_h^0, we rewrite (3.30) as

$$(L^{-1}\,{}^tB\psi, f) = [\psi, p - p_h]\,, \tag{3.31}$$

that holds for any p_h in M_h. Using again the equivalence between the norms $[\,.\,]$ and $|\,.\,|_M$, (3.31) yields

$$(L^{-1}\,{}^tB\psi, f) \leqslant C\,|\psi|_M |p - p_h|_M \tag{3.32}$$

that holds for any p_h in M_h. We then use (3.9) to derive from (3.32) the following

$$(L^{-1}\,{}^tB\psi, f) \leqslant Ch|\psi|_M|{}^tBp|\,. \tag{3.33}$$

We then use the regularity argument (2.15) to deduce from (3.33) the following

$$(L^{-1}\,{}^tB\psi, f) \leqslant Ch|\psi|_M|f|\,. \tag{3.34}$$

(3.34) holds for any f in H. Therefore (3.23) is proved.

The proof of (3.24) is also based on an Aubin-Nitsche argument. Let f be in H. Let w be in $D(L)$ such that

$$Lw = f\,. \tag{3.35}$$

Let z be in W_h^0. We then have, for any y_h in W_h, the following

$$(z, f) = (z, Lw) = ((z, w)) = ((z, w - y_h))\,. \tag{3.36}$$

Using (3.8) and (3.36), we obtain

$$\begin{aligned}(z, f) &\leqslant C \|z\| \|w - y_h\| \\ &\leqslant C \|z\| h |Lw| .\end{aligned} \tag{3.37}$$

Recall $Lw = f$; hence (3.37) that holds for any f in H implies (3.24).

We now prove another technical lemma that will be useful in the next section. ■

LEMMA 2 : *Let us define* ρ_h *as the orthogonal projector in M endowed with scalar product* $[\,.\,,\,.\,]$ *onto* M_h. *If we still denote by* ρ_h *the following application*

$$\begin{aligned}\rho_h : \dot{H}^1(\Omega) &\to M_h \\ p_h &\mapsto \rho_h(p_h) ,\end{aligned} \tag{3.38}$$

then ρ_h is bounded as an operator acting on $\dot{H}^1(\Omega)$, independently of h.

Proof : For the sake of convenience, we introduce the elliptic projector $\tilde{\rho}_h$ from $\dot{H}^1(\Omega)$ into M_h defined as follows

$$({}^tB\tilde{\rho}_h p, {}^tBq_h) = ({}^tBp, {}^tBq_h) , \quad \forall q_h \in M_h . \tag{3.39}$$

We recall the following result (see Th. A.2 in Appendix A in [12]), that holds for spaces corresponding to uniformly regular triangulation of $\bar{\Omega}$.

$$|p - \tilde{\rho}_h p|_M \leqslant C \,.\, h \,.\, |{}^tBp| , \tag{3.40}$$

for p in $\dot{H}^1(\Omega)$. Hence we write, for p as above

$$|{}^tB\rho_h p| \leqslant |{}^tB(\rho_h p - \tilde{\rho}_h p)| + |{}^tB\tilde{\rho}_h p| . \tag{3.41}$$

On the other hand, recalling the following standard inverse inequality (see the Appendix A in [12])

$$|{}^tBq_h| \leqslant \frac{C}{h} |q_h|_M , \tag{3.42}$$

for any q_h in M_h, we rewrite (3.41) as

$$|{}^tB\rho_h p| \leqslant \frac{C}{h} |\rho_h(p) - \tilde{\rho}_h(p)|_M + |{}^tB\tilde{\rho}_h p| . \tag{3.43}$$

We then observe that, since $|\,.\,|_M$ and $[\,.\,]$ define equivalent norms,

$$|\rho_h(p) - \tilde{\rho}_h(p)|_M \leqslant C |p - \tilde{\rho}_h(p)|_M . \tag{3.44}$$

On the other hand, we obtain from (3.39)

$$|{}^tB\tilde{\rho}_h p| \leq |{}^tBp| . \tag{3.45}$$

Hence inequalities (3.40), (3.43), (3.44) and (3.45) imply

$$|p - \tilde{\rho}_h p|_M \leq \frac{C}{h} \cdot h \cdot |{}^tBp| + |{}^tBp| , \tag{3.46}$$

that concludes the proof of the lemma. ■

3.3 Error estimates

Let (u, p) be the solution of (2.4)-(2.5). This can be approximated by (u_h, p_h) defined as follows ; p_h is the solution of (3.17), and u_h is defined from p_h and u_h^*, that is solution of (3.10), as

$$u_h = u_h^* - L^{-1}\, {}^tBp_h . \tag{3.47}$$

First we state, and we prove, some error estimates when f is smooth, i.e. f belongs to $\mathbb{L}^2(\Omega)$. Then we describe some error estimates when f belongs only to $\mathbb{H}^{-1}(\Omega)$.

PROPOSITION 4 : *Let f be in $\mathbb{L}^2(\Omega)$. Let u, p, u_h, p_h be as above. Then the following estimates hold :*

- $\|u - u_h\| \leq C \,.\, h \,.\, |f|$, (3.48)
- $|u - u_h| \leq C \,.\, h^2 \,.\, |f|$, (3.49)
- $|p - p_h|_M \leq C \,.\, h \,.\, |f|$, (3.50)
- $|\operatorname{div} u_h|_M \leq C \,.\, h \,.\, |f|$. (3.51)

Remark : Let us reinterpret Proposition 4 in terms of convergence results for series (3.15) and (3.20) ; moreover we first define $\mathcal{C}_{h_j}$ as

$$\mathcal{C}_{h_j} = z_{h_j} - L^{-1}\, {}^tB\psi_{h_j} , \tag{3.52}$$

for z_{h_j} and ψ_{h_j} as above. This gives the following expansion of u as

$$u = u_{h_0} + \sum_{j=0}^{+\infty} \mathcal{C}_{h_j} . \tag{3.53}$$

We observe that, since the linear operator $L^{-1}\, {}^tB$ is bounded from M into W, and since (3.15) holds in W and (3.20) holds in M, then (3.53) holds in W.

Moreover (3.48) is an improvement of (3.53), since it features an error estimate for the convergence of series (3.53) in W. A similar remark holds for (3.50) and (3.20). On the other hand, combining (2.46), (3.47), (3.48) and (3.50)

$$\|u^* - u_h^*\| \leqslant C \cdot h \cdot |f| , \tag{3.54}$$

that provides an error estimate for the convergence of series (3.15).

Proof of Proposition 4 : Let us first prove (3.50). Thanks to (2.30) we have

$$|p - p_h|_M \leqslant C [p - p_h] . \tag{3.55}$$

We then use the $[.,.]$-orthogonal decomposition (3.20) to write

$$[p - p_h]^2 = \sum_{k=0}^{+\infty} [\psi_{h_k}]^2 , \tag{3.56}$$

where $h_0 = h$ in (3.56). From (2.28), (3.22) and Lemma 2, we also derive

$$[\psi_{h_k}] \leqslant C \left|\psi_{h_k}\right|_M \leqslant C \cdot h_k \cdot \left|{}^tB\psi_{h_k}\right| \leqslant C \cdot h_k \cdot |{}^tBp| . \tag{3.57}$$

We recall $h_k = \dfrac{h}{2^k}$, and then obtain from (3.56), (3.57) that

$$[p - p_h]^2 \leqslant C \cdot |{}^tBp|^2 \cdot \sum_{k=0}^{+\infty} \frac{h^2}{4^k} \leqslant C \cdot h^2 \cdot |{}^tBp|^2 . \tag{3.58}$$

Hence (2.15), (3.55) and (3.58) yield (3.50).

Let us now prove (3.49). From (2.46) and (3.47) we have

$$|u - u_h| \leqslant |u^* - u_h^*| + \left|L^{-1}\,{}^tB(p - p_h)\right| . \tag{3.59}$$

On the one hand, using (3.20) as above, we obtain

$$\left|L^{-1}\,{}^tB(p - p_h)\right| \leqslant \sum_{k=0}^{+\infty} \left|L^{-1}\,{}^tB\psi_{h_k}\right| . \tag{3.60}$$

Moreover, due to (3.23), we rewrite (3.60) as

$$\left|L^{-1}\,{}^tB(p - p_h)\right| \leqslant C \sum_{k=0}^{+\infty} \frac{h}{2^k} \left|\psi_{h_k}\right| . \tag{3.61}$$

Using (2.30) and straightforward computations, we obtain

$$\begin{aligned} \left|L^{-1}\,{}^tB(p - p_h)\right| &\leqslant C \sum_{k=0}^{+\infty} \frac{h}{2^k} [\psi_{h_k}] \\ &\leqslant C \left(\sum_{k=0}^{+\infty} \frac{1}{4^k}\right)^{1/2} h \left(\sum_{k=0}^{+\infty} [\psi_{h_k}]^2\right)^{1/2} . \end{aligned} \tag{3.62}$$

Hence due to (3.56)

$$\left|L^{-1}\,{}^tB(p-p_h)\right| \leqslant C\,.\,h\,.\,[p-p_h]\,, \tag{3.63}$$

and using (3.58) we have

$$\left|L^{-1}\,{}^tB(p-p_h)\right| \leqslant C\,.\,h^2\,.\,|{}^tBp|\,. \tag{3.64}$$

On the other hand, using (3.15), we write

$$|u^* - u_h^*| \leqslant \sum_{k=0}^{+\infty} |z_{h_k}|\,. \tag{3.65}$$

We then apply (3.24) to derive from (3.65)

$$\begin{aligned} |u^* - u_h^*| &\leqslant C \sum_{k=0}^{+\infty} \frac{h}{2^k}\,\|z_{h_k}\| \\ &\leqslant Ch\left(\sum_{k=0}^{+\infty}\frac{1}{4^k}\right)^{1/2}\left(\sum_{k=0}^{+\infty}\|z_{h_k}\|^2\right)^{1/2}. \end{aligned} \tag{3.66}$$

In other words

$$|u^* - u_h^*| \leqslant C\,.\,h\,.\,\|u^* - u_h^*\|\,, \tag{3.67}$$

since the expansion (3.15) is orthogonal in W. We observe also that u_h^* satisfies

$$\|u^* - u_h^*\| = \inf_{\tilde{y}_h \in W_h} \|u^* - \tilde{y}_h\|\,, \tag{3.68}$$

since u_h^* is the projection in W of u^* into W_h. Therefore, we derive from (3.8), (3.67) and (3.68) that

$$|u^* - u_h^*| \leqslant C\,.\,h^2\,.\,|Lu^*|\,. \tag{3.69}$$

Hence (3.49) follows from (3.64), (3.69) and (2.15), observing that

$$|Lu^*| \leqslant |Lu| + |{}^tBp|\,. \tag{3.70}$$

The proof of (3.48) is similar. We write

$$\|u - u_h\| \leqslant \|u^* - u_h^*\| + [p - p_h]\,, \tag{3.71}$$

recalling $[q] = |L^{-1/2}\,{}^tBq|$. Therefore (3.48) follows from (3.8), (3.58), (3.68) and (3.71).

To conclude the proof of Proposition 4, using $Bu = 0$, we write

$$|\operatorname{div} u_h|_M = |B(u - u_h)|_M \leqslant \sqrt{2}\,\|u - u_h\| . \tag{3.72}$$

Hence (3.51) merely follows from (3.48) and (3.72).

We now consider the case where f belongs only to $\mathbb{H}^{-1}(\Omega)$.

PROPOSITION 5 : *Let f be in $\mathbb{H}^{-1}(\Omega)$. Let u, p, u_h, p_h be as above. Then the following estimates hold*

$$|u - u_h| \leqslant C\,.\,h\,.\,\|f\|_* , \tag{3.73}$$

$$|L^{-1}\,{}^tB(p - p_h)| \leqslant C\,.\,h\,.\,\|f\|_* . \tag{3.74}$$

Remark : Let us observe that series (3.15) and (3.52) still converge in W, and that series (3.20) converges in M. But we no longer have error estimates for these topologies. On the other hand, we reinterpret (3.73) and (3.74) as error estimates for the convergences of these series for weaker topologies.

Proof of Proposition 5 : We rewrite (3.63) as

$$|L^{-1}\,{}^tB(p - p_h)| \leqslant C\,.\,h\,.\,[p - p_h] \leqslant C\,.\,h\,.\,[p] , \tag{3.75}$$

observing that p_h is the $[\,.\,,\,.\,]$ projection of p into M_h. Using (2.28), and the following classical regularity result (see [23]),

$$\|u\| + |p|_M \leqslant C\,\|f\|_* , \tag{3.76}$$

we obtain (3.74). On the other hand, using

$$\|u^* - u_h^*\| \leqslant \|u^*\| = \|f\|_* , \tag{3.77}$$

we derive from (3.67) that

$$|u^* - u_h^*| \leqslant C\,.\,h\,.\,\|f\|_* . \tag{3.78}$$

Then (3.59), (3.74) and (3.78) provide (3.73). ■

4. AN ALGORITHM TO APPROXIMATE THE STOKES PROBLEM

Having described the theorical multiscale analysis of the Stokes problem, the next task is to address the practical computation of the approximation (u_h, p_h) defined above. In a first subsection, we describe a three-step algorithm. Then we give error estimates in a subsequent paragraph.

4.1. A three-step algorithm

Let us first describe the strategy of the algorithm on the continuous problem. First we compute u^* that is solution of (2.48), i.e. solution of the Dirichlet problem

$$\begin{aligned} - \Delta u^* &= f \quad \text{in} \quad \Omega , \\ u^* &= 0 \quad \text{on} \quad \partial \Omega . \end{aligned}$$

We observe that u^* does not by any mean approximate u, in particular u^* does not satisfy the incompressibility condition (2.5). Next we compute p that is solution of (2.49), i.e. solution of the problem

$$(BL^{-1}\, {}^tB) p = - \operatorname{div} u^* \quad \text{in} \quad \Omega .$$

The last step is to compute $u - u^*$ that is solution of the following variational problem

$$((u - u^*, w)) = - ({}^tBp, w) , \quad \forall w \in W . \tag{4.1}$$

First step : The first step is devoted to the computation of u_h^*, that is solution of (3.10), and that approximates u^*. Actually u_h^* is solution of the standard elliptic problem (3.11), and the computations can be carried out using the regular multiscale finite element process ; for instance we refer to [25] where this last question is addressed.

Second step : This step should be the computation of p_h that is solution of (3.16) ; nevertheless p_h is actually solution of the elliptic equation

$$[p_h, q_h] = (Bu^*, q_h)_M , \quad \forall q_h \in M_h , \tag{4.2}$$

and we do not know a priori u^*. We then replace p_h by $\tilde{p}_h$ defined as follows

$$[\tilde{p}_h, q_h] = (Bu_h^*, q_h)_M , \quad \forall q_h \in M_h . \tag{4.3}$$

We shall prove in the next section that $\tilde{p}_h$ is a suitable approximation of p_h, and then a suitable approximation of p as well.

Let us give now a remark concerning the practical computation of $\tilde{p}_h$. $\tilde{p}_h$ defined above is the solution of an elliptic problem, and the computations can be carried out using standard multiscale technics ; nevertheless Proposition 2 asserts that scalar products $[.,.]$ and $(.,.)_M$ provide equivalent norms on M. This implies that the Gram matrix of the usual nodal basis of M_h for the $[.,.]$ scalar product has a condition number that is independent of h, since this result holds for the $(.,.)_M$ scalar product (see Lemma 3 hereafter). Therefore computations of $\tilde{p}_h$ can be carried out without multiscale process.

Third step : The aim of this step is to compute u_h, or at least one approximation of u_h ; we rewrite (3.47) as

$$((u_h - u_h^*, y)) = -(By, p_h)_M, \quad \forall y \in W. \tag{4.4}$$

A suitable approximation for u_h (this point will be addressed subsequently) is $\tilde{u}_h$ defined as follows

$$((\tilde{u}_h - u_h^*, y_h)) = -(By_h, \tilde{p}_h)_M, \quad \forall y_h \in W_h, \tag{4.5}$$

for $\tilde{p}_h$ defined above. Since $\tilde{p}_h$ is known, problem (4.5) is also an elliptic problem for the Laplacian, and the computations can be carried out by standard multiscale process, as in the first step.

LEMMA 3 : *Let us consider a uniformly regular triangulation of $\bar{\Omega}$. Then the condition number of the L^2-Gram matrix of the usual nodal basis of M_h behaves like $O(1)$ with respect to h.*

Proof : Because this lemma is classical, we just sketch the proof for P_1 elements. For a node N of the triangulation, we define the corresponding nodal function σ_N as the unique function in M_h that enjoys the following

$$\sigma_N(N') = \delta_{N,N'}, \tag{4.6}$$

for N' a node of the triangulation. Let y be

$$y(x) = \sum_N y(N)\,\sigma_N(x), \tag{4.7}$$

where we sum over all the nodes of the triangulation. Actually, the fact that we have a uniformly regular triangulation means that we have, for any T triangle

$$c_1 \cdot \int_T |y(x)|^2\,dx \leq h^2 \cdot \sum_{N \in T} |y(N)|^2 \leq c_2 \int_T |y(x)|^2\,dx. \tag{4.8}$$

We then sum over all the triangles T to conclude the proof of the lemma. ■

4.2. Error estimates

In this section we assume (without loss of generality) that the data f belongs to $\mathbb{L}^2(\Omega)$.

PROPOSITION 6 : *Let u, p, $\tilde{u}_h$, $\tilde{p}_h$ be as above. The following inequalities hold*

$$|\tilde{p}_h - p|_M \leq C \cdot h \cdot |f|, \tag{4.9}$$

$$\|u - \tilde{u}_h\| \leq C \cdot h \cdot |f|. \tag{4.10}$$

Proof : Let us prove (4.9). Due to (3.50), (4.9) follows as soon as we prove

$$|\tilde{p}_h - p_h|_M \leq C \, . \, h \, . \, |f| \, . \tag{4.11}$$

On the other hand, thanks to (2.30), we just have to prove

$$[\tilde{p}_h - p_h] \leq C \, . \, h \, . \, |f| \, . \tag{4.12}$$

We then substract (4.3) from (4.2), we take $q_h = p_h - \tilde{p}_h$, and we obtain

$$[p_h - \tilde{p}_h]^2 = (B(u^* - u_h^*), p_h - \tilde{p}_h)_M \, . \tag{4.13}$$

Using (2.12) and (2.30), we derive from (4.13) that

$$[p_h - \tilde{p}_h] \leq C \, . \, \|u^* - u_h^*\| \, . \tag{4.14}$$

Therefore (3.8), (3.68) and (4.14) yield

$$[p_h - \tilde{p}_h] \leq C \, . \, h \, . \, |Lu^*| \, . \tag{4.15}$$

Hence (4.9) holds since $|Lu^*| = |f|$.

We now prove (4.10). We take $y = y_h$ in (4.4), we substract this equality to (4.5) to obtain

$$((u_h - \tilde{u}_h, y_h)) = - (By_h, p_h - \tilde{p}_h)_M \, . \tag{4.16}$$

Let us define P_1 as the orthogonal projector in W onto W_h. We infer from (4.16) that

$$\|P_1(u_h - \tilde{u}_h)\| \leq C \, |p_h - \tilde{p}_h|_M \, . \tag{4.17}$$

Therefore, using (4.11), we obtain

$$\|P_1(u_h - \tilde{u}_h)\| \leq C \, . \, h \, . \, |f| \, . \tag{4.18}$$

Then (4.10) follows as soon as we prove the following

$$\|(I - P_1)(u_h - \tilde{u}_h)\| \leq C \, . \, h \, . \, |f| \, . \tag{4.19}$$

First we observe that, since $(I - P_1)\,\tilde{u}_h = 0$, we just have to estimate $\|(I - P_1)\,u_h\|$. Let z belong to the orthogonal complementary of W_h in W, i.e. $\bigoplus_{k=0}^{+\infty} W_{h_k}^0$. We take $y = z$ in (4.4) to obtain

$$((u_h, z)) = (-Bz, p_h)_M = - (z, {}^tBp_h) \, . \tag{4.20}$$

Therefore

$$\|(I - P_1) u_h\| = \sup_{\|z\| = 1} ((u_h, z))$$
$$\leqslant \left(\sup_{\|z\| = 1} |z| \right) |{}^t B p_h| . \tag{4.21}$$

On the other hand, we use the following expansion of z (see (3.15))

$$z = \sum_{k=0}^{+\infty} z_k, \quad z_k \in W_{h_k} . \tag{4.22}$$

Writting (3.24) to each z_k, i.e.

$$|z_k| \leqslant C \cdot \frac{h}{2^k} \cdot \|z_k\| , \tag{4.23}$$

we derive (as in (3.65), (3.66))

$$|z| \leqslant \sum_{k=0}^{+\infty} |z_k| \leqslant C \cdot h \cdot \sum_{k=0}^{+\infty} 2^{-k} \cdot \|z_k\|$$
$$\leqslant C \cdot h \cdot \left(\sum_k \|z_k\|^2 \right)^{1/2} \left(\sum_{k=0}^{+\infty} 4^{-k} \right)^{1/2}$$
$$\leqslant C \cdot h \cdot \|z\| , \tag{4.24}$$

since $\|z\|^2 = \sum_k \|z_k\|^2$ accordingly to (3.16). On the other hand, due to Lemma 2, we have the following

$$|{}^t B p_h| \leqslant C \, |{}^t B p| , \tag{4.25}$$

since p_h is the $[\cdot,\cdot]$-projection of p onto M_h, i.e. $p_h = \rho_h(p)$. We summarize (4.19), (4.21), (4.24) and (4.25) in

$$\|u_h - \tilde{u}_h\| \leqslant C \,.\, h \,.\, (|f| + |{}^t B p|) . \tag{4.26}$$

Hence (2.15) and (4.25) yield

$$\|u_h - \tilde{u}_h\| \leqslant C \,.\, h \,.\, |f| , \tag{4.27}$$

and (4.10) follows from (3.48) and (4.27). ■

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