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B-RATIONAL CURVES AND REPARAMETRIZATION : THE QUADRATIC CASE

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Abstract. — *The notion of massic vector, previously introduced, allows us to represent any rational curve as a* B*-rational* (*BR*) *curve, such a BR-curve being completely determined by its massic polygon. For computational and stability purposes we are led to determine the massic polygon resulting from a one-to-one quadratic change of parameter which allows a description of the whole support with a parameter belonging to* [0, 1[.

Keywords : Rational curves, *BR*-curves, quadratic change of parameter, homogeneous coordinates, massic polygons, ρ-reciprocity, Bézier curves.

Résumé. — *La notion de vecteur massique, récemment introduite, nous permet de représenter toute courbe rationnelle comme une courbe* B*-rationnelle (BR), une telle courbe (BR) étant complètement déterminée par son polygone massique. Dans l'objectif de calculer une courbe rationnelle de manière stable nous sommes amenés à déterminer le polygone massique résultant d'un changement quadratique biunivoque du paramètre. Ce nouveau polygone massique permet alors une description de tout le support avec un paramètre parcourant* [0, 1[.

1. INTRODUCTION

In [5, 6, 7] a general framework is given to represent any rational curve by a massic polygon. Such a representation is called a B-rational (*BR*) curve, it includes the Bézier (*BP*) curves [1, 2], i.e., polynomial curves defined over a bounded interval and the rational Bézier curves [4, 3]. *BR*-curves were introduced and analysed to enable rational curves to be exploited in computer

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science, CAGD and CAM. Example 1 following proposition 1 illustrates the process in order to obtain a massic polygon of a BR-curve.

The problem of change of parameter crops up in the study of parametrized curves in order to give equivalent representation of their supports, and more significantly for computational and stability purposes. In [8] we studied the homographic (and affine) change of parameter, for BR-curves. This paper is devoted to the quadratic case. Usually a rational curve is parametrized over $\tilde{\mathbb{R}}$. We want to parametrize the support of such a curve over a finite interval which will be $[0, 1[$. The simplest function allowing us to make this change is a special quadratic function introduced in [6, section 2.3] which provides a one-to-one correspondence between $\tilde{\mathbb{R}}$ and $[0, 1[$. Hence the support of a BR-curve of length n with parameter $t \in \tilde{\mathbb{R}}$ is identical to that of a BR-curve of length $2\,n$ with parameter $u \in [0, 1[$. In [6, section 2.3] such a representation is given for the circle, the folium of Descartes, the skew cubic and the window of Viviani. In this paper we propose a systematic study of the problem.

We are led to treat this problem for computational and stability purposes. For instance the question of overflow arises when calculating points of a rational curve which are at finite distance, image of parameter values belonging to a neighbourhood of infinity. The proposed quadratic change of parameter bringing back the parameter in $[0, 1[$ provides an answer to this question. Moreover with a parameter belonging to $[0, 1[$ we can calculate the curve by extensions of the De Casteljau's algorithm, called ALBR1 and ALBR2 [6, sections 3.3 and 3.4], by handling affine combinations that are convex, which would not be the case with a curve defined over $\tilde{\mathbb{R}}$. This property offers a guarantee of stability.

The paper is organized as follows. In the first section we recall the general framework and some results concerning the BR-curves. This material comes from [6]. In the second section we make, for a BR-curve of length n, a one-to-one quadratic change of parameter. The new parameter belongs to $[0, 1[$. The curve obtained is again a rational curve, of length $2\,n$. The new massic vectors are determined as linear combinations of the previous ones. The new massic polygon presents a kind of symmetry called ρ-reciprocity. These $(2\,n + 1)$ massic vectors are completely determined by the first $(n + 1)$. The third section gives an algorithm to calculate the triangular matrix defining the new massic vectors from the old ones. In the fourth section we apply this process to the polynomial curves. Three examples show the simplicity of it. Another development of this work is done in [9, 10]. This paper is written so as to be self-contained.

2. GENERAL FRAMEWORK. *BR*-FORM OF A RATIONAL CURVE.

According to [5, 6, 7] we recall some basic definitions and results concerning the representation of rational curves by BR-curves.

Let us define $\mathscr{E}$ (resp. $\mathscr{F}$) a real affine space, $\vec{\mathscr{E}}$ (resp. $\vec{\mathscr{F}}$) its associated linear vector space such that $\mathscr{E}$ (resp. $\vec{\mathscr{E}}$) is an hyperplane of $\mathscr{F}$ (resp. $\vec{\mathscr{F}}$) and $\tilde{\mathscr{E}}$ the projective completion of $\mathscr{E}$; In general, $\mathscr{E}$ is a 2 or 3-dimensional affine space.

Let Ω be a point of $\mathscr{F}$ not belonging to $\mathscr{E}$.

We define the linear vector space $\hat{\mathscr{E}} = (\mathscr{E} \times \mathbb{R}^*) \cup \vec{\mathscr{E}}$ called the *massic vector space* [6] : $\theta \in \hat{\mathscr{E}}$ is called a *massic vector*, it is either a weighted point or a pure vector.

The isomorphism $\hat{\Omega} : \hat{\mathscr{E}} \to \vec{\mathscr{F}}$ defined by $\hat{\Omega}(P, \alpha) = \alpha \overrightarrow{\Omega P}$, $\hat{\Omega}(\vec{u}) = \vec{u}$ induces an addition operator and an external multiplication in $\hat{\mathscr{E}}$, respectively denoted by $\oplus$ and $*$, such that $\hat{\mathscr{E}}$ is a linear space and $\hat{\Omega}$ is an isomorphism.

For any θ and $\theta' \in \hat{\mathscr{E}}$ and $\lambda \in \mathbb{R}$: $\theta \oplus \theta' = \hat{\Omega}^{-1}(\hat{\Omega}(\theta) + \hat{\Omega}(\theta'))$ and $\lambda * \theta = \hat{\Omega}^{-1}(\lambda \,.\, \hat{\Omega}(\theta))$. Proposition 1.2.1.6 in [6] gives the rules of use of $\oplus$ and $*$ operators. We consider the linear form $\chi : \hat{\mathscr{E}} \to \mathbb{R}$; $\chi(P\,;\alpha) = \alpha$, $\chi(\vec{u}) = 0$; $\chi(\theta)$ is called the *mass* of θ.

Let $\Pi\Omega : \vec{\mathscr{F}} - \{\Omega\} \to \tilde{\mathscr{E}}$ be the conic projection of apex Ω and $\Pi : \hat{\mathscr{E}} - \{\vec{0}\} \to \tilde{\mathscr{E}}$ be the natural projection : $\Pi(P\,;\alpha) = P$, $\Pi(\vec{u}) = (\vec{u})_\infty$. They are linked by the relation $\Pi = \Pi\Omega \circ \hat{\Omega}$ [6, proposition 1.2.2.3]. We have : $\forall \lambda \neq 0$, $\Pi(\lambda\theta) = \Pi(\theta)$.

DEFINITION 1 OF A B-RATIONAL CURVE : *Let $\theta_0, \theta_1, \ldots, \theta_n$ be $(n+1)$ massic vectors not simultaneously null ; a* B-*rational (BR)-curve of $\tilde{\mathscr{E}}$, of controlling massic polygon $\theta = (\theta_0, \theta_1, \ldots, \theta_n)$, denoted by $BR[\theta_0, \theta_1, \ldots, \theta_n]$ or $BR[\theta]$, is described by the point :*

$$BR[\theta_0, \theta_1, \ldots, \theta_n]\,(t) = \Pi(BP\,[\theta_0, \theta_1, \ldots, \theta_n]\,(t))\,;$$

$BP\,[\theta_0, \theta_1, \ldots, \theta_n] = \sum_{i=0}^{n} B_i^n(t) * \theta_i$ *is the Bézier curve in $\hat{\mathscr{E}}$, n is called the length of the (BR)-curve ; $B_i^n(t) = \binom{n}{i}(1-t)^{n-i}\,t^i$, $i = 0, 1, \ldots, n$ are the Bernstein basis polynomials relatively to* $[0, 1]$.

EQUIVALENT DEFINITION 2 : *Considering $R_i \in \mathscr{F}$ defined by $\overrightarrow{\Omega R_i} = \hat{\Omega}(\theta_i)$ we have :*

$$BR[\theta_0, \theta_1, \ldots, \theta_n]\,(t) = \Pi\Omega\,(BP\,[R_0, R_1, \ldots, R_n]\,(t))\,.$$

EXPLICIT FORM : *Define* $I = \{i : \theta_i \in \mathscr{E} \times \mathbb{R}^*,\ \theta_i = (P_i ; \beta_i)\}$ *and* $\bar{I} = \left\{i : \theta_i \in \vec{\mathscr{E}},\ \theta_i = \vec{U}_i\right\}$. *We have* $I \cup \bar{I} = \{0, 1, \ldots, n\}$, $I \cap \bar{I} = \varnothing$. *From Definition 1 above and preliminary results [[6] proposition 1.3] we obtain the explicit form of a (BR) curve :*

$$BR[\theta](t) = \frac{\sum_{i \in I} \beta_i B_i^n(t) P_i}{\beta(t)} + \frac{\sum_{i \in \bar{I}} B_i^n(t) \vec{U}_i}{\beta(t)} \tag{1}$$

if $\beta(t) \neq 0$ *and where* $\beta(t) = \sum_{i \in I} \beta_i B_i^n(t)$.

$$BR[\theta](t) = \left(\sum_{i \in I} \beta_i B_i^n(t) \vec{P}_i + \sum_{i \in \bar{I}} B_i^n(t) \vec{U}_i\right)_\infty \tag{2}$$

if $\beta(t) = 0$ *and* $\vec{V}(t) = \sum_{i \in I} \beta_i B_i^n(t) \vec{P}_i + \sum_{i \in \bar{I}} B_i^n(t) \vec{U}_i \neq \vec{0}$

$$BR[\theta](t_0) = \lim_{t \to t_0} BR[\theta](t) \text{ if } \beta(t_0) = 0 \text{ and } \vec{V}(t_0) = \vec{0}\,. \tag{3}$$

Remark 1 :

(i) When $\bar{I} = \varnothing$, the BR-curve is reduced to a rational Bézier curve [4, 3].

(ii) When $\bar{I} = \varnothing$ and $\theta_i = (P_i ; c)$, $i = 0, 1, \ldots, n$, c being a constant, then $\beta(t) = c$ and the BR-curve is reduced to a polynomial Bézier (BP) curve [1, 2].

THEOREM 1 [6, proposition 2.2.2.2] : *Any rational (or unicursal) curve is a BR-curve and conversely.*

3. QUADRATIC CHANGE OF PARAMETER FOR A *BR*-CURVE. BASIC RESULTS

Let $BR[\omega]$ be a BR-curve of length n with massic polygon $\omega = (\omega_0, \omega_1, \ldots, \omega_n)$.

We consider the quadratic change of parameter introduced in [6, section 2.3] : $t = \Phi(u)$

$$\Phi(u) = \frac{\alpha B_0^2(u) + \beta B_1^2(u) + \gamma B_2^2(u)}{B_1^2(u)},$$

α, β, γ being three arbitrary constants with $\alpha\gamma < 0$.

This last condition ensures that whatever β, Φ is a one-to-one correspondence between $[0, 1[$ and $\tilde{\mathbb{R}}$ (Φ is increasing if $\alpha < 0$, decreasing if $\alpha > 0$). We notice that $\Phi(0) = \Phi(1) = \infty$. Then the curve described by

$BR[\omega](t)$ with $t \in \tilde{\mathbb{R}}$ is identical to that described by $BR[\omega](\Phi(u))$ with $u \in [0, 1[$. We will see that $BR[\omega](\Phi(u))$ is a BR-curve of length $2n$:

$$BR[\omega](\Phi(u)) = BR[\theta_0, \theta_1, \ldots, \theta_{2n}](u).$$

The aim of this paper is to calculate the new massic polygon θ as a function of ω. Let us recall that if θ describes a BR-curve and $\lambda \in \mathbb{R}^*$, then $\lambda\theta = (\lambda\theta_1, \lambda\theta_2, \ldots, \lambda\theta_{2n})$ describes the same curve.

LEMMA 1 : *Let $F(T_1, T_2)$ be a homogeneous polynomial (vector-valued or not) of degree n in T_1, T_2 and let $P_1(U_1, U_2)$, $P_2(U_1, U_2)$ be two homogeneous polynomials of degree 2 in U_1, U_2 such that*

$$P_1(\gamma U_1, \alpha U_2) = \alpha\gamma P_1(U_2, U_1)$$
$$P_2(\gamma U_1, \alpha U_2) = \alpha\gamma P_2(U_2, U_1).$$

Then $G(U_1, U_2) = F(P_1(U_1, U_2), P_2(U_1, U_2))$ is a homogeneous polynomial of degree $2n$ in U_1, U_2. It can be written :

$$G(U_1, U_2) = \sum_{i=0}^{2n} b_i U_1^{2n-i} U_2^i$$

with b_i satisfying $b_{2n-i} = \rho^{n-i} b_i$, $i = 0, 1, \ldots, n$, where $\rho = \dfrac{\gamma}{\alpha}$.

Proof : Successively we have :

$$\begin{aligned} G(\gamma U_1, \alpha U_2) &= F(P_1(\gamma U_1, \alpha U_2), P_2(\gamma U_1, \alpha U_2)) \\ &= F(\alpha\gamma P_1(U_2, U_1), \alpha\gamma P_2(U_2, U_1)) \\ &= \alpha^n \gamma^n F(P_1(U_2, U_1), P_2(U_2, U_1)) \\ G(\gamma U_1, \alpha U_2) &= \alpha^n \gamma^n G(U_2, U_1). \end{aligned}$$

Replacing G by $\sum_{i=0}^{2n} b_i U_1^{2n-i} U_2^i$ on both sides of this last equality we obtain the result.

DEFINITION 3 : *A sequence $b = (b_0, b_1, \ldots, b_{2n})$ satisfying :*

$$b_{2n-i} = \rho^{n-i} b_i, \quad i = 0, 1, \ldots, n, \quad \rho \neq 0,$$

is said to be ρ-reciprocal.

THEOREM 2 : *Let $BR[\omega]$ be a BR-curve of length n and let Φ be the quadratic function defined as above. Then $BR[\omega](\Phi(u))$ can be written in the following form :*

$$BR[\omega](\Phi(u)) = BR[\theta_0, \theta_1, \ldots, \theta_{2n}](u).$$

The massic polygon $\theta = (\theta_0, \theta_1, \ldots, \theta_{2n})$ *is* $\frac{\gamma}{\alpha}$-reciprocal :

$$\theta_{2n-i} = \left(\frac{\gamma}{\alpha}\right)^{n-i} \theta_i\,, \quad i = 0, 1, \ldots, n\,.$$

Proof : By (T_1, T_2) we denote the homogeneous cartesian coordinates of t relatively to the cartesian projective frame of reference $(\infty, 0, 1)$ of the projective line $\tilde{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$:

$$t = \frac{T_1}{T_2} \quad \text{if} \quad T_2 \neq 0\,, \quad t = \infty \quad \text{for} \quad T_2 = 0\,.$$

By (U_1, U_2) we denote the homogeneous barycentric coordinates of u relatively to the barycentric projective frame of reference $\left(0, 1, \frac{1}{2}\right)$ of $\tilde{\mathbb{R}}$:

$$u \in \mathbb{R} : \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \lambda \begin{pmatrix} 1-u \\ u \end{pmatrix}, \quad u = \infty : \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \lambda \neq 0\,.$$

The quadratic change of parameter $t = \Phi(u)$ can be written again in the following form :

$$\begin{aligned} T_1 &= \alpha U_1^2 + 2\,\beta U_1 U_2 + \gamma U_2^2 \\ T_2 &= 2\, U_1 U_2\,. \end{aligned}$$

By $T_1(U_1, U_2)$ and $T_2(U_1, U_2)$ we denote the two above-mentioned polynomials. Δ being the symbol for forward difference operator, successively we have :

$$\begin{aligned} BR[\omega](t) &= \Pi\left(\sum_{i=0}^{n} B_i^n(t)\, \omega_i\right) \\ &= \Pi\left(\sum_{i=0}^{n} \binom{n}{i}\, t^i\, \Delta^i \omega_0\right) \\ &= \Pi\left(\frac{1}{T_2^n} \sum_{i=0}^{n} \binom{n}{i}\, T_1^i\, T_2^{n-i}\, \Delta^i \omega_0\right) \\ &= \Pi\left(\sum_{i=0}^{n} \binom{n}{i}\, T_1^i\, T_2^{n-i}\, \Delta^i \omega_0\right). \end{aligned}$$

Let us define $F(T_1, T_2) = \sum_{i=0}^{n} \binom{n}{i}\, T_1^i\, T_2^{n-i}\, \Delta^i \omega_0$. F is a vector-valued homogeneous polynomial of degree n in T_1, T_2.

$T_1(U_1, U_2)$ and $T_2(U_1, U_2)$ are homogeneous polynomials of degree 2 in U_1, U_2 having the following property :

$$T_1(\gamma U_1, \alpha U_2) = \alpha \gamma T_1(U_2, U_1)$$
$$T_2(\gamma U_1, \alpha U_2) = \alpha \gamma T_2(U_2, U_1) .$$

From F we define $G(U_1, U_2) = F(T_1(U_1, U_2), T_2(U_1, U_2))$, so G is a homogeneous polynomial of degree $2n$ in U_1, U_2. It can be written

$$G(U_1, U_2) = \sum_{j=0}^{2n} \binom{2n}{j} U_1^{2n-j} U_2^j \theta_j \quad \text{with} \quad \theta_j \in \hat{\mathscr{E}} .$$

We have :

$$\begin{aligned} BR[\omega](\Phi(u)) &= \Pi(F(T_1(U_1, U_2), T_2(U_1, U_2))) \\ &= \Pi\left(\sum_{j=0}^{2n} \binom{2n}{j} U_1^{2n-j} U_2^j \theta_j\right) \\ &= \Pi\left(\lambda^{2n} \sum_{j=0}^{2n} B_j^{2n}(u)\, \theta_j\right) \\ &= \Pi\left(\sum_{j=0}^{2n} B_j^{2n}(u)\, \theta_j\right) \end{aligned}$$

$$BR[\omega](\Phi(u)) = BR[\theta_0, \theta_1, \ldots, \theta_{2n}](u) .$$

The polynomials $F(T_1, T_2)$, $T_1(U_1, U_2)$, $T_2(U_1, U_2)$ satisfying the hypothesis of lemma 1, it follows that coefficients of G verify the relations

$$\binom{2n}{2n-i} \theta_{2n-i} = \left(\frac{\gamma}{\alpha}\right)^{n-i} \binom{2n}{i} \theta_i , \quad i = 0, 1, \ldots, n$$
$$\text{i.e., } \theta_{2n-i} = \left(\frac{\gamma}{\alpha}\right)^{n-i} \theta_i , \quad i = 0, 1, \ldots, n .$$

COROLLARY 1 : *The support of* $BR[\theta](u)$ *when* $u \in [0, 1[$ *is identical to that of* $BR[\omega](t)$ *when* $t \in \tilde{\mathbb{R}}$.

Proof : It comes from $BR[\omega](u) = BR[\omega](\Phi(u))$ and the properties of Φ.

Remark 2 : Theorem 2 states that it is sufficient to know $\theta_0, \theta_1, \ldots, \theta_n$ to determine entirely $\theta = (\theta_0, \theta_1, \ldots, \theta_{2n})$.

PROPOSITION 1 : *With the notation above, for* $i = 0, 1, \ldots, n$ *we let*

$$T_1^i T_2^{n-i} = \sum_{j=0}^{2n} a_i^j U_1^{2n-j} U_2^j .$$

By K we denote the $(2n+1) \times (n+1)$ *matrix with* a_i^j *as entries at row* j *and column* i *and by* $\hat{K}$ *the submatrix of* K *reduced to its* $n+1$ *first rows. Then it follows :*

(i) *The sequence of the* $2n+1$ *rows of* K *are* $\frac{\gamma}{\alpha}$*-reciprocal* ($\hat{K}$ *determines* K *entirely),*

(ii) *the entries of* $\hat{K}$ *above the secondary diagonal are null* ($a_i^j = 0$ *for* $i+j = n-1, n-2, \dots, 0$),

(iii) *the rising secondary diagonal entries of* $\hat{K}$ *are* $(2^n, \alpha 2^{n-1}, \alpha^2 2^{n-2}, \dots, \alpha^n)$,

(iv) *determinant of* $\hat{K} = (-2\alpha)^{\frac{n(n+1)}{2}}$,

(v) *the massic vectors* $\theta_0, \theta_1, \dots, \theta_n$ *are related to* $\omega_0, \omega_1, \dots, \omega_n$ *by the following matrix relation :*

$$\begin{pmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_n \end{pmatrix} = D_1^{-1} \hat{K} D_2 \begin{pmatrix} \omega_0 \\ \Delta\omega_0 \\ \vdots \\ \Delta^n \omega_0 \end{pmatrix}$$

with

$$D_1 = \operatorname{diag}\left(\binom{2n}{0}, \binom{2n}{1}, \dots, \binom{2n}{n} \right)$$

and

$$D_2 = \operatorname{diag}\left(\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n} \right).$$

Proof : (i) $T_1^i T_2^{n-i}$ is a homogeneous polynomial of degree n in T_1, T_2 ; $T_1(U_1, U_2)$ and $T_2(U_1, U_2)$ are homogeneous polynomials of degree 2 in U_1, U_2. According to the lemma 1 we deduce that $T_1^i T_2^{n-i} = \sum_{j=0}^{2n} a_i^j U_1^{2n-j} U_2^j$ with $a_i^{2n-j} = \left(\frac{\gamma}{\alpha}\right)^{n-j} a_i^j$, $j = 0, 1, \dots, n$.

The sequence $(a_i^0, a_i^1, \dots, a_i^n, \dots, a_i^{2n})$ is the i-th column of K and is $\frac{\gamma}{\alpha}$-reciprocal for each $i = 0, 1, \dots, n$.

It follows that the sequence of rows of $K = (K^0, K^1, \dots, K^n, \dots, K^{2n})$ is $\frac{\gamma}{\alpha}$-reciprocal : $K^{2n-j} = \left(\frac{\gamma}{\alpha}\right)^{n-j} K^j$ for $j = 0, 1, \dots, n$.

(ii) Expanding $T_1^i T_2^{n-i}$ in decreasing powers of U_1 :

$$\begin{aligned} T_1^i T_2^{n-i} &= (\alpha U_1^2 + 2\beta U_1 U_2 + \gamma U_2^2)^i \, 2^{n-i} U_1^{n-i} 2^{n-i} \\ &= 2^{n-i} \alpha^i U_1^{n+i} U_2^{n-i} + \dots , \end{aligned}$$

it gives : $a_i^0 = a_i^1 = \cdots = a_i^{n-i-1} = 0$ for $i = 0, 1, \ldots, n-1$ and $a_i^{n-i} = 2^{n-i}\,\alpha^i$ for $i = 0, 1, \ldots, n$.

(iii) From above the rising secondary diagonal is $(2^n, 2^{n-1}\,\alpha, \ldots, 2^{n-i}\,\alpha^i, \ldots, \alpha^n)$.

(iv) Determinant of $\hat{K}$ is equal to $(-1)^{\frac{n(n+1)}{2}}$ times the product of the rising secondary diagonal entries i.e., determinant of $\hat{K} = (-2\,\alpha)^{\frac{n(n+1)}{2}}$.

(v) We know (previous proof) that :

$$BR[\omega](t) = \Pi\left(\sum_{i=0}^{n} \binom{n}{i} \Delta^i \omega_0 T_1^i T_2^{n-i}\right),$$

then

$$\begin{aligned} BR[\omega](\Phi(u)) &= \Pi\left(\sum_{i=0}^{n} \binom{n}{i} \Delta^i \omega_0 \left(\sum_{j=0}^{2n} a_i^j U_1^{2n-j} U_2^j\right)\right) \\ &= \Pi\left(\sum_{j=0}^{2n} \left(\sum_{i=0}^{n} \binom{n}{i} a_i^j \Delta^i \omega_0\right) U_1^{2n-j} U_2^j\right) \\ &= \Pi\left(\sum_{j=0}^{2n} \binom{2n}{j} U_1^{2n-j} U_2^j \theta_j\right) \end{aligned}$$

with

$$\theta_j = \frac{1}{\binom{2n}{j}} \sum_{i=0}^{n} \binom{n}{i} a_i^j \Delta^i \omega_0, \quad j = 0, 1, \ldots, 2n.$$

By the $\frac{\gamma}{\alpha}$-reciprocal property of the θ_i, $i = 0, 1, \ldots, 2n$, we only calculate the $(n+1)$ first massic vectors θ_i. They are related with $\omega_0, \omega_1, \ldots, \omega_n$ by the matrix relation given in the proposition with

$$\hat{K} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 & a_n^0 \\ 0 & 0 & \cdots & 0 & \cdots & a_{n-1}^1 & a_n^1 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a_i^{n-i} & \cdots & a_{n-1}^{n-i} & a_n^{n-i} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & a_1^{n-1} & \cdots & a_i^{n-1} & \cdots & a_{n-1}^{n-1} & a_n^{n-1} \\ a_0^n & a_1^n & \cdots & a_i^n & \cdots & a_{n-1}^n & a_n^n \end{pmatrix}$$

Remark 3. — Determination of matrix $D_1^{-1}\,\hat{K}\,D_2$: The entries of the i-th columm of matric $\hat{K}$ are the $(n+1)$ first coefficients of the expansion of

$T_1^{\iota} T_2^{n-\iota} = (\alpha U_1^2 + 2\beta U_1 U_2 + \gamma U_2^2)^{\iota} (2 U_1 U_2)^{n-\iota}$ with respect to U_1^{2n}, $U_1^{2n-1} U_2, \ldots, U_1^n U_2^n, \ldots, U_2^{2n}$.

For example we obtains, as matrix $D_1^{-1} \hat{K} D_2$,

n=2 :

$$\begin{pmatrix} 0 & 0 & \alpha^2 \\ 0 & \alpha & \alpha\beta \\ \frac{2}{3} & \frac{4}{3}\beta & \frac{1}{3}(2\beta^2 + \alpha\gamma) \end{pmatrix}$$

$n = 3$:

$$\begin{pmatrix} 0 & 0 & 0 & \alpha^3 \\ 0 & 0 & \alpha^2 & \alpha^2\beta \\ 0 & \frac{4}{5}\alpha & \frac{8}{5}\alpha\beta & \frac{1}{5}(4\alpha\beta^2 + \alpha^2\gamma) \\ \frac{2}{5} & \frac{6}{5}\beta & \frac{3}{5}(2\beta^2 + \alpha\gamma) & \frac{1}{5}(3\alpha\beta\gamma + 2\beta^3) \end{pmatrix}$$

$n = 4$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \alpha^4 \\ 0 & 0 & 0 & \alpha^3 & \alpha^3\beta \\ 0 & 0 & \frac{6}{7}\alpha^2 & \frac{12}{7}\alpha^2\beta & \frac{1}{7}(6\alpha^2\beta^2 + \alpha^3\gamma) \\ 0 & \frac{4}{7}\alpha & \frac{12}{7}\alpha\beta & \frac{1}{7}(12\alpha\beta^2 + 3\alpha^2\gamma) & \frac{1}{7}(4\alpha\beta^3 + 3\alpha^2\beta\gamma) \\ \frac{8}{35} & \frac{32}{35}\beta & \frac{8}{35}(6\beta^2 + 3\alpha\gamma) & \frac{8}{35}(4\beta^3 + 6\alpha\beta\gamma) & \frac{8}{35}\left(\beta^4 + 3\alpha\beta^2\gamma + \frac{3}{8}\alpha^2\gamma^2\right) \end{pmatrix}.$$

For greater values of n, the expansion of $T_1^{\iota} T_2^{n-\iota}$ and matrix $D_1^{-1} \hat{K} D_2$ can be obtained using computer algebra systems.

Propositions 2 and 3 below lead to an algorithm for another calculation of $\hat{K}$.

Example 1 : Folium of Descartes defined over $[0, 1[$

Suppose the plane $\mathscr{E}$ and the space $\mathscr{F}$ have a cartesian frame of reference respectively $(\Omega_1, \vec{i}, \vec{j})$ and $(\Omega, \vec{i}, \vec{j}, \vec{k} = \overrightarrow{\Omega\Omega_1})$.

The Folium of Descartes (1638) is the curve represented parametrically by the point $M(t)$ with coordinates :

$$x(t) = \frac{3\,t}{1+t^3}, \quad t \in \tilde{\mathbb{R}}$$
$$y(t) = \frac{3\,t^2}{1+t^3}$$

or in homogeneous cartesian coordinates :

$$X(t) = 3\,t$$
$$Y(t) = 3\,t^2$$
$$Z(t) = 1 + t^3\,.$$

We know [6, proposition 2.1.4.1] that

$$\begin{aligned} M(t) &= \Pi\Omega\,(X(t)\,\vec{i} + Y(t)\,\vec{j} + Z(t)\,\vec{k}) \\ &= \Pi\Omega\,(\vec{k} + 3\,t\vec{i} + 3\,t^2\,\vec{j} + t^3\,\vec{k})\,. \end{aligned}$$

As $\Pi\Omega = \Pi \circ \hat{\Omega}^{-1}$ and $\hat{\Omega}^{-1}(\vec{k}) = (\Omega_1\,;1)$, $\hat{\Omega}^{-1}(\vec{i}) = \vec{i}$, $\hat{\Omega}^{-1}(\vec{j}) = \vec{j}$, it becomes

$$M(t) = \Pi((\Omega_1\,;1) \oplus 3\,t\vec{i} \oplus 3\,t^2\,\vec{j} \oplus t^3(\Omega_1\,;1))\,.$$

Therefore there exists a massic polygon $\omega = (\omega_0, \omega_1, \omega_2, \omega_3)$ such that :

$$\omega_0 = (\Omega_1\,;1)$$
$$\Delta\omega_0 = \vec{i}$$
$$\Delta^2\omega_0 = \vec{j}$$
$$\Delta^3\omega_0 = (\Omega_1\,;1)\,.$$

Then

$$\begin{aligned} M(t) &= \Pi(\omega_0 \oplus 3\,t\,\Delta\omega_0 \oplus 3\,t^2\,\Delta^2\omega_0 \oplus t^3\,\Delta^3\,\omega_0) \\ &= \Pi\left(\sum_{i=0}^{3} B_i^3(t)\,\omega_i\right) \\ &= BR\,[\omega_0, \omega_1, \omega_2, \omega_3]\,(t) \end{aligned}$$

with : $\omega_0 = (\Omega_1\,;1)$, $\omega_1 = (\Omega_1 + \vec{i}\,;1)$, $\omega_2 = (\Omega_1 + 2\,\vec{i} + \vec{j}\,;1)$, $\omega_3 = \left(\Omega_1 + \frac{3}{2}\,(\vec{i} + \vec{j}\,)\,;2\right)$.

Defining $P = \Omega_1 + \vec{i}$, $Q = \Omega_1 + 2\vec{i} + \vec{j}$, $R = \Omega_1 + \frac{3}{2}(\vec{i} + \vec{j})$, we obtain explicitly :

$$M(t) = \frac{B_0^3(t)\,\Omega_1 + B_1^3(t)\,P + B_2^3(t)\,Q + 2\,B_3^3(t)\,R}{B_0^3(t) + B_1^3(t) + B_2^3(t) + 2\,B_3^3(t)}\,, \quad t \in \tilde{\mathbb{R}}\,.$$

Again we find the massic polygon given in [6, section 2.3.2]. The image of [0, 1] is the anticlockwise arc (Ω_1, R), (*fig.* 1).

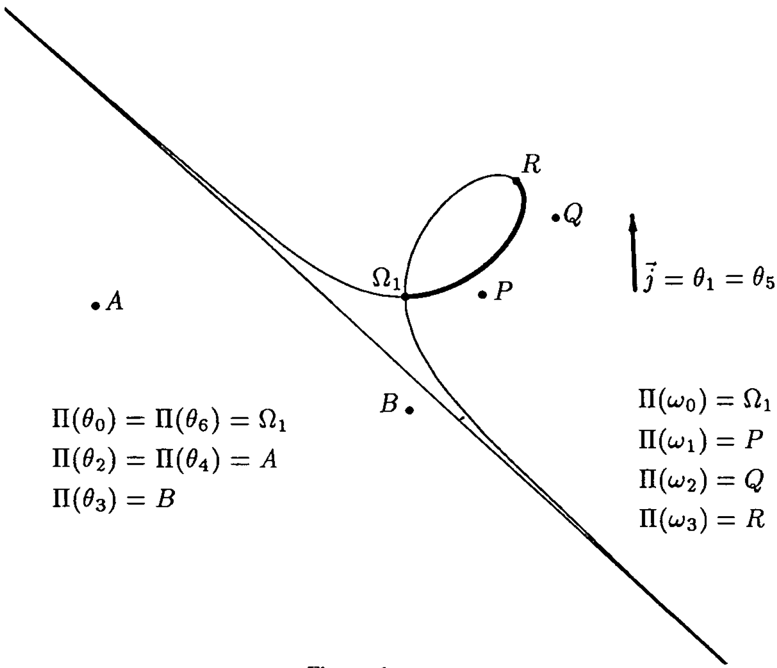

Figure 1.

We make the quadratic change of parameter $t = \Phi(u)$. We obtain

$$M \circ \Phi = BR[\omega] \circ \Phi = BR[\theta_0, \theta_1, ,\theta_2, \theta_3, \theta_4, \theta_5, ,\theta_6]$$

with $\theta = (\theta_0, \theta_1, \ldots, ,\theta_6)$, a ρ-reciprocal massic polygon such that

$$\begin{pmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = D_1^{-1}\,\hat{K} D_2 \begin{pmatrix} \omega_0 \\ \Delta\omega_0 \\ \Delta^2\omega_0 \\ \Delta^3\omega_0 \end{pmatrix}.$$

From remark 3 ($n = 3$), within the specific case $\alpha = -1$, $\beta = 0$, $\gamma = 1$ ($\rho = -1$) we obtain

$$\begin{pmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{4}{5} & 0 & \frac{1}{5} \\ \frac{2}{5} & 0 & -\frac{3}{5} & 0 \end{pmatrix} \begin{pmatrix} (\Omega_1 ; 1) \\ \vec{i} \\ \vec{j} \\ (\Omega_1 ; 1) \end{pmatrix}$$

and finally :

$$\begin{aligned}
\theta_0 &= (\Omega_1 ; -1) \\
\theta_1 &= \vec{j} \\
\theta_2 &= \left(\Omega_1 - 4\vec{i} ; \frac{1}{5}\right) \\
\theta_3 &= \left(\Omega_1 - \frac{3}{2}\vec{j} ; \frac{2}{5}\right) \\
\theta_4 &= -\theta_2 \\
\theta_5 &= \theta_1 \\
\theta_6 &= -\theta_0 .
\end{aligned}$$

Defining $A = \Omega_1 - 4\vec{i}$, $B = \Omega_1 - \frac{3}{2}\vec{j}$, we obtain explicitly

$$M(u) = \frac{-B_0^6(u)\,\Omega_1 + \frac{1}{5}B_2^6(u)\,A + \frac{2}{5}B_3^6(u)\,B - \frac{1}{5}B_4^6(u)\,A + B_6^6(u)\,\Omega_1}{-B_0^6(u) + \frac{1}{5}B_2^6(u) + \frac{2}{5}B_3^6(u) - \frac{1}{5}B_4^6(u) + B_6^6(u)}$$

$$+ \frac{B_1^6(u)\,\vec{j} + B_5^6(u)\,\vec{j}}{-B_0^6(u) + \frac{1}{5}B_2^6(u) + \frac{2}{5}B_3^6(u) - \frac{1}{5}B_4^6(u) + B_6^6(u)}, \quad u \in [0, 1[.$$

In [6, section 2.3.2] we give a slightly different massic polygon corresponding to $\alpha = 1$, $\beta = 0$, $\gamma = -1$.

The anticlockwise loop (*fig.* 1) corresponds to $u \in \left[\frac{1}{2}, 1\right[$ (resp. to $t \in [0, +\infty[$), segment on the second quadrant to $u \in \left[1 - \frac{\sqrt{2}}{2}, \frac{1}{2}\right]$ (resp. to $t \in [-1, 0]$) and the segment on the fourth quadrant to $u \in \left]0, 1 - \frac{\sqrt{2}}{2}\right]$ (resp. to $t \in]-\infty, -1]$).

COROLLARY 2 : *Denoting the mass of a massic vector* θ by $\chi(\theta)$ *we have the relation*

$$\begin{pmatrix} \chi(\theta_0) \\ \chi(\theta_1) \\ \vdots \\ \chi(\theta_n) \end{pmatrix} = D_1^{-1} \hat{K} D_2 \begin{pmatrix} \chi(\omega_0) \\ \Delta\chi(\omega_0) \\ \vdots \\ \Delta^n \chi(\omega_0) \end{pmatrix}.$$

The massic sequence $(\chi(\theta_0), \chi(\theta_1), \dots, \chi(\theta_n), \dots, \chi(\theta_{2n}))$ *is* $\frac{\gamma}{\alpha}$*-reciprocal.*

Proof : We know [6, proposition 1.2.2.8] that χ is a linear form : $\hat{\mathscr{E}} \to \mathbb{R}$, the relation ($\nu$) of proposition 1 giving the θ_i as a function of ω_i remains true by considering the masses, on both sides of the equality.

COROLLARY 3 : *Let* $\theta = (\theta_0, \theta_1, \dots, \theta_{2n})$ *be a* ρ*-reciprocal massic polygon with* $\rho < 0$. *Choosing* α *and* γ *such that* $\rho = \frac{\gamma}{\alpha}$, *then there exists a massic polygon* $\omega = (\omega_0, \omega_1, \dots, \omega_n)$ *such that*

$$BR[\omega] \circ \Phi = BR[\theta]$$

Φ *is the quadratic function previously given* (β *is arbitrary*). *It is sufficient to define the massic polygon* ω *by the relation :*

$$\begin{pmatrix} \omega_0 \\ \Delta\omega_0 \\ \vdots \\ \Delta^n \omega_0 \end{pmatrix} = D_2^{-1} \hat{K}^{-1} D_1 \begin{pmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}.$$

Proof: The scalar α is different from zero, so $\hat{K}$ is invertible and we deduce $(\omega_0, \omega_1, \dots, \omega_n)$ from equation (ν) proposition 1. The condition $\rho < 0$ guarantees that the quadratic correspondence $t = \Phi(u)$ is a one-to-one correspondence between $[0, 1[$ and $\tilde{\mathbb{R}}$.

Example 2 : Curve of Agnesi

We consider the curve of Agnesi (1748) also named versiera, was first considered by Fermat (1666), is given parametrically by the point $M(u)$ with cartesian coordinates :

$$x(u) = \frac{2u - 1}{2u(1-u)}$$
$$y(u) = \frac{4u^2(1-u)^2}{4u^4 - 8u^3 + 8u^2 - 4u + 1}, \quad u \in [0, 1[.$$

Its homogeneous cartesian coordinates are :

$$X(u) = 8u^5 - 20u^4 + 24u^3 - 16u^2 + 6u - 1$$
$$Y(u) = -8u^6 + 24u^5 - 24u^4 + 8u^3$$
$$Z(u) = -8u^6 + 24u^5 - 32u^4 + 24u^3 - 10u^2 + 2u\,.$$

We obtain the following BR-form of this rational curve by the method illustrated in example 1 or by that of [6, section 2.2.2.3] :

$$M(u) = BR[\theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6](u)$$

with $\theta_0 = \vec{V}_0$, $\vec{V}_0 = (-1, 0)$; $\theta_1 = \left(Q_1 ; \frac{1}{3}\right)$, $Q_1 = (0, 0)$, $\theta_2 = \vec{V}_2$, $\vec{V}_2 = \left(-\frac{1}{15}, 0\right)$, $\theta_3 = \left(Q_1 ; \frac{1}{5}\right)$, $Q_3 = (0, 2)$; $\theta_4 = -\theta_2$, $\theta_5 = \theta_1$, $\theta_6 = -\theta_0$. In figure 2 the length of pure vectors θ_0, θ_2, θ_4, θ_6 has been multiplied by 2.

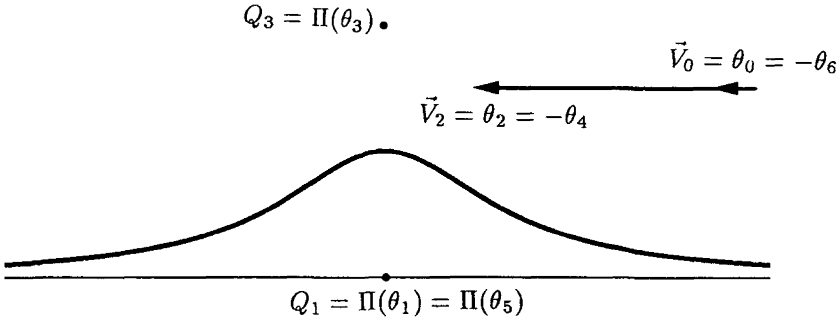

Figure 2.

Explicitly we have :

$$M(u) = \frac{\frac{1}{3}B_1^6(u)\,Q_1 + \frac{1}{5}B_3^6(u)\,Q_3 + \frac{1}{3}B_5^6(u)\,Q_1}{\frac{1}{3}B_1^6(u) + \frac{1}{5}B_3^6(u) + \frac{1}{3}B_5^6(u)}$$
$$+ \frac{B_0^6(u)\,\vec{V}_0 + B_2^6(u)\,\vec{V}_2 - B_4^6(u)\,\vec{V}_2 - B_6^6(u)\,\vec{V}_0}{\frac{1}{3}B_1^6(u) + \frac{1}{5}B_3^6(u) + \frac{1}{3}B_5^6(u)}, \quad u \in [0, 1[\,.$$

This massic polygon being ρ-reciprocal with $\rho = -1$ then by corollary 3 this BR-curve comes from a BR-curve of length 3 by a quadratic change of parameter. More precisely there exists a massic polygon $\omega = (\omega_0, \omega_1, \omega_2, \omega_3)$ such that

$$BR[\omega] \circ \Phi = BR[\theta].$$

Taking $\alpha = -1$, $\beta = 0$, $\gamma = 1$, $(\rho = -1)$ we obtain :

$$D_2^{-1} \hat{K}^{-1} D_1 = \begin{pmatrix} 0 & \frac{3}{2} & 0 & \frac{5}{2} \\ -\frac{1}{4} & 0 & -\frac{5}{4} & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

which gives ω_0, $\Delta\omega_0$, $\Delta^2\omega_0$, $\Delta^3\omega_0$ and finally :

$\omega_0 = (P_0 ; 1)$, $P_0 = (0, 1)$; $\omega_1 = (P_1 ; 1)$, $P_1 = \left(\frac{1}{3}, 1\right)$; $\omega_2 = \left(P_2 ; \frac{4}{3}\right)$, $P_2 = \left(\frac{1}{2}, \frac{3}{4}\right)$; $\omega_3 = (P_3 ; 2)$, $P_3 = \left(1, \frac{1}{2}\right)$, (*fig.* 3).

Explicitly $M(\Phi^{-1}(t)) = BR[\omega]\,(t)$ is given by :

$$BR[\omega]\,(t) = \frac{B_0^3(t)\,P_0 + B_1^3(t)\,P_1 + \frac{4}{3}B_2^3(t)\,P_2 + 2\,B_3^3(t)\,P_3}{B_0^3(t) + B_1^3(t) + \frac{4}{3}B_2^3(t) + 2\,B_3^3(t)}, \quad t \in \tilde{\mathbb{R}}.$$

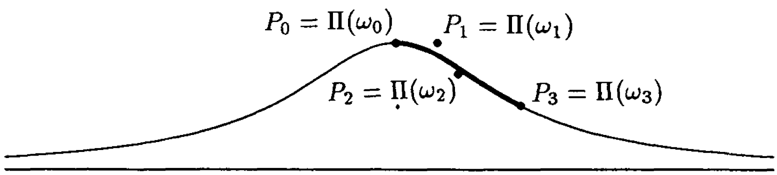

Figure 3.

4. ALGORITHM FOR CALCULATING MATRIX $\hat{K}$

PROPOSITION 2 :

(i) T_1^i *must be written*

$$T_1^i = \sum_{j=0}^{2i} \alpha_j^i \, U_1^{2i-j} \, U_2^j .$$

(ii) (α_j^i), $j = 0, 1, \ldots, 2i$, *are* $\frac{\gamma}{\alpha}$*-reciprocal i.e.,* $\alpha_{2i-k}^i = \left(\frac{\gamma}{\alpha}\right)^{i-k} \alpha_k^i$, $k = 0, 1, \ldots, i$.

(iii) *The following algorithm gives the coefficients* (α_j^i) *of polynomials* T_1^i, $i = 2, 3, \ldots, n$ $(T_1^1 = \alpha U_1^2 + 2\beta U_1 U_2 + \gamma U_2^2)$: *initialization :* $\alpha_{-2}^1 = \alpha_{-1}^1 = 0$, $\alpha_0^1 = \alpha$, $\alpha_1^1 = 2\beta$, $\alpha_2^1 = \gamma$, $\alpha_3^1 = 0$, $\rho = \dfrac{\gamma}{\alpha}$;

$$\begin{aligned}
&for\ i = 2, 3, \ldots, n\\
&for\ j = 0, 1, \ldots, i\ do\\
&\qquad \alpha_j^i = \alpha\alpha_j^{i-1} + 2\beta\alpha_{j-1}^{i-1} + \gamma\alpha_{j-2}^{i-1}\\
&\quad \alpha_{2i-j}^i = \rho^{i-j}\alpha_j^i\ (if\ j < i)\\
&\quad \alpha_{-2}^i = \alpha_{-1}^i = 0\,.
\end{aligned}$$

Proof :

(i) Expanding $T_1^i = (\alpha U_1^2 + 2\beta U_1 U_2 + \gamma U_2^2)^i$, T_1^i can be written

$$T_1^i = \sum_{i=0}^{2i} \alpha_j^i U_1^{2i-j} U_2^j\,.$$

(ii) T_1^i is a homogeneous polynomial of degree i in T_1, T_2, according to lemma 1, T_1^i has $\dfrac{\gamma}{\alpha}$-reciprocal coefficients.

(iii)

$$\begin{aligned}
T_1^{i-1} &= \sum_{j=0}^{2i-2} \alpha_j^{i-1} U_1^{2i-2-j} U_2^j\\
T_1^i &= T_1 T_1^{i-1} = (\alpha U_1^2 + 2\beta U_1 U_2 + \gamma U_2^2) T_1^{i-1}\\
T_1^i &= \sum_{j=0}^{2i-2} \alpha\alpha_j^{i-1} U_1^{2i-j} U_2^j + \sum_{j=0}^{2i-2} 2\beta\alpha_j^{i-1} U_1^{2i-1-j} U_2^{j+1} +\\
&\quad + \sum_{j=0}^{2i-2} \gamma\alpha_j^{i-1} U_1^{2i-2-j} U_2^{j+2}\,.
\end{aligned}$$

We define $\alpha_{-2}^{i-1} = \alpha_{-1}^{i-1} = \alpha_{2i-1}^{i-1} = \alpha_{2i}^{i-1} = 0$. By some direct change of indices, T_1^i becomes :

$$T_1^i = \sum_{j=0}^{2i} (\alpha\alpha_j^{i-1} + 2\beta\alpha_{j-1}^{i-1} + \gamma\alpha_{j-2}^{i-1}) U_1^{2i-j} U_2^j\,.$$

Then $\alpha_j^i = \alpha\alpha_j^{i-1} + 2\beta\alpha_{j-1}^{i-1} + \gamma\alpha_{j-2}^{i-1}$, $j = 0, 1, \ldots, 2i$.

The (α_j^i) being $\dfrac{\gamma}{\alpha}$-reciprocal, we calculate (α_j^i), for $j = 0, 1, \ldots, i$, by this relation and we deduce $\alpha_{2i-j}^i = \left(\dfrac{\gamma}{\alpha}\right)^{i-j} \alpha_j^i$, $j = 0, 1, \ldots, i-1$.

PROPOSITION 3 : *The entries a_i^j of the $(n+1)\times(n+1)$ matrix $\hat{K}$ are given by :*

$$a_i^j = 0 \quad for \quad i + j = n-1, n-2, \dots, 0$$

$$a^{n-i+j} = 2^{n-i}\,\alpha_j^i, \quad i = 0, 1, \dots, n, \quad j = 0, 1, \dots, i$$

$$(\alpha_0^0 = 1).$$

Proof: The first result was proved proposition 1(ii).

From $T_1^i = \sum_{j=0}^{2i} \alpha_j^i U_1^{2i-j} U_2^j$ we obtain

$$T_1^i T_2^{n-i} = \left(\sum_{j=0}^{2i} \alpha_j^i U_1^{2i-j} U_2^j\right)(2\,U_1 U_2)^{n-i}$$
$$= \sum_{j=0}^{2i} 2^{n-i}\alpha_j^i U_1^{n-j+i} U_2^{n+j-i}. \tag{I}$$

In proposition 1 we have defined $T_1^i T_2^{n-i} = \sum_{j=0}^{2n} a_i^j U_1^{2n-j} U_2^j$. From (i) and (ii) of this proposition we deduce :

$$T_1^i T_2^{n-i} = \sum_{k=n-i}^{n+i} a_i^k U_1^{2n-k} U_2^k.$$

By changing the indices : $j = k - n + i$, it becomes

$$T_1^i T_2^{n-i} = \sum_{j=0}^{2i} a_i^{n+j-i} U_1^{n-j+i} U_2^{n+j-i}.$$

By comparing with equality (I) we deduce :

$$a_i^{n+j-i} = 2^{n-i}\alpha_j^i, \quad i = 0, 1, \dots, n, \quad j = 0, 1, \dots, i.$$

In proposition 1 (iii) we have seen that $a_0^n = 2^n$, so we define $\alpha_0^0 = 1$. Then we deduce the following algorithm giving the non-zero entries of $\hat{K}$.

ALGORITHM : *Initialization :* $\alpha_{-2}^1 = \alpha_{-1}^1 = 0$, $\alpha_0^1 = \alpha$, $\alpha_1^1 = 2\beta$, $\alpha_2^1 = \gamma$, $a_0^n = 2^n$, $a_1^{n-1} = \alpha 2^{n-1}$, $a_1^n = \beta 2^n$;

$$for \quad i = 2, 3, \dots, n$$

$$for \quad j = 0, 1, \dots, i \; do$$

$$\alpha_j^i = \alpha\alpha_j^{i-1} + 2\beta\alpha_{j-1}^{i-1} + \gamma\alpha_{j-2}^{i-1}$$

$$a_i^{n-i+j} = 2^{n-i}\alpha_j^i$$

$$\alpha_{i+1}^i = \frac{\gamma}{\alpha}\alpha_{i-1}^i$$

$$\alpha_{-2}^i = \alpha_{-1}^i = 0.$$

5. REPRESENTATION OF A COMPLETE POLYNOMIAL CURVE BY A *BR*-CURVE

If we have to calculate a complete polynomial curve, i.e., for parameter $t \in \tilde{\mathbb{R}}$, calculating with its Bézier representation can be inaccurate. Actually for points corresponding to values of t not belonging to $[0, 1]$, we use, in the De Casteljau's algorithm, affine combinations of points which are not convex and coefficients of these combinations tend to infinity. Proposition 4 below allows us to describe a Bézier curve as a *BR*-curve with parameter $u \in [0, 1[$. Hence calculating any point of this *BR*-curve by algorithm ALBR1 or ALBR2 [6, sections 3.3 and 3.4] (algorithms related to De Casteljau's), this time we manipulate convex affine combinations of points or massic vectors which offers a guarantee of stability.

PROPOSITION 4 : *If* $\omega_i = (P_i ; c)$, $i = 0, 1, \ldots, n$, *c being the common mass, then*

(i) $BR[\omega] = BP[P]$, *i.e., a Bézier curve,*

(ii) $BP[P] \circ \Phi(u) = BR[\theta](u)$ *where the* θ_i *are all pure vectors* $(\theta_i \in \vec{\mathscr{E}})$ *except* θ_n *which is a weighted point of mass equal to* $\dfrac{2^n}{\binom{2n}{n}} c$.

More precisely

$$\theta_j = \frac{1}{\binom{2n}{j}} \sum_{i=n-j}^{n} \binom{n}{i} a_i^j \, \Delta^i P_0, \quad j = 0, 1, \ldots, n-1$$

$$\theta_n = \left(P_0 + \frac{1}{2^n c} \sum_{i=1}^{n} \binom{n}{i} a_i^n \, \Delta^i P_0 ; \frac{2^n}{\binom{2n}{n}} c \right).$$

The massic polygon θ *being* $\left(\dfrac{\gamma}{\alpha}\right)$-reciprocal.

(iii) *The support of* $BR[\theta](u)$ *when* $u \in [0, 1[$ *is identical to that of* $BP[P](t)$ *when* $t \in \tilde{\mathbb{R}}$.

(iv) *Conversely if the massic polygon* $\theta = (\theta_0, \theta_1, \ldots, \theta_{2n})$ *is* ρ*-reciprocal and if all* θ_i *are pure vectors except* θ_n *then there exist points of* $\mathscr{E}$: $P_0, P_1, \ldots, P_n$, *such that :*

$$BP[P_0, P_1, \ldots, P_n](\Phi(u)) = BR[\theta](u)$$

with Φ *a quadratic function defined as previously, the constants* α *and* γ *satisfying* $\dfrac{\gamma}{\alpha} = \rho$, β *arbitrary.*

Proof :

(i) We have $\omega_i = (P_i ; c)$, $i = 0, 1, \ldots, n$. From remark 1 (ii) : $BR[\omega](t) = BP[P](t)$.

(ii) The ω_i have same mass then $\Delta\chi(\omega_0) = \cdots = \Delta^n\chi(\omega_0) = 0$. Corollary 2 implies $\chi(\theta_0) = \cdots = \chi(\theta_{n-1}) = 0$ i.e., $\theta_0, \theta_1, \ldots, \theta_{n-1}$ are pure vectors and $\chi(\theta_n) = \dfrac{2^n}{\binom{2n}{n}} \chi(\theta_0)$.

From proposition 1 (ν) we deduce that

$$\theta_j = \frac{1}{\binom{2n}{j}} \sum_{i=n-j}^{n} \binom{n}{i} a_i^j \Delta^i \omega_0 , \quad j = 0, 1, \ldots, n .$$

For $i = 1, 2, \ldots, n$: $\Delta^i \omega_0 = \Delta^i P_0$. Distinguishing the case $j = n$ and $j = 0, 1, \ldots, n-1$ we obtain the announced result.

(iii) is identical to corollary 1.

(iv) Conversely let $\theta = (\theta_0, \theta_1, \ldots, \theta_{2n})$ be a ρ-reciprocal massic polygon with all θ_i being pure vectors except θ_n. Following Corollary 3 we define $\omega = (\omega_0, \omega_1, \ldots, \omega_n)$ by :

$$\begin{pmatrix} \omega_0 \\ \Delta\omega_0 \\ \vdots \\ \Delta^n \omega_0 \end{pmatrix} = D_2^{-1} \hat{K}^{-1} D_1 \begin{pmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}$$

with D_1, D_2, $\hat{K}$ defined in proposition 1 $\left(\rho = \dfrac{\gamma}{\alpha}, \ \beta \text{ arbitrary}\right)$. We have $BR[\omega](\Phi(u)) = BR[\theta](u)$ and

$$\begin{pmatrix} \chi(\omega_0) \\ \Delta\chi(\omega_0) \\ \vdots \\ \Delta^{n-1}\chi(\omega_0) \\ \Delta^n \chi(\omega_0) \end{pmatrix} = D_2^{-1} \hat{K}^{-1} D_1 \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \chi(\theta_n) \end{pmatrix} .$$

But matrices $\hat{K}^{-1}$ and $D_2 \hat{K}^{-1} D_1$ have zero entries under the secondary diagonal i.e., if b_i^j denote the entries of $\hat{K}^{-1}$ we have $b_i^j = 0$ for $i + j = n + 1, n + 2, \ldots, 2n$.

Consequently $\chi(\omega_0) \neq 0$ and $\Delta\chi(\omega_0) = \cdots = \Delta^n \chi(\omega_0) = 0$ i.e., $\chi(\omega_i) = \chi(\omega_0) = c$; ω_i can be written $(P_i ; c)$, $P_i \in \mathscr{E}$ and $BR[\omega] = BP[P]$ i.e., a Bézier curve of length n.

Example 3. — *Cubic of Tschirnausen*

The cubic of Tschirnausen (1690) also named trisectrix of Catalan (1501-1576) or L'Hospital's cubic (1696) is the polynomial curve given parametrically by the point $M(t)$ with cartesian coordinates :

$$x(t) = 3(t^2 - 3)$$
$$y(t) = t(t^2 - 3), \quad t \in \tilde{\mathbb{R}} .$$

Its Bézier representation follows directly :

$$M(t) = BP\,[P_0, P_1, P_2, P_3]\,(t), \quad t \in \tilde{\mathbb{R}} ,$$

with $P_0 = (-9, 0)$, $P_1 = (-9, -1)$, $P_2 = (-8, -2)$, $P_3 = (-6, -2)$ and $BP\,[P] = BR\,[\omega]$ with $\omega_i = (P_i\,; 1)$, $i = 0, 1, 2, 3$ (*fig.* 4).

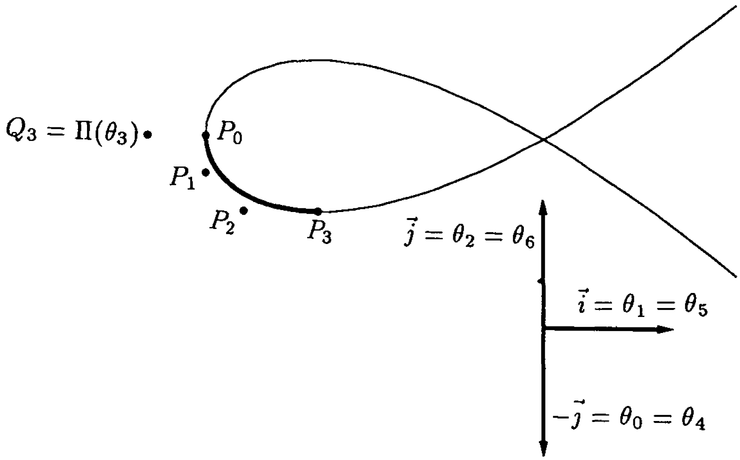

Figure 4.

By making the quadratic change of parameter $t = \Phi(u)$ we obtain :

$$BP\,[P_0, P_1, P_2, P_3] \circ \Phi(u) = BR\,[\theta_0, \theta_1, \dots, \theta_6]\,(u) .$$

From $\omega_i = (P_i\,; 1)$ we deduce

$$\omega_0 = (P_0\,; 1), \quad P_0 = (-9, 0)\,; \quad \Delta\omega_0 = \Delta P_0 = -\vec{j}\,;$$
$$\Delta^2\omega_0 = \Delta^2 P_0 = \vec{i}\,; \quad \Delta^3\omega_0 = \Delta^3 P_0 = \vec{j} .$$

We take again $\alpha = -1$, $\beta = 0$, $\gamma = 1$ $(\rho = -1)$ in the definition of

Φ, then proposition 1 (ν) and matrix $D_1^{-1}\hat{K}D_2$ of example 1 give :

$$\begin{pmatrix}\theta_0\\ \theta_1\\ \theta_2\\ \theta_3\end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & -\frac{4}{5} & 0 & \frac{1}{5}\\ \frac{2}{5} & 0 & -\frac{3}{5} & 0\end{pmatrix}\begin{pmatrix}(P_0\,;\,1)\\ -\vec{j}\\ \vec{i}\\ \vec{j}\end{pmatrix}$$

$$\begin{aligned}\theta_0 &= -\vec{j}\\ \theta_1 &= \vec{i}\\ \theta_2 &= \vec{j}\\ \theta_3 &= \left(P_0 - \frac{3}{2}\vec{i}\,;\,\frac{2}{5}\right)\\ \theta_4 &= -\theta_2\\ \theta_5 &= \theta_1\\ \theta_6 &= -\theta_0\,.\end{aligned}$$

Defining $Q_3 = P_0 - \frac{3}{2}\vec{i}$, $Q_3 = \left(-\frac{21}{2}, 0\right)$. For more clarity in figure 4, the length of vectors $\vec{i}$ and $\vec{j}$ has been multiplied by 3.

$$BR[\theta]\ (u) = Q_3 + \\ + \frac{-B_0^6(u)\vec{j} + B_1^6(u)\vec{i} + B_2^6(u)\vec{j} - B_4^6(u)\vec{j} + B_5^6(u)\vec{i} + B_6^6(u)\vec{j}}{B_3^6(u)}\,,$$

$$u \in [0, 1[\,.$$

6. CONCLUSION

For a complete description of a rational curve or *a fortiori* of a polynomial curve we use its BR-representation. To describe its whole support we have resorted to special quadratic change of parameter which is a one-to-one correspondence between $\tilde{\mathbb{R}}$ and $[0, 1[$. We have determined a new massic polygon from the old one by linear relations. This new massic polygon presents a kind of symmetry called ρ-reciprocity which reduces the computation. It enables us to store the whole support as BR-curve and to compute it by convex affine combinations of points or massic vectors.

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