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APPROXIMATION PROPERTIES OF PERIODIC INTERPOLATION BY TRANSLATES OF ONE FUNCTION (*)

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Communicated by P. J. LAURENT

To the memory of Professor Dr. Lothar COLLATZ

Abstract. — *In the paper [5] Golomb has derived a Hilbert space approach to periodic splines of odd degree on uniform meshes which were studied systematically for the first time by Quade and Collatz [9]. Golomb's approach has been extended to more general methods of periodic interpolation by translates of a given periodic function g [1, 2, 3, 8]. It is the objective of this paper to investigate approximation properties of these interpolation methods in spaces of periodic functions which are closely related to g and extend the results of [4]. As an application approximation properties of periodic splines of even degree are obtained.*

Résumé. — *Dans l'article [3] Golomb a présenté une construction hilbertienne des fonctions-spline périodiques de degré impair dans un réseau uniforme, fonctions étudiées systématiquement pour la première fois par Quade et Collatz [7]. La construction de Golomb a été généralisée pour les méthodes d'interpolation par les translates d'une fonction périodique g. Le but de cet article est d'étudier des propriétés d'approximation des méthodes d'interpolation par les translates de g dans des espaces des fonctions périodiques qui sont associées à la fonction g. En application, nous déduisons les propriétés d'approximation des fonctions-spline périodiques de degré pair.*

1. THE INTERPOLATION METHOD

Let g be a fixed real valued periodic function from the Wiener algebra $\mathscr{A}_{2\pi}$ of those continuous periodic functions from $\mathscr{C}_{2\pi}$ which possess an absolutely convergent Fourier series. The inner product of $f,\ g \in \mathscr{C}_{2\pi}$ is defined by

$$(f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, g(t)^*\, dt\ .$$

The exponential functions are denoted by $e_k(t) = \exp(ikt)$, $k \in \mathbb{Z}$. If

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$g \in \mathscr{A}_{2\pi}$, then

$$g(t) = \sum_{k=-\infty}^{\infty} (g, e_k) e_k(t), \quad \sum_{k=-\infty}^{\infty} |(g, e_k)| < \infty .$$

Let $t_j = 2\pi j/n$, $j \in \mathbb{Z}$, be a uniform mesh depending on $n \in \mathbb{N}$. The discrete inner product is given by

$$\langle f, g \rangle = \frac{1}{n} \sum_{k=0}^{n-1} f(t_k) g(t_k)^* .$$

The discrete Fourier transform and the finite Fourier transform are related by *aliasing :*

$$\langle f, e_j \rangle = \sum_s (f, e_{j+sn}) \tag{1}$$

where $j \in \mathbb{Z}$ and $f \in \mathscr{A}_{2\pi}$.

The *basis functions* b_j, $0 < j < n$, are obtained via the discrete Fourier transform from the generating function g :

$$b_j(t) = \langle g(t - \cdot\,), e_{-j} \rangle = \sum_s (g, e_{j+sn}) e_{j+sn}(t) . \tag{2}$$

We have the fundamental relations

$$b_j(t_k) = e_j(t_k) b_j(0) \quad (j, k \in \mathbb{Z}) . \tag{3}$$

We assume that

$$b_j(0) \neq 0 (0 < j < n) \tag{4}$$

and define

$$Q_n(f)(t) = \langle f, e_0 \rangle + \sum_{k=1}^{n-1} \langle f, e_k \rangle b_k(t)/b_k(0) \tag{5}$$

for any $f \in \mathscr{C}_{2\pi}$.

Discrete Fourier transform yields the interpolation properties

$$Q_n(f)(t_j) = f(t_j) \quad (0 \leq j < n) . \tag{6}$$

The space of interpolants is given by (see [2, 3])

$$V_n(g) = \langle 1, g(\cdot - t_1) - g, \dots, g(\cdot - t_{n-1}) - g \rangle . \tag{7}$$

We assume that the finite Fourier transform of the generating function

g possesses certain properties such that the *existence conditions* $b_j(0) \neq 0$ are valid and certain approximation properties can be established.

Let d_k, $k \in \mathbb{Z}$, be a sequence of real numbers satisfying the following relations

$$d_{-k} = d_k , \quad d_k > d_{k+1} > 0 \quad (k \in \mathbb{N}) .$$

Moreover we assume that

$$d_{rm} \leq \alpha_r d_m (r, m \in \mathbb{N}) , \quad \alpha := \sum_{r=1}^{\infty} \alpha_r < \infty .$$

We consider mainly two cases of generating functions g :

(I) $$g(t) = \sum_k d_k e_k(t) .$$

(II) $$g(t) = \sum_k - i \cdot \operatorname{sgn}(k) d_k e_k(-\pi/n) e_k(t) .$$

The generating function g defining the interpolation process is independent of the number n of interpolation points in case I while it is dependent in case II.

Trigonometric interpolation is characterized by the choice

(III) $$g(t) = \sum_{k=-m}^{m} e_k(t) (m = [n/2]) .$$

In this case g depends also on n. We refer to [3] for the proof of the validity of the existence condition.

Two examples are given for possible choices of the sequence (d_k).

Example a :

$$d_k = k^{-q} \quad (k \in \mathbb{N}) .$$

Here q is a real number greater than 1. If q is *even,* case I yields periodic odd degree spline interpolation on uniform mesh as studied by Golomb [3]. The generating function is the well known Bernoulli function $P_q(t)$ up to a factor

$$g(t) = \sum_{k=1}^{\infty} k^{-2r} \cos(kt) = \frac{(-1)^r}{2} P_{2r}(t) .$$

The function g is a spline of degree $2r$ with deficiency 1, while $g(\cdot - t_k) - g$ is a spline of degree $2r - 1$.

If q is odd, then case II yields periodic even degree midpoint spline interpolation on a uniform mesh [3]. The generating function is the *shifted*

Bernoulli function $P_q(t - \pi/n)$ up to a factor

$$g(t) = \sum_{k=1}^{\infty} k^{-2r-1} \sin(k(t - \pi/n)) = \frac{(-1)^r}{2} P_{2r+1}(t - \pi/n).$$

Example b :

$$d_k = e^{-bk} \quad (k \in \mathbb{N}).$$

Here b is a positive real number. Both cases I and II yield rational trigonometric interpolation processes (see [2, 3]).

In case I the generating function is related to the Poisson kernel

$$P_b(t) = \frac{\sinh(b)}{2(\cosh(b) - \cos(t))},$$

$$g(t) = \sum_{k=1}^{\infty} e^{-kb} \cos(kt) = P_b(t) - \frac{1}{2}.$$

In case II the generating function is related to the *shifted conjugate* Poisson kernel

$$Q_b(t) = \frac{\sin(t)}{2(\cosh(b) - \cos(t))},$$

$$g(t) = \sum_{k=1}^{\infty} e^{-kb} \sin(k(t - \pi/n)) = Q_b(t - \pi/n).$$

2. UNIFORM BOUNDEDNESS OF THE INTERPOLATION PROJECTORS

The Wiener algebra $\mathscr{A}_{2\pi}$ is a dense subalgebra of $\mathscr{C}_{2\pi}$. $\mathscr{A}_{2\pi}$ is a Banach space of continuous periodic functions with respect to the norm (see [6])

$$\|f\|_a = \sum_{k=-\infty}^{\infty} |(g, e_k)|.$$

Moreover, we have

$$\|f\|_\infty \leq \|f\|_a \quad (f \in \mathscr{A}_{2\pi})$$

where

$$\|f\|_\infty = \sup \{|f(t)| : t \in \mathbb{R}\}.$$

We will first investigate approximation properties of the Fourier partial sum projector which is in some sense a universal approximation operator. The

Fourier partial sum projector F_n is defined by

$$F_n(f) = \sum_{k=-m}^{m} (f, e_k) e_k \quad (m = [n/2]) .$$

Clearly, F_n is a bounded linear operator on $\mathscr{C}_{2\pi}$. On the other hand, it is well known that the norms $\|F_n\|$, $n \in \mathbb{N}$, are not uniformly bounded. According to the Banach Steinhaus principle [7], $F_n(f)$, $n \in \mathbb{N}$, is not convergent for every continuous periodic function. Therefore it is natural to consider F_n as a bounded linear operator from $\mathscr{A}_{2\pi}$ into $\mathscr{C}_{2\pi}$. It follows from the definition of the norms that

$$\|F_n(f)\|_\infty \leq \|f\|_a \quad (f \in \mathscr{A}_{2\pi}) .$$

Thus, the norms $\|F_n\|$, $n \in \mathbb{N}$, are uniformly bounded. Since

$$F_n(e_k) = e_k(|k| < n/2) ,$$

an application of the Banach Steinhaus principle yields uniform convergence

$$\lim_{n \to \infty} \|f - F_n(f)\|_\infty \quad (f \in \mathscr{A}_{2\pi}) .$$

Replacing $\mathscr{C}_{2\pi}$ by $\mathscr{A}_{2\pi}$ is not a severe restriction from the practical point of view since $\mathscr{A}_{2\pi}$ contains all Lipschitz-continuous periodic functions [6] and in particular those functions from $\mathscr{C}_{2\pi}$ which are piecewise $\mathscr{C}^1$ [10].

Next we will establish an analogous result for the interpolation projectors Q_n, $n \in \mathbb{N}$. The interpolation projector Q_n is a bounded linear projector from $\mathscr{A}_{2\pi}$ into $\mathscr{C}_{2\pi}$ with $Q_n(f) \in \mathscr{A}_{2\pi}$ for $f \in \mathscr{A}_{2\pi}$.

THEOREM 1 : *There is a positive constant c independent of n such that*

$$\|Q_n(f)\|_\infty \leq c \|f\|_a \quad (f \in \mathscr{A}_{2\pi}) ,$$

i.e., the interpolation projectors Q_n *are uniformly bounded as projectors from* $\mathscr{A}_{2\pi}$ *into* $\mathscr{C}_{2\pi}$.

Proof: We start with case I :

$$g(t) = \sum_k d_k e_k(t) .$$

Recall that

$$b_j(t) = \sum_s (g, e_{j+sn}) e_{j+sn}(t) = \sum_s d_{j+sn} e_{j+sn}(t)$$

where $0 < j < n$. Since $d_{j+sn} > 0$ we obtain

$$|b_j(t)| \leq b_j(0) , \quad 0 < j < n .$$

Next remember that

$$Q_n(f)(t) = \langle f, e_0 \rangle + \sum_{k=1}^{n-1} \langle f, e_k \rangle \, b_k(t)/b_k(0).$$

This implies

$$|Q_n(f)(t)| \leq \sum_{k=0}^{n-1} | \langle f, e_k \rangle | .$$

Since $\langle f, e_j \rangle = \sum_s (f, e_{j+sn})$, we can conclude

$$|Q_n(f)(t)| \leq \sum_{k=0}^{n-1} \left| \sum_s (f, e_{k+sn}) \right| \leq \sum_{k=0}^{n-1} \sum_s |(f, e_{k+sn})| = \|f\|_a .$$

Thus the assertion holds for case I.

Next we consider case II. Recall that

$$b_j(t) = \sum_s (g, e_{j+sn}) \, e_{j+sn}(t) .$$

Since

$$g(t) = \sum_k - i \cdot \operatorname{sgn}(k) \, d_k \, e_k(\pi/n) \, e_k(t) ,$$

we obtain

$$|b_j(t)| = \left| \sum_{s=0}^{\infty} d_{j+sn}(-1)^s e_{sn}(t) - \sum_{s=1}^{\infty} d_{-j+sn}(-1)^s e_{-sn}(t) \right| ,$$

$$|b_j(0)| = \sum_{s=0}^{\infty} (d_{j+sn} + d_{n-j+sn})(-1)^s$$

where $0 < j < n$.

Due to the properties of the sequence $d = (d_k)$ we obtain

$$|b_j(0)| \geq d_j - d_n \geq d_j(1 - \alpha_2) ,$$

$$|b_j(t)| \leq d_j + \sum_{s=1}^{\infty} d_{j+sn} + \sum_{s=1}^{\infty} d_{-j+sn} \leq d_j(1 + \alpha)$$

where $0 < j < n$. This implies

$$|b_j(t)/b_j(0)| \leq (1 + \alpha)/(1 - \alpha_2)$$

where $t \in \mathbb{R}$ and $0 < t < n$. Proceeding as in the proof for case I we can conclude

$$|Q_n(f)(t)| \leq (1+\alpha)/(1-\alpha_2)\|f\|_u .$$

Thus the assertion holds also for case II.

For completeness we consider also case III (trigonometric interpolation). Recall that

$$g(t) = \sum_{k=-m}^{m} e_k(t) \quad (m = [n/2]) ,$$

$$b_j(t) = \sum_s (g, e_{j+sn}) e_{j+sn}(t) .$$

This implies

$$b_j(t) = e_j(t) , \quad 0 < j < n/2 , \quad b_j(t) = e_{-n+j}(t) , \quad n/2 < j < n$$

for odd n and

$$b_j(t) = e_j(t) , \quad 0 < j < n/2 , \quad b_j(t) = e_{-n+j}(t) , \quad n/2 < j < n ,$$

$$b_{n/2}(t) = e_{n/2}(t) + e_{-n/2}(t)$$

for even n. Again we have

$$|b_j(t)/b_j(0)| \leq 1 , \quad 0 < j < n .$$

Thus the assertion holds also for case III.

To establish uniform convergence of $Q_n(f)$ for $f \in \mathscr{A}_{2\pi}$ via the Banach-Steinhaus principle [7] we have to verify uniform convergence of $Q_n(f)$ on a dense subset of $\mathscr{A}_{2\pi}$. This will be done in the following section.

3. APPROXIMATION OF THE EXPONENTIALS

The approximation properties of Q_n will first be investigated for the exponential functions e_j.

THEOREM 2 : *For any $k \in \mathbb{Z}$ the asymptotic error relation*

$$\|e_k - Q_n(e_k)\|_\infty = \mathcal{O}(d_{[n/2]}) \quad (n \to \infty)$$

holds.

Proof : We consider case I. Then we have

$$g(t) = \sum_k d_k e_k(t) = d_0 + \sum_{k=1}^{\infty} 2 d_k \cos(kt) ,$$

i.e., $g(t)$ is real valued. This implies $Q_n(\bar{f}) = \overline{Q_n(f)}$.

Since $\bar{e}_k = e_{-k}$ and $Q_n(e_0) = e_0$ we may assume $0 < k \leq n/2$. Recall that

$$b_k(t) = \sum_s d_{k+sn}\, e_{k+sn}(t) = e_k(t) \sum_s d_{k+sn}\, e_{sn}(t)\,.$$

Then we obtain

$$\begin{aligned}|e_k(t) - Q_n(e_k)(t)| &= |e_k(t) - b_k(t)/b_k(0)| = \\ &= |b_k(0) - b_k(t)\, e_{-k}(t)| / |b_k(0)| \\ &\leq \sum_{s \neq 0} 2\, d_{k+sn}/d_k \leq \sum_{s=1}^{\infty} 2\, d_{s[n/2]}/d_k \leq (2\,\alpha/d_k)\, d_{[n/2]}\,.\end{aligned}$$

Thus, the assertation holds for case I.

We consider now case II. Then we have

$$g(t) = \sum_k -i\,.\,\mathrm{sgn}\,(k)\, d_k\, e_k(-\pi/n)\, e_k(t) = \sum_{k=1}^{\infty} 2\, d_k \sin\,(k(t - \pi/n))\,,$$

i.e., $g(t)$ is real valued which implies $Q_n(\bar{f}) = \overline{Q_n(f)}$. Since $\bar{e}_k = e_{-k}$ and $Q_n(e_0) = e_0$, again we may assume $0 < k \leq n/2$. Recall that

$$b_k(t) = -i\,.\,e_k(t) \left(\sum_{s=0}^{\infty} d_{k+sn}(-1)^s\, e_{sn}(t) - \sum_{s=1}^{\infty} d_{-k+sn}(-1)^s\, e_{-sn}(t) \right)\,.$$

Then we obtain

$$\begin{aligned}|e_k(t) - Q_n(e_k)(t)| &= |e_k(t) - b_k(t)/b_k(0)| = \\ &= |b_k(0) - b_k(t)\, e_{-k}(t)| / |b_k(0)| \\ &\leq \left| \sum_{s=1}^{\infty} d_{k+sn}(-1)^s(1 - e_{sn}(t)) - \sum_{s=1}^{\infty} d_{-k+sn}(-1)^s(1 - e_{-sn}(t)) \right| / |b_k(0)| \\ &\leq (2\,\alpha/((1 - \alpha_2)\, d_k))\, d_{[n/2]}\,.\end{aligned}$$

Thus, the assertation holds also for case II.

For the sake of completeness we note that Theorem 2 is trivially true for case III (*trigonometric interpolation*) since in this case we have

$$e_k = Q_n(e_k) \quad (|k| < [n/2])\,.$$

THEOREM 3 : *For any* $f \in \mathcal{A}_{2\pi}$ $Q_n(f)$ *converges uniformly to* f *as* n *tends to infinity :*

$$\|f - Q_n(f)\|_{\infty} \to 0 \quad (n \to \infty)\,.$$

Proof: Note that the exponentials form a dense subalgebra of $\mathscr{A}_{2\pi}$. In view of Theorem 1 and Theorem 2 the Banach Steinhaus principle is applicable and yields a proof of Theorem 3.

4. QUANTITATIVE ERROR BOUNDS

We use the sequence $d = (d_k)$ to introduce a *smooth* subspace $\mathscr{A}_{2\pi}^d$ of the Wiener algebra $\mathscr{A}_{2\pi}$:

$$\mathscr{A}_{2\pi}^d := \left\{ f \in \mathscr{A}_{2\pi} : \sum_{k \neq 0} |(f, e_k)|/d_k < \infty \right\}.$$

Then we can introduce a linear operator $D : \mathscr{A}_{2\pi}^d \to \mathscr{A}_{2\pi}$ being defined by

$$D(f) = \sum_{k \neq 0} d_k^{-1} (f, e_k) e_k .$$

The space $\mathscr{A}_{2\pi}^d$ and the operator D are related to the generating function g and the interpolation projector Q_n. It is useful to investigate first the approximation properties in $\mathscr{A}_{2\pi}^d$ of the Fourier partial sum projector F_n.

THEOREM 4 : *If* $f \in \mathscr{A}_{2\pi}^d$ *then*

$$\|f - F_n(f)\|_\infty = o(d_{[n/2]}) \quad (n \to \infty).$$

Proof: Note first that

$$F_n(D(f)) = D(F_n(f)) \quad (f \in \mathscr{A}_{2\pi}^d).$$

Then we have

$$|f(t) - F_n(f)(t)| \leq \sum_{|k| > [n/2]} |(f, e_k)| = \|f - F_n(f)\|_a =$$

$$= \sum_{|k| > [n/2]} |(D(f) f, e_k)| \, d_k \leq \left(\sum_{|k| > [n/2]} |(D(f) f, e_k)| \right) d_{[n/2]}$$

$$= \|D(f) - F_n(D(f))\|_a \cdot d_{[n/2]},$$

i.e., we have

$$\|f - F_n(f)\|_\infty \leq \|f - F_n(f)\|_a \leq \|D(f) - F_n(D(f))\|_a \cdot d_{[n/2]}.$$

Since $D(f) \in \mathscr{A}_{2\pi}$, $\|D(f) - F_n(D(f))\|_a = o(1)(n \to \infty)$, and the proof of Theorem 4 is complete.

THEOREM 5 : *If* $f \in \mathscr{A}^d_{2\pi}$ *then*

$$\|f - Q_n(f)\|_\infty = \mathcal{O}(d_{[n/2]}) \quad (n \to \infty) .$$

Proof : We have

$$|f(t) - Q_n(f)(t)| \le |f(t) - F_n(f)(t)| + |F_n(f)(t) - Q_n(F_n(f))(t)| + |Q_n(F_n(f))(t) - Q_n(f)(t)| .$$

In view of Theorem 1 we have

$$|Q_n(F_n(f))(t) - Q_n(f)(t)| \le c \|f - F_n(f)\|_a .$$

This implies

$$\begin{aligned} |f(t) - Q_n(f)(t)| &\le C \|f - F_n(f)\|_a + |F_n(f)(t) - Q_n(F_n(f))(t)| \\ &\le C \|f - F_n(f)\|_a + \sum_{|k| \le [n/2]} |(f, e_k)| \cdot \|e_k - Q_n(e_k)\|_\infty \end{aligned}$$

where $C > 0$ is a constant.

In view of Theorem 2 we can conclude

$$\begin{aligned} &|f(t) - Q_n(f)(t)| \le \\ &\le C \|f - F_n(f)\|_a + \left(\sum_{0 < |k| \le [n/2]} |(f, e_k)| \right) \cdot (2\,\alpha/((1 - \alpha_2)\, d_k))\, d_{[n/2]} \\ &\le C \|f - F_n(f)\|_a + (2\,\alpha/(1 - \alpha_2))\, d_{[n/2]}\, \|D(f)\|_a . \end{aligned}$$

Now an application of Theorem 4 completes the proof of Theorem 5.

The approximation properties of the trigonometric interpolation projector are more closely related to those of the Fourier partial sum projector.

THEOREM 6 : *Let* Q_n *be the projector of trigonometric interpolation. Then for any* $f \in \mathscr{A}^d_{2\pi}$ *the asymptotic error relation*

$$\|f - Q_n(f)\|_\infty = o(d_{[n/2]}) \quad (n \to \infty)$$

holds.

Proof : As in the proof of Theorem 5 we have

$$|f(t) - Q_n(f)(t)| \le |f(t) - F_n(f)(t)| + |F_n(f)(t) - Q_n(F_n(f))(t)| + |Q_n(F_n(f))(t) - Q_n(f)(t)| .$$

Since Q_n is the trigonometric interpolation projector, we have

$$F_n(f) = Q_n(F_n(f)) .$$

Taking into account Theorem 1 we obtain

$$|f(t) - Q_n(f)(t)| \leqslant C \, \|f - F_n(f)\|_a$$

where $C > 0$ is a constant. Now an application of Theorem 4 completes the proof.
As an application we consider

$$d_k = k^{-q} \quad (k \in \mathbb{N},\ q \in \mathbb{N},\ q > 1\ ).$$

Then we have

$$\mathscr{A}^d_{2\pi} := \left\{ f \in \mathscr{A}_{2\pi} : \sum_{k \neq 0} |k|^q \, |(f, e_k)| < \infty \right\},$$

i.e.,

$$\mathscr{A}^d_{2\pi} = \{ f \in \mathscr{C}^q_{2\pi} : D^q f \in \mathscr{A}_{2\pi} \}$$

where $D^q f$ denotes the q-th derivative of f.

THEOREM 7 : *For any $f \in \mathscr{C}^q_{2\pi}$ with $D^q f \in \mathscr{A}_{2\pi}$ the asymptotic error relation*

$$\|f - Q_n(f)\|_\infty = \mathcal{O}(n^{-q}) \quad (n \to \infty)$$

holds.

If $q = 2r$, $r \in \mathbb{N}$, case I yields the error estimate of Golomb [5] for odd degree periodic splines.

If $q = 2r + 1$, $r \in \mathbb{N}$, case II extends the error estimate of Golomb to even degree periodic midpoint splines.

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