M2AN - Modélisation mathématique et analyse numérique

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M2AN - Modélisation mathématique et analyse numérique, tome 27, nº 2 (1993), p. 157-182
<http://www.numdam.org/item?id=M2AN_1993__27_2_157_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE
(Vol. 27, n° 2, 1993, p. 157 à 182)

CONTROL/FICTITIOUS DOMAIN METHOD FOR SOLVING OPTIMAL SHAPE DESIGN PROBLEMS (*)

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Communicated by O. PIRONNEAU

Abstract. — *Combining a fictitious domain and an optimal control approach, we present a new method for the numerical realization of optimal shape design problems. This approach enables us to perform all calculations on a fixed domain with a fixed grid. Finite element approximation is studied.*

Résumé. — *On présente une méthode nouvelle pour la résolution numérique des problèmes de l'optimisation de la forme. Cette méthode est basée sur la combinaison d'une méthode des domaines fictifs avec une méthode de contrôle optimal. L'avantage de cette approche est le fait qu'on peut réaliser tous les calculs sur un domaine et un maillage fixe.*

Shape optimization is a branch of the optimal control theory, in which the control variable is connected with the geometry of the problem. Mathematical analysis, including the approximation theory and the numerical realization, has been widely discussed in [4, 7]. The numerical realization of optimal shape design problems has specific features. One of them is the fact that the state problem is solved many times on the domain, changing during the computation. For domains with complicated shapes this requires the use of mesh generators. Moreover, data, defining the finite dimensional approximation (stiffness matrix, etc.) have to be recomputed again and again. As a result, the whole procedure is time consuming and hence expensive.

To overcome this difficulty, the control/fictitious domain method is proposed. The method has been used in [1] for the numerical solution of the Helmholtz equation. Nevertheless, the same approach can be used in the

(*) Received for publication April 29, 1991.

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fram of the shape optimization. Introducing the new control variable on the right-hand side of the state problem, one can transform the original optimal shape design problem into a new one, in which the state problem is formulated and solved on a given, fixed domain $\hat{\Omega}$ with the same differential operators. The advantage of this approach is obvious : choosing the simple shape of $\hat{\Omega}$, the construction of triangulation of $\hat{\Omega}$ is elementary and what is more important, all computations are realized with the same stiffness matrix A. Using the factorization of A, one can solve very efficiently the corresponding system of algebraic equations. An alternative approach, based on the control on the boundary is suggested in [9].

The present paper deals with the mathematical analysis and the approximation of the control/fictitious method for the numerical solution of the optimal shape design problems, with the state given by the homogeneous Dirichlet problem in the first part. The second part of the paper deals with the numerical realization of the method. For the sake of simplicity, numerical experiments are performed for the state, given by an ordinary differential equation. The experiments for partial differential equations, as well as the mathematical analysis of the control/fictitious method for the state, given by variational inequalities, will be presented in subsequent papers.

1. INTRODUCTION

Let $\Omega(\alpha) \subset \mathbb{R}^2$ be a bounded domain, defined as follows :

$$\Omega(\alpha) = \left\{[x_1, x_2] \in \mathbb{R}^2 \,|\, 0 < x_1 < \alpha(x_2),\ x_2 \in (0, 1)\right\},$$

α is a design variable, belonging to the set U_{ad}, where

$$U_{ad} = \{\alpha \in C^{0,1}([0, 1]) \,|\, 0 < C_0 \leq \alpha(x_2) \leq C_1,\ |\alpha'(x_2)| \leq C_2 \text{ a.e. in } (0, 1), \text{ meas } \Omega(\alpha) = C_3\}.$$

$C_0, \ldots, C_3$ are given positive constants such that $U_{ad} \neq \emptyset$. Let $\hat{\Omega} = (0, 2C_1) \times (0, 1)$. Note that $\hat{\Omega} \supset \Omega(\alpha)\ \forall \alpha \in U_{ad}$.

On any $\Omega(\alpha)$, $\alpha \in U_{ad}$, we shall consider the following homogeneous Dirichlet boundary value problem :

$$\begin{cases} -\Delta u(\alpha) = f & \text{in } \Omega(\alpha) \\ u(\alpha) = 0 & \text{in } \partial\Omega(\alpha) \end{cases} \tag{1.1}$$

or in the meak form

$$(\mathscr{P}(\alpha)) \quad \begin{cases} \text{Find } u \equiv u(\alpha) \in V_1(\alpha) \text{ such that} \\ (\nabla u, \nabla\varphi)_{\Omega(\alpha)} = (f, \varphi)_{\Omega(\alpha)} \quad \forall \varphi \in V_1(\alpha). \end{cases}$$

Here $V_1(\alpha) = H_0^1(\Omega(\alpha))$, $f \in L^2(\hat{\Omega})$ and $(\cdot,\cdot)_{\Omega(\alpha)}$ stands for the usual scalar product in $L^2(\Omega(\alpha))$.

Let $J : [\alpha, y] \to \mathbb{R}^1$, $\alpha \in U_{ad}$, $y \in H^1(\Omega(\alpha))$, be a cost functional. The optimal shape design problem is defined as follows :

$$
(\mathbf{P}) \qquad \begin{cases} \text{Find } \alpha^* \in U_{ad} \text{ such that} \\ J(\alpha^*, u(\alpha^*)) \leq J(\alpha, u(\alpha)) \quad \forall \alpha U_{ad}, \end{cases}
$$

with $u(\alpha) \in V_1(\alpha)$ being the solution of $(\mathscr{P}(\alpha))$.

In order to guarantee the existence of a solution of (**P**), we make the following assumption.

J is *lower semicontinuous* in the following sense :

$$
\left.\begin{aligned} &\alpha_n \Rightarrow \alpha \ \text{(uniformly) in } [0, 1],\ \alpha_n, \alpha \in U_{ad} \\ &y_n \rightharpoonup y \ \text{(weakly) in } \hat{V},\ y_n, y \in \hat{V} \end{aligned}\right\} \Rightarrow \tag{1.2}
$$

$$
\Rightarrow \liminf_{n \to \infty} J(\alpha_n, y_n|_{\Omega_n}) \geq J(\alpha, y|_{\Omega(\alpha)}).
$$

Here and in what follows $\hat{V} = H_0^1(\hat{\Omega})$ and $\Omega_n = \Omega(\alpha_n)$.

PROPOSITION 1.1 : *If* (1.2) *is satisfied, then there exists at least one solution of* (**P**) (*see* [4]).

2. METHOD OF FICTITIOUS DOMAINS USED FOR THE SOLUTION OF (P)

Let

$$
\begin{aligned} \Xi(\alpha) &= \hat{\Omega} \setminus \bar{\Omega}(\alpha), \\ V_2(\alpha) &= \{ y \in H^1(\Xi(\alpha)) | y = 0 \text{ on } \partial\Xi(\alpha) \setminus \Gamma(\alpha) \}. \end{aligned}
$$

where

$$
\Gamma(\alpha) = \{ [x_1, x_2] \in \mathbb{R}^2 | x_1 = \alpha(x_2), x_2 \in (0, 1) \}, \quad \alpha \in U_{ad}.
$$

Recall that $V_1(\alpha) = H_0^1(\Omega(\alpha))$, $\hat{V} = H_0^1(\hat{\Omega})$. Symboles $\|\cdot\|_{V_1(\alpha)}$, $\|\cdot\|_{V_2(\alpha)}$ and $\|\cdot\|_{\hat{V}}$ stand for norms in $V_1(\alpha)$, $V_2(\alpha)$ and $\hat{V}$, respectively. The corresponding dual spaces are denoted by $V_1'(\alpha)$, $V_2'(\alpha)$ and $\hat{V}'$. The duality pairings between $V_2'(\alpha)$ and $V_2(\alpha)$, $\hat{V}'$ and $\hat{V}$ are denoted by $\langle \cdot,\cdot \rangle_\alpha$, $\langle \cdot,\cdot \rangle$, respectively, and $\|\omega\|_{*,\alpha}$ stands for the dual norm of $\omega \in V_2'(\alpha)$.

With any $\alpha \in U_{ad}$ the following set will be associated :

$$
Q(\alpha) = \{ v = (v_1, v_2) | v_1 \in L^2(\Omega(\alpha)), v_2 \in V_2'(\alpha) \}
$$

and we write

$$Q = \bigcup_{\alpha \in U_{ad}} Q(\alpha)$$

Instead of (1 1), we shall consider the homogeneous Dirichlet boundary value problem on $\hat{\Omega}$

$$\begin{cases} -\Delta \hat{u}(v) = v & \text{in } \hat{\Omega} \\ \hat{u}(v) = 0 & \text{on } \partial\hat{\Omega} , \end{cases} \tag{2 1}$$

where $v \in Q$ will play the role of the control variable

If we prove that for any $\alpha \in U_{ad}$ there exists $\bar{v}_2 \in V_2'(\alpha)$ such that the restriction on $\Omega(\alpha)$ of the solution $\hat{u}(v)$ of (2 1) with $v = (f, \bar{v}_2)$ solves $(\mathscr{P}(\alpha))$, the following idea arises instead of the state problem $(\mathscr{P}(\alpha))$, which is defined on the variable domain $\Omega(\alpha)$, we shall use (2 1) as the state problem for a new optimal control problem We shall prove that solutions of the original optimal shape design problem can be approached by solutions of the new problem The advantage is obvious The state problem (2 1) is still solved on the same domain This is of a great importance for the numerical realization All calculations are performed on a given, fixed domain

Let us introduce some notations and results, which will be useful in what follows If $v \in Q(\alpha)$, $v = (v_1, v_2)$, $\alpha \in U_{ad}$ and $\varphi \in \hat{V}$, we denote

$$[v, \varphi]_\alpha \overset{\text{def}}{\equiv} (v_1, \varphi)_{\Omega(\alpha)} + \langle v_2, \varphi \rangle_\alpha \tag{2 2}$$

For the sake of simplicity of notations, we use the symbol φ on the right-hand side of (2 2), instead of $\varphi|_{\Omega(\alpha)}$ and $\varphi|_{\Xi(\alpha)}$ It is readily seen that the formula

$$\varphi \mapsto [v, \varphi]_\alpha , \quad \varphi \in \hat{V} , \quad v \in Q(\alpha)$$

defines the linear, continuous functional on $\hat{V}'$, i e $Q(\alpha) \subset \hat{V}'$ Hence, if $v \in Q(\alpha)$, $\alpha \in U_{ad}$, we have

$$\langle v, \varphi \rangle = [v, \varphi]_\alpha \quad \forall \varphi \in \hat{V}$$

Let $v \in Q$, i e , there exists $\alpha \in U_{ad}$ such that $v \in Q(\alpha)$ We shall consider the weak form of (2 1)

$$(\hat{\mathscr{P}}(\alpha, v)) \quad \begin{cases} \text{Find } \hat{u}(v) \in \hat{V} \text{ such that} \\ (\nabla \hat{u}(v), \nabla \varphi)_{\Omega} = [v, \varphi]_\alpha \quad \forall \varphi \in \hat{V} \end{cases}$$

In what follows, we establish the continuous dependence of $\hat{u}$ with respect to variations of $v \in Q$, $\alpha \in U_{ad}$. To this end we need :

PROPOSITION 2.1 : *Let* α_n, $\alpha \in U_{ad}$ *be such that* $\alpha_n \rightrightarrows \alpha$ *in* $[0, 1]$ *and let* $v_n = (v_{n1}, v_{n2}) \in Q(\alpha_n)$ *be such that :*

$$\text{the sequence } \left\{ \| v_{n1} \|_{\Omega_n} \right\} \text{ is bounded ;} \tag{2.3}$$

$$v_n \rightharpoonup v \quad \text{in } \hat{V}' . \tag{2.4}$$

Then $v \in Q(\alpha)$ *and* $[v, \varphi]_\alpha = \lim_{n \to \infty} [v_n, \varphi]_{\alpha_n} \quad \forall \varphi \in \hat{V}$.

Proof : Let $\varphi \in \mathscr{D}(\Omega(\alpha))$ be given. Then $\varphi \in \mathscr{D}(\Omega_n)$ for n sufficiently large and

$$\langle v_n, \varphi \rangle = [v_n, \varphi]_{\alpha_n} = (v_{n1}, \varphi)_{\Omega_n} = (\tilde{v}_{n1}, \varphi)_{\Omega(\alpha)} \to \langle v, \varphi \rangle . \tag{2.5}$$

The symbol $\tilde{}$ denotes the extension of the corresponding function by zero from its domain of definition on $\hat{\Omega}$. From (2.5) it follows that there exists $v_1 \in L^2(\Omega(\alpha))$ such that

$$\langle v, \varphi \rangle = (v_1, \varphi)_{\Omega(\alpha)} \quad \forall \varphi \in \mathscr{D}(\Omega(\alpha)) . \tag{2.6}$$

Now, let $\varphi \in \hat{V}$ be given. Then

$$\langle v_n, \varphi \rangle = (v_{n1}, \varphi)_{\Omega_n} + \langle v_{n2}, \varphi \rangle_{\alpha_n} \to \langle v, \varphi \rangle .$$

Taking into account (2.5) and (2.6), we see that

$$\lim_{n \to \infty} \langle v_{n2}, \varphi \rangle_{\alpha_n} = \langle v, \varphi \rangle - (v_1, \varphi)_{\Omega(\alpha)} ,$$

i.e., the limit is finite. Hence, the formula

$$\varphi |_{\Xi(\alpha)} \mapsto \lim_{n \to \infty} \langle v_{n2}, \varphi \rangle_{\alpha_n}$$

defines the linear functional v_2 on $V_2(\alpha)$. Let us show that $v_2 \in V_2'(\alpha)$, i.e., v_2 is continuous. Let us assume that we have already proved that the sequence $\left\{ \| v_{n2} \|_{*, \alpha_n} \right\}$ is bounded. As $v_n \in V_2'(\alpha_n)$, then

$$\left| \langle v_{n2}, \varphi \rangle_{\alpha_n} \right| \leq \| v_{n2} \|_{*, \alpha_n} \| \varphi \|_{V_2(\alpha_n)} \leq c \| \varphi \|_{V_2(\alpha_n)} \text{ (1)} . \tag{2.7}$$

(1) In what follows, the symbol c denotes a generic positive constant.

Passing to the limit with $n \to \infty$ in (2.7), we see that

$$\left| \lim_{n \to \infty} \langle v_{n2}, \varphi \rangle_{\alpha_n} \right| \leq c \| \varphi \|_{V_2(\alpha)} .$$

Let us prove that the norms $\|v_{n2}\|_{*, \alpha_n}$ remain bounded. As the family $\{\Xi(\alpha), \alpha \in U_{\text{ad}}\}$ possesses the uniform extension property, there exists an extension mapping $\pi : V_2(\alpha) \to \hat{V}$, the norm of which does not depend on $\alpha \in U_{\text{ad}}$, i.e., if $\varphi \in V_2(\alpha)$, the $\pi \varphi \in \hat{V}$ and

$$\begin{cases} \pi \varphi = \varphi \quad \text{in } \Xi(\alpha) \\ \| \pi \varphi \|_{\hat{V}} \leq \tilde{c} \| \varphi \|_{V_2(\alpha)} , \end{cases} \tag{2.8}$$

with a constant $\tilde{c} > 0$ independent on $\alpha \in U_{\text{ad}}$. Let $\varphi \in V_2(\alpha_n)$. Then there exists a constant $c > 0$, which does not depend on n and

$$\left| \langle v_{n2}, \varphi \rangle_{\alpha_n} \right| = \left| \langle v_n, \pi \varphi \rangle - (v_{n1}, \pi \varphi)_{\Omega_n} \right| \leq c \| \varphi \|_{V_2(\alpha_n)}$$

making use of (2.3), (2.4) and $(2.8)_2$. □

PROPOSITION 2.2 : *Let* α_n, $\alpha \in U_{\text{ad}}$, $v_n \in Q(\alpha_n)$, $v \in Q(\alpha)$ *be such that*

$$\alpha_n \rightrightarrows \alpha \quad in\ [0, 1] ,$$
$$v_n \rightharpoonup v \quad in\ \hat{V}' ,$$

and (2.3) *be satisfied. Let* $\hat{u}_n \in \hat{V}$ *be solutions of* $(\hat{\mathscr{P}}(\alpha_n, v_n))$. *Then there exists a subsequence* $\{\hat{u}_{n'}\}$ *of* $\{\hat{u}_n\}$ *such that*

$$\hat{u}_{n'} \rightharpoonup \hat{u} \quad in\ \hat{V}$$

and $\hat{u}$ *solves* $(\hat{\mathscr{P}}(\alpha, v))$.

Proof : The sequence $\{\hat{u}_n\}$ is bounded in $\hat{V}$. One can find its subsequence, satisfying (2.9). The fact that $\hat{u}$ solves $(\hat{\mathscr{P}}(\alpha, v))$ is obvious, taking into account Proposition 2.1. □

We introduce further notations :

$$Q^f(\alpha) = \{ v = (v_1, v_2) \in Q(\alpha) | v_1 = f \text{ in } \Omega(\alpha) \} ,$$
$$Q^f = \bigcup_{\alpha \in U_{\text{ad}}} Q^f(\alpha) ,$$
$$Q^f_{\text{ad}}(\alpha) = \left\{ v = (v_1, v_2) \in Q^f(\alpha) | \, \|v_2\|_{*, \alpha} \leq C \right\} ,$$
$$Q^f_{\text{ad}} = \bigcup_{\alpha \in U_{\text{ad}}} Q^f_{\text{ad}}(\alpha) .$$

Here C denotes the positive constant, which does not depend on $\alpha \in U_{ad}$.

Let $J : [\alpha, y] \to \mathbb{R}^1$, $\alpha \in U_{ad}$, $y \in H^1(\Omega(\alpha))$ be a cost functional, satisfying (1.2).

For any $\varepsilon > 0$ we define :

$$E_\varepsilon(\alpha, v) \equiv J(\alpha, \hat{u}(v)|_{\Omega(\alpha)}) + \frac{1}{\varepsilon}\int_0^1 (\hat{u}(v))^2\, dx_2 ,$$

where $\hat{u}(v) \in \hat{V}$ is the solution of $(\hat{\mathscr{P}}(\alpha, v))$ with $v \in Q^f_{ad}(\alpha)$. The last integral is defined as follows :

$$\int_0^1 (\hat{u}(v))^2\, dx_2 \equiv \int_0^1 (\hat{u}(v)(\alpha(x_2), x_2))^2\, dx_2 .$$

In what follows, we analyze the problem

$$(\hat{\mathbf{P}})_\varepsilon \quad \begin{cases} \text{Find } (\alpha^*_\varepsilon, v^*_\varepsilon) \in U_{ad} \times Q^f_{ad}(\alpha^*_\varepsilon) \text{ such that} \\ E_\varepsilon(\alpha^*_\varepsilon, v^*_\varepsilon) \leq E_\varepsilon(\alpha, v) \quad \forall (\alpha, v) \in U_{ad} \times Q^f_{ad}(\alpha) . \end{cases}$$

We shall prove :

(j) the *existence* of at least one solution $(\alpha^*_\varepsilon, v^*_\varepsilon)$ of $(\hat{\mathbf{P}})_\varepsilon$;

(jj) the *mutual relation* between (**P**) and $(\hat{\mathbf{P}})_\varepsilon$ if $\varepsilon \to 0^+$.

PROPOSITION 2.3 : *For any* $\varepsilon > 0$ *there exists at least one solution* $(\alpha^*_\varepsilon, v^*_\varepsilon)$ *of* $(\hat{\mathbf{P}})_\varepsilon$.

Proof : Let $(\alpha_n, v_n) \in U_{ad} \times Q^f_{ad}(\alpha_n)$ be a minimizing sequence of the problem :

$$E_\varepsilon(\alpha_n, v_n) \to q \equiv \inf_{U_{ad} \times Q^f_{ad}} E_\varepsilon(\alpha, v) . \tag{2.10}$$

Taking into account the definition of U_{ad} and Q^f_{ad}, we may assume that

$$\begin{cases} \alpha_n \rightrightarrows \bar{\alpha} & \text{in } [0, 1] \\ v_n \rightharpoonup \bar{v} & \text{in } \hat{V}' . \end{cases}$$

Clearly $\bar{\alpha} \in U_{ad}$ and $\bar{v} = (\bar{v}_1, \bar{v}_2) \in Q(\bar{\alpha})$ (see Proposition 2.1). The fact that $\bar{v}_1 = f$ in $\Omega(\bar{\alpha})$ and $\|v_2\|_{*, \alpha} \leq C$ is obvious. Hence $\bar{v} \in Q^f_{ad}(\bar{\alpha})$. Let us show that $(\bar{\alpha}, \bar{v})$ is a solution of $(\hat{\mathbf{P}})_\varepsilon$. Denoting by $\hat{u}_n$, $\hat{u}$ solutions of $(\hat{\mathscr{P}}(\alpha_n, v_n))$, $(\hat{\mathscr{P}}(\bar{\alpha}, \bar{v}))$, respectively, we have a subsequence of $\{\hat{u}_n\}$ (denoted by the same symbol) such that

$$\hat{u}_n \rightharpoonup \hat{u} \quad \text{in } \hat{V} .$$

This, (2.11) and (1.2) yield :

$$\liminf_{n \to \infty} J(\alpha_n, \hat{u}_n|_{\Omega_n}) \geq J(\bar{\alpha}, \hat{u}|_{\Omega(\bar{\alpha})}) . \tag{2.12}$$

Using the similar approach as in [4], Lemma 1.1, one can prove

$$\int_0^1 (\hat{u}_n)^2 \, dx_2 \to \int_0^1 (\hat{u})^2 \, dx_2 .$$

This and (2.12) give the assertion of Proposition. □

In Appendix we prove that for any $\bar{\alpha} \in U_{\text{ad}}$ there exists $\bar{v} \in Q^f(\bar{\alpha})$ such that the solution $\hat{u}(\bar{v})$ of $(\hat{\mathscr{P}}(\bar{\alpha}, \bar{v}))$ restricted on $\Omega(\bar{\alpha})$ solves $(\mathscr{P}(\bar{\alpha}))$. Moreover, $\bar{v}$ lies in a ball, radius of which depends solely on $\|f\|_{L^2(\hat{\Omega})}$ (see Appendix, (A.4)). Next, *the constant C appearing in the definition of Q^f_{ad} will be greater or equal to the number on the right-hand side of* (A.4). Hence, $\bar{v}$ with the above mentioned property is the element of $Q^f_{\text{ad}}(\bar{\alpha})$.

Next, we analyze the relation between (**P**) and $(\hat{\mathbf{P}})_\varepsilon$ if $\varepsilon \to 0^+$. To this end, let $\{\varepsilon_k\}$ be a sequence of the penalty parameters tending to zero. By $(\hat{\mathbf{P}})_k$ we denote the problem $(\hat{\mathbf{P}})_\varepsilon$ with $\varepsilon = \varepsilon_k$. We show that these problems are closed in some sense.

PROPOSITION 2.4 : *Let $(\alpha_k^*, v_k^*) \in U_{\text{ad}} \times Q^f_{\text{ad}}(\alpha_k^*)$ be a solution of $(\hat{\mathbf{P}})_k$ and $\hat{u}_k^* \in \hat{V}$ the solution of $(\hat{\mathscr{P}}(\alpha_k^*, v_k^*))$. Then there exist subsequences $\{\alpha_{k_j}^*\}$, $\{v_{k_j}^*\}$, $\{\hat{u}_{k_j}^*\}$ and elements $\alpha^* \in U_{\text{ad}}$, $v^* \in Q^f_{\text{ad}}(\alpha^*)$ and $\hat{u}^* \in \hat{V}$ satisfying :*

$$\begin{cases} \alpha_{k_j}^* \rightrightarrows \alpha^* & \text{in } [0, 1] , \\ v_{k_j}^* \rightharpoonup v^* & \text{in } \hat{V}' , \\ \hat{u}_{k_j}^* \rightharpoonup \hat{u}^* & \text{in } \hat{V} . \end{cases} \tag{2.13}$$

Moreover, α^* is a solution of (**P**) *and $\hat{u}^*|_{\Omega(\alpha^*)}$ solves $(\mathscr{P}(\alpha^*))$.*

Proof : Let $\{\alpha_{k_j}^*\}$, $\{v_{k_j}^*\}$ and $\{\hat{u}_{k_j}^*\}$ be sequences, satisfying (2.13) (their construction is obvious). On the basis of Proposition 2.2 we know that $\hat{u}^*$ solves $(\hat{\mathscr{P}}(\alpha^*, v^*))$. The definition of $(\hat{\mathbf{P}})_k$ yields : $(\Omega_{k_j} \equiv \Omega(\alpha_{k_j}^*))$

$$J\left(\alpha_{k_j}^*, \hat{u}_{k_j}^*|_{\Omega_{k_j}}\right) \leq E_{\varepsilon_{k_j}}(\alpha_{k_j}^*, v_{k_j}^*) \leq$$

$$\leq E_{\varepsilon_{k_j}}(\alpha, v) \quad \forall (\alpha, v) \in U_{\text{ad}} \times Q^f_{\text{ad}}(\alpha) . \tag{2.14}$$

Let us fix $\bar{\alpha} \in U_{ad}$. According to Proposition A.1 (see Appendix), there exists $\bar{v} \in Q^f_{ad}(\bar{\alpha})$ such that $\hat{u}(\bar{v})|_{\Omega(\bar{\alpha})}$ solves $(\mathscr{P}(\bar{\alpha}))$ and especially $\hat{u}(\bar{v}) = 0$ on $\partial\Omega(\bar{\alpha})$ (let us recall that $\hat{u}(v)$ with $v \in Q_{ad}(\alpha)$ denotes the unique solution of $(\hat{\mathscr{P}}(\alpha, v))$). Substituting $(\bar{\alpha}, \bar{v})$ into the right-hand side of (2.14), we have

$$E_{\varepsilon_{k_j}}(\alpha^*_{k_j}, v^*_{k_j}) \leq E_{\varepsilon_{k_j}}(\bar{\alpha}, \bar{v}) = J(\bar{\alpha}, \hat{u}(\bar{v})|_{\Omega(\bar{\alpha})}) .$$

Hence

$$0 \leq \int_0^1 (\hat{u}^*_{k_j})^2 \, dx_2 \leq \varepsilon_{k_j} \left(J(\bar{\alpha}, \hat{u}(\bar{v})|_{\Omega(\bar{v})}) - J\left(\alpha^*_{k_j}, \hat{u}^*_{k_j}|_{\Omega_{k_j}} \right) \right) \to 0 . \quad (2.15)$$

At the same time

$$\int_0^1 (\hat{u}^*_{k_j})^2 \, dx_2 \to \int_0^1 (\hat{u}^*)^2 \, dx_2 .$$

Comparing this with (2.15) we see that $\hat{u}^* = 0$ on $\Gamma(\alpha^*)$, i.e., $\hat{u}^*|_{\Omega(\alpha^*)}$ solves $(\mathscr{P}(\alpha^*))$.

Let $\tilde{\alpha} \in U_{ad}$ be arbitrary and $u(\tilde{\alpha}) \in V_1(\tilde{\alpha})$ be the solution of $(\mathscr{P}(\tilde{\alpha}))$. Then there exists $\tilde{v} \in Q^f_{ad}(\tilde{\alpha})$ such that $\hat{u}(\tilde{v})|_{\Omega(\tilde{\alpha})} = u(\tilde{\alpha})$ (see Appendix, Remark A.1). Substituting $\tilde{\alpha}$, $\tilde{v}$ and $\hat{u}(\tilde{\alpha})|_{\Omega(\tilde{\alpha})}$ into the right-hand side of (2.14), we obtain

$$J\left(\alpha^*_{k_j}, \hat{u}^*_{k_j}|_{\Omega_{k_j}} \right) \leq J(\tilde{\alpha}, \hat{u}(\tilde{\alpha})|_{\Omega(\tilde{\alpha})}) = J(\tilde{\alpha}, u(\tilde{\alpha})) .$$

Passing to the limit with $j \to \infty$, we finally get

$$J(\alpha^*, \hat{u}^*|_{\Omega(\alpha^*)}) \leq J(\tilde{\alpha}, u(\tilde{\alpha}))$$

making use of (1.2). As $\hat{u}^*|_{\Omega(\alpha^*)}$ solves $(\mathscr{P}(\alpha^*))$, $\alpha^* \in U_{ad}$ and $\tilde{\alpha} \in U_{ad}$ is arbitrary, we arrive at the assertion of Proposition. □

Remark 2.1 : We may use an alternative approach. Up to now, we have considered that $v = (v_1, v_2) \in Q^f_{ad}$. This means, among others, that $v_1 = f$ on $\Omega(\alpha)$ for some $\alpha \in U_{ad}$. We may regard the previous equality to be the constraint, which can be treated by a penalty method again.

Let

$$Q_{ad}(\alpha) = \left\{ v = (v_1, v_2) \in Q(\alpha) | \; \|v_1\|_{\Omega(\alpha)} \leq \bar{C}_1, \; \|v_2\|_{*,\alpha} \leq \bar{C}_2 \right\} ,$$

where $\bar{C}_1$, $\bar{C}_2$ are given positive constants independent on $\alpha \in U_{ad}$ and where $\|\cdot\|_{\Omega(\alpha)}$ stands for the $L^2(\Omega(\alpha))$-norm Let us define

$$\mathscr{F}_\varepsilon(\alpha, v) \equiv J(\alpha, \hat{u}(v)|_{\Omega(\alpha)}) + \frac{1}{\varepsilon}\|v_1 - f\|^2_{\Omega(\alpha)} + \frac{1}{\varepsilon}\int_0^1 (\hat{u}(v))^2\, dx_2\,,$$

where $(\alpha, v) \in U_{ad} \times Q_{ad}(\alpha)$ and $\hat{u}(v)$ is the solution of $(\hat{\mathscr{P}}(\alpha, v))$.

We shall consider the following problem .

$$(\mathbf{R})_\varepsilon \qquad \begin{cases} \text{Find } (\alpha_\varepsilon^*, v_\varepsilon^*) \in U_{ad} \times Q_{ad}(\alpha_\varepsilon^*) \text{ such that} \\ \mathscr{F}_\varepsilon(\alpha_\varepsilon^*, v_\varepsilon^*) \leq \mathscr{F}_\varepsilon(\alpha, v) \quad \forall (\alpha, v) \in U_{ad} \times Q_{ad}(\alpha)\,. \end{cases}$$

PROPOSITION 2.5 : *For any $\varepsilon > 0$ there exists at least one solution $(\alpha_\varepsilon^*, v_\varepsilon^*)$ of* $(\mathbf{R})_\varepsilon$.

Proof is almost identical to that of Proposition 2.3. We only have to show that

$$\begin{cases} \alpha_n \rightrightarrows \alpha & \text{in } [0, 1] \\ v_n \rightharpoonup v & \text{in } \hat{V}'\,, \quad v_n = (v_{n1}, v_{n2}) \in Q(\alpha_n)\,, \quad v = (v_1, v_2) \in Q(\alpha) \end{cases} \tag{2 16}$$

imply

$$\liminf_{n\to\infty} \|v_{n1} - f\|^2_{\Omega_n} \geq \|v_1 - f\|^2_{\Omega(\alpha)}\,. \tag{2.17}$$

Indeed, from $(2.16)_2$ it follows that

$$\tilde{v}_{n1} \to v_1 \quad \text{in} \quad L^2(\Omega(\alpha))\,, \tag{2.18}$$

where $\tilde{v}_{n1}$ denotes the function v_{n1} extended by zero outside of Ω_n Let χ_n, χ be the characteristic functions of Ω_n, $\Omega(\alpha)$, respectively. As a consequence of $(2.16)_1$ we have

$$\chi_n \to \chi \quad \text{in } L^2(\hat{\Omega})\,. \tag{2.19}$$

Now

$$\begin{aligned} \|v_{n1} - f\|^2_{\Omega_n} &= \|\chi_n(\tilde{v}_{n1} - f)\|^2_{\Omega} \\ &= \|(\chi_n - \chi)(\tilde{v}_{n1} - f)\|^2_{\Omega} + \|\tilde{v}_{n1} - f\|^2_{\Omega(\alpha)}\,. \end{aligned}$$

Hence

$$\liminf_{n\to\infty} \|v_{n1} - f\|^2_{\Omega_n} \geq \|v_1 - f\|^2_{\Omega(\alpha)}$$

taking into account (2.18) and (2.19) □

Also, the result parallel to Proposition 2.4 can be established. To this end it is necessary to choose the constant $\bar{C}_1$, appearing in the definition of $Q_{ad}(\alpha)$ greater or equal to $\|f\|_{L^2(\Omega)}$.

3. APPROXIMATION OF $(\hat{\mathbf{P}})_\varepsilon$

The aim of the present section is to define and to analyze the discretization $(\hat{\mathbf{P}})_{\varepsilon h}$ of the problem $(\hat{\mathbf{P}})_\varepsilon$ with $\varepsilon > 0$ fixed. As the next step, the relation between $(\hat{\mathbf{P}})_\varepsilon$ and $(\hat{\mathbf{P}})_{\varepsilon h}$ for $h \to 0^+$ will be established. Let us note that we are not able to analyze the simultaneous limit procedure for ε, $h \to 0^+$.

We shall start with notations and definitions of finite dimensional approximations of sets, introduced in the previous section.

Let $\{D_H\}$, $H \to 0^+$ be a family of partitions of $[0, 1]$, $D_H : 0 = a_0 < a_1 < \dots < a_{N(H)} = 1$, such that $H \leq \beta_0 H_{\min}$, where $\beta_0 > 0$ does not depend on

$$H = \max_{i=0,\dots,N-1} |a_{i+1} - a_i| , \quad H_{\min} = \min_{i=0,\dots,N-1} |a_{i+1} - a_i| .$$

With any D_H, the following set will be associated :

$$U_{ad}^H = \left\{ \alpha_H \in C([0, 1]) | \alpha_H|_{a_i a_{i+1}} \in P_1(a_i\, a_{i+1}),\ i = 0, \dots, N-1 \right\} \cap U_{ad}$$

i.e., U_{ad}^H contains all functions from U_{ad} which are piecewise linear on D_H.

By $\{\mathcal{T}_h\}$, $h \to 0^+$ we denote a regular family of triangulations of $\bar{\hat{\Omega}}$. With any $\mathcal{T}_h$ the set of continuous, piecewise linear functions on $\hat{\Omega}$ and vanishing on $\partial\hat{\Omega}$ will be associated :

$$\hat{V}_h = \left\{ y_h \in C(\bar{\hat{\Omega}}) | y_h|_{T_i} \in P_1(T_i)\ \forall T_i \in \mathcal{T}_h,\ y_h = 0 \text{ on } \partial\hat{\Omega} \right\} .$$

U_{ad}^H, $\hat{V}_h$ are the finite dimensional approximations of U_{ad}, $\hat{V}$, respectively.

Mesh sizes h and H are independent each other, however, we shall assume that there exist positive constants τ_1, τ_2 such that $\tau_1 \leq h/H \leq \tau_2$. This means that discrete models can be characterized by one parameter, say h.

Let us recall that

$$\Omega(\alpha_H) = \left\{ [x_1, x_2] \in \mathbb{R}^2 | 0 < x_1 < \alpha_H(x_2),\ x_2 \in (0, 1) \right\} , \quad \alpha_H \in U_{ad}^H .$$

By $\Omega_h(\alpha_H)$ we denote the set $\Omega(\alpha_H)$, the partition of which is done by the restriction of $\mathcal{T}_h$ on $\Omega(\alpha_H)$, i.e., $\Omega_h(\alpha_H) = \operatorname{int}\left(\bigcup \overline{T_i \cap \Omega(\alpha_H)}\right)$. Similarly, $\Xi_h(\alpha_H) = \operatorname{int}\left(\bigcup \overline{T_i \cap \Xi(\alpha_H)}\right)$.

Further, let

$$L_h(\alpha_H) = \{v_h \in L^2(\Xi_h(\alpha_H)) | v_h|_{T_i} \in P_0(T_i)\ \forall T_i \text{ such that } T_i \cap \Xi_h(\alpha_H) \neq \emptyset\},$$

i.e., $L_h(\alpha_H)$ contains all piecewise constant functions on $\mathcal{T}_h$, restricted on $\Xi_h(\alpha_H)$ and set

$$Q_h(\alpha_H) = \left\{\mu_h = (v, w_{h0}, w_{h1}, w_{h2}) | v \in L^2(\Omega(\alpha_H)), (w_{h0}, w_{h1}, w_{h2}) \in (L_h(\alpha_H))^3\right\}.$$

If $\varphi \in \hat{V}$ and $\mu_h \in Q_h(\alpha_H)$, we define

$$[\mu_h, \varphi]_{\alpha_H} \equiv (v, \varphi)_{\Omega(\alpha_H)} + (w_{h0}, \varphi)_{\Xi_h(\alpha_H)} + \sum_{j=1}^{2}\left(w_{hj}, \frac{\partial\varphi}{\partial x_j}\right)_{\Xi_h(\alpha_H)}.$$

Remark 3.1 : $Q_h(\alpha_H)$ can be viewed to be the approximation of $Q(\alpha)$. To see that, let us recall that $v = (v_1, v_2) \in Q(\alpha)$ if and only if $v_1 \in L^2(\Omega(\alpha))$ and $v_2 \in V_2'(\alpha)$. The following representation of v_2 is known (see [5]) : there exist functions $f_0, f_1, f_2 \in L^2(\Xi(\alpha))$ such that

$$\langle v_2, \varphi\rangle_\alpha = (f_0, \varphi)_{\Xi(\alpha)} + \sum_{j=1}^{2}\left(f_j, \frac{\partial\varphi}{\partial x_j}\right)_{\Xi(\alpha)} \quad \forall\varphi \in V_2(\alpha) \tag{3.1}$$

and

$$\|(f_0, f_1, f_2)\|_{(L^2(\Xi(\alpha)))^3} = \|v_2\|_{*,\alpha}. \tag{3.2}$$

In other words, the functions f_0, f_1, f_2 characterizing $v_2 \in V_2'(\alpha)$ through (3.1) are approximated by means of piecewise constant functions.

Finally, let

$$Q_h^f(\alpha_H) = \left\{\mu_h = (v, w_{h0}, w_{h1}, w_{h2}) \in Q_h(\alpha_H) | v = f \text{ in } \Omega(\alpha_H)\right\}$$

and

$$Q_{\mathrm{ad},h}^f(\alpha_H) = \left\{\mu_h = (f, w_{h0}, w_{h1}, w_{h2}) \in Q_h^f(\alpha_H) \,\middle|\, \|(w_{h0}, w_{h1}, w_{h2})\|_{(L^2(\Xi_h(\alpha_H)))^3} \leq C\right\},$$

where $C > 0$ is the constant appearing in the definition of $Q_{\mathrm{ad}}^f(\alpha)$.

Instead of the state problem $(\hat{\mathscr{P}}(\alpha, v))$, $\alpha \in U_{ad}$, $v \in Q^f_{ad}(\alpha)$, we consider its *finite dimensional approximation :*

$$(\hat{\mathscr{P}}(\alpha_H, \mu_h))_h \quad \begin{cases} \text{Find } \hat{u}_h(\mu_h) \in \hat{V}_h \text{ such that} \\ (\nabla \hat{u}_h, \nabla \varphi_h)_{\hat{\Omega}} = [\mu_h, \varphi_h]_{\alpha_H} \quad \forall \varphi_h \in \hat{V}_h , \end{cases}$$

where $\alpha_H \in U^H_{ad}$, $\mu_h \in Q^f_{ad,\,h}(\alpha_H)$. For any $\varepsilon > 0$ we define

$$E_\varepsilon(\alpha_H, \mu_h) \equiv J(\alpha_H, \hat{u}_h(\mu_h)\big|_{\Omega_h(\alpha_H)}) + \frac{1}{\varepsilon}\int_0^1 (\hat{u}_h(\mu_h))^2\, dx_2 ,$$

where $\hat{u}_h(\mu_h) \in \hat{V}_h$ solves $(\hat{\mathscr{P}}(\alpha_H, \mu_h))_h$.

By the *approximation of* $(\hat{\mathbf{P}})_\varepsilon$, $\varepsilon > 0$ fixed, we mean the problem

$$(\hat{\mathbf{P}})_{\varepsilon h} \quad \begin{cases} \text{Find } (\alpha^*_{\varepsilon H}, \mu^*_{\varepsilon h}) \in U^H_{ad} \times Q^f_{ad,\,h}(\alpha^*_{\varepsilon H}) \text{ such that} \\ E_\varepsilon(\alpha^*_{\varepsilon H}, \mu^*_{\varepsilon h}) \leq E_\varepsilon(\alpha_H, \mu_h) \quad \forall (\alpha_H, \mu_h) \in U^H_{ad} \times Q^f_{ad,\,h}(\alpha_H) . \end{cases}$$

Concerning the solution of $(\hat{\mathbf{P}})_{\varepsilon h}$ we have

PROPOSITION 3.1 : *If* (1.2) *is satisfied,* $(\hat{\mathbf{P}})_{\varepsilon h}$ *has at least one solution* $(\alpha^*_{\varepsilon H}, \mu^*_{\varepsilon h})$.

Proof is parallel to the proof of Proposition 2.3. □

Next, we analyze the mutual relation between $(\hat{\mathbf{P}})_\varepsilon$ and $(\hat{\mathbf{P}})_{\varepsilon h}$, assuming $h \to 0^+$. We shall show that under additional assumptions on J, the problems are closed each other. Before doing that, let us summarize some basic results needed in what follows.

PROPOSITION 3.2 : *For any* $\alpha \in U_{ad}$ *there exists a sequence* $\{\alpha_H\}$, $\alpha_H \in U^H_{ad}$ *such that*

$$\alpha_H \rightrightarrows \alpha , \quad H \to 0^+ \quad \text{in } [0, 1] . \tag{3.4}$$

Proof : See [2].

PROPOSITION 3.3 : *For any* $y \in \hat{V}$ *there exists a sequence* $\{y_h\}$, $y_h \in \hat{V}_h$ *such that*

$$y_h \to y , \quad h \to 0^+ \quad \text{in } \hat{V} . \tag{3.5}$$

PROPOSITION 3.4 : *Let* $\alpha_H \rightrightarrows \alpha$ *in* $[0, 1]$, $\alpha_H \in U^H_{ad}$, $\alpha \in U_{ad}$, $\mu_h \in Q^f_{ad,\,h}(\alpha_H)$ *be such that*

$$\mu_h \rightharpoonup \mu \quad in\ \hat{V}' .$$

Then $\mu \in Q^f_{ad}(\alpha)$.

Proof : Let $\mu_h = (f, w_{h0}, w_{h1}, w_{h2}) \in Q^f_{\mathrm{ad},h}(\alpha_H)$ and $\varphi \in \hat{V}$. Then

$$[\mu_h, \varphi]_{\alpha_H} = (f, \varphi)_{\Omega(\alpha_H)} + (w_{h0}, \varphi)_{\Xi_h(\alpha_H)} + \sum_{j=1}^{2} \left(w_{hj}, \frac{\partial \varphi}{\partial x_j} \right)_{\Xi_h(\alpha_H)} \to \langle \mu, \varphi \rangle . \tag{3.6}$$

Let the symbol ~ denote the extension by zero of the corresponding function outside of its domain of definition. From the definition of $Q^f_{\mathrm{ad},h}(\alpha_H)$ it follows that there exist subsequences of $\{\tilde{w}_{h0}\}$, $\{\tilde{w}_{h1}\}$ and $\{\tilde{w}_{h2}\}$ that converge weakly in $L^2(\hat{\Omega})$ to functions $\tilde{w}_0$, $\tilde{w}_1$ and $\tilde{w}_2$. It is easy to see that $\tilde{w}_0 = \tilde{w}_1 = \tilde{w}_2 \equiv 0$ on $\Omega(\alpha)$. Let $w_j = \tilde{w}_j\big|_{\Xi(\alpha)}$, $j = 0, 1, 2$. One can easily verify that

$$(f, \varphi)_{\Omega(\alpha_H)} \to (f, \varphi)_{\Omega(\alpha)}$$

$$(w_{h0}, \varphi)_{\Xi_h(\alpha_H)} \to (w_0, \varphi)_{\Xi(\alpha)}, \quad \text{etc.}$$

Hence,

$$\langle \mu, \varphi \rangle = (f, \varphi)_{\Omega(\alpha)} + (w_0, \varphi)_{\Xi(\alpha)} + \sum_{j=1}^{2} \left(w_j, \frac{\partial \varphi}{\partial x_j} \right)_{\Xi(\alpha)} = [\mu, \varphi]_{\alpha} .$$

As $\| (w_0, w_1, w_2) \|_{(L^2(\Xi(\alpha)))^3} \leq C$, $\mu \in Q^f_{\mathrm{ad}}(\alpha)$.

PROPOSITION 3.5 : *Let* $\alpha_H \rightarrow \alpha$ *in* $[0, 1]$, $\alpha_H \in U^H_{\mathrm{ad}}$, $\alpha \in U_{\mathrm{ad}}$, $\mu_h \in Q^f_{\mathrm{ad},h}(\alpha_H)$ *be such that*

$$\mu_h \rightharpoonup \mu \quad \text{in } \hat{V}' .$$

If $\hat{u}_h$ *denotes the solution of* $(\hat{\mathcal{P}}(\alpha_H, \mu_h))_h$, *then there exist a subsequence* $\{\hat{u}_{h'}\} \subset \{\hat{u}_h\}$ *such that*

$$\hat{u}_{h'} \rightharpoonup \hat{u} \quad \text{in } \hat{V} \tag{3.7}$$

and $\hat{u} \in \hat{V}$ *solves* $(\hat{\mathcal{P}}(\alpha, \mu))$.

Proof : The construction of a subsequence, satisfying (3.7) is obvious. It remains to prove that $\hat{u}$ solves $(\hat{\mathcal{P}}(\alpha, \mu))$. But this is a direct consequence of Propositions 3.3 and 3.4.

PROPOSITION 3.6 : *Let* $\alpha \in U_{\mathrm{ad}}$, $\mu \in Q^f_{\mathrm{ad}}(\alpha)$ *be given. Then there exist* $\alpha_H \in U^H_{\mathrm{ad}}$ *and* $\mu_h \in Q^f_{\mathrm{ad},h}(\alpha_H)$ *such that :*

$$\alpha_H \rightrightarrows \alpha , \quad H \to 0^+ \quad \text{in } [0, 1] ; \tag{3.8}$$

$$\mu_h \to \mu \ (\text{strongly}) \quad \text{in } \hat{V}' . \tag{3.9}$$

Proof : According to Proposition 3.2, there exists $\alpha_H \in U^H_{ad}$ satisfying (3.8). Let $\mu \in Q^f_{ad}(\alpha)$ be given,

$$[\mu, \varphi]_\alpha = (f, \varphi)_{\Omega(\alpha)} + (w_0, \varphi)_{\Xi(\alpha)} + \sum_{j=1}^{2} \left(w_j, \frac{\partial \varphi}{\partial x_j} \right)_{\Xi(\alpha)},$$

with $f \in L^2(\Omega(\alpha))$, $w_j \in L^2(\Xi(\alpha))$, $j = 0, 1, 2$. Recall that the symbol $\tilde{}$ stands for the extension of functions by zero from the domain of their definition on $\hat{\Omega}$. Let $q_{hj} = P\tilde{w}_j$, $j = 0, 1, 2$, where P is the orthogonal projection of functions from $L^2(\hat{\Omega})$ on

$$L_h(\hat{\Omega}) = \left\{ \varphi \in L^2(\hat{\Omega}) \mid \varphi|_T \in P_0(T)\ \forall T \in \mathcal{T}_h \right\}.$$

Then the quadruple

$$\mu_h = (f|_{\Omega(\alpha_H)}, w_{h0}, w_{h1}, w_{h2}),$$

where $w_{hj} = q_{hj}|_{\Xi(\alpha_H)}$ satisfies (3.9). Indeed :

$$\left\| \widetilde{(f|_{\Omega(\alpha_H)})} - \widetilde{(f|_{\Omega(\alpha)})} \right\|_{\hat{\Omega}} \to 0, \quad H \to 0^+,$$

$$\| \tilde{w}_{hj} - \tilde{w}_j \|_{\hat{\Omega}} \to 0, \quad h \to 0^+, \quad j = 0, 1, 2.$$

The fact that $\mu_h \in Q^f_{ad,h}(\alpha_H)$ is obvious.

In order to establish the mutual relation between $(\hat{\mathbf{P}})_\varepsilon$ and $(\hat{\mathbf{P}})_{\varepsilon h}$ for $h \to 0^+$, we shall need, besides of (1.2), the following assumption, concerning the continuity of J :

$$\left.\begin{cases} \text{if } \alpha_H \rightrightarrows \alpha \text{ in } [0,1],\ \alpha_H \in U^H_{ad},\ \alpha \in U_{ad} \\ \text{and} \\ \text{if } y_h \to y \text{ (strongly) in } \hat{V},\ y_h \in \hat{V}_h,\ y \in \hat{V} \end{cases}\right\} \Rightarrow$$

$$\Rightarrow \lim_{h \to 0^+} J(\alpha_H, y_h|_{\Omega(\alpha_H)}) = J(\alpha, y|_{\Omega(\alpha)}). \quad (3.10)$$

The *main* result of this section is

PROPOSITION 3.7 : *Let* (3.10) *be satisfied. If* $(\alpha^*_{\varepsilon H}, \mu^*_{\varepsilon h})$ *is a solution of* $(\hat{\mathbf{P}})_{\varepsilon h}$ *and* $u^*_{\varepsilon h} \in \hat{V}_h$ *is the corresponding solution of* $(\mathscr{P}(\alpha^*_{\varepsilon H}, \mu^*_{\varepsilon h}))_h$, *there* *exist subsequences* $\{\alpha^*_{\varepsilon H'}\} \subset \{\alpha^*_{\varepsilon H}\}$, $\{\mu^*_{\varepsilon h'}\} \subset \{\mu^*_{\varepsilon h}\}$, $\{u^*_{\varepsilon h'}\} \subset \{u^*_{\varepsilon h}\}$ *and elements* $\alpha^*_\varepsilon \in U_{\text{ad}}$, $\mu^*_\varepsilon \in Q^f_{\text{ad}}(\alpha^*_\varepsilon)$, $u^*_\varepsilon \in \hat{V}$ *such that*

$$\begin{cases} \alpha^*_{\varepsilon H'} \rightrightarrows \alpha^*_\varepsilon , & H' \to 0^+ , \quad in\ [0, 1] ; \\ \mu^*_{\varepsilon h'} \rightharpoonup \mu^*_\varepsilon , & h' \to 0^+ , \quad in\ \hat{V}' ; \\ u^*_{\varepsilon h'} \rightharpoonup u^*_\varepsilon & in\ \hat{V} \end{cases} \tag{3.11}$$

and $(\alpha^*_\varepsilon, \mu^*_\varepsilon)$ *solves* $(\hat{\mathbf{P}})_\varepsilon$ *and* u^*_ε *is the solution of* $(\hat{\mathscr{P}}(\alpha^*_\varepsilon, \mu^*_\varepsilon))$.

Proof : The existence of subsequences satisfying (3.11) is obvious as well as the fact that $\alpha^*_\varepsilon \in U_{\text{ad}}$, $\mu^*_\varepsilon \in Q^f_{\text{ad}}(\alpha^*_\varepsilon)$ and u^*_ε solves $(\hat{\mathscr{P}}(\alpha^*_\varepsilon, \mu^*_\varepsilon))$. Taking into account (1.2), we see that

$$E_\varepsilon(\alpha^*_\varepsilon, \mu^*_\varepsilon) \leq \liminf_{h' \to 0^+} E_\varepsilon(\alpha^*_{\varepsilon H'}, \mu^*_{\varepsilon h'}) . \tag{3.12}$$

Let $(\bar{\alpha}, \bar{\mu}) \in U_{\text{ad}} \times Q^f_{\text{ad}}(\bar{\alpha})$ be given. According to Proposition 3.6, there exist $\bar{\alpha}_H \in U^H_{\text{ad}}$, $\bar{\mu}_h \in Q^f_{\text{ad}, h}(\bar{\alpha}_H)$ such that

$$\begin{cases} \bar{\alpha}_H \rightrightarrows \bar{\alpha} , & H \to 0 \quad \text{in } [0, 1] \\ \bar{\mu}_h \to \bar{\mu} , & h \to 0^+ \quad \text{in } \hat{V}' . \end{cases} \tag{3.13}$$

If $\bar{u}_h$ denotes the solution of $(\hat{\mathscr{P}}(\bar{\alpha}_H, \bar{\mu}_h))_h$, from Proposition 3.5 and $(3.13)_2$ we obtain

$$\bar{u}_h \to \bar{u} \text{ (strongly)} \quad \text{in } \hat{V} ,$$

with $\bar{u}$ being the solution of $(\hat{\mathscr{P}}(\bar{\alpha}, \bar{\mu}))$. Now, the definition of $(\hat{\mathbf{P}})_{\varepsilon h}$ and (3.10) yield :

$$\liminf_{h' \to 0^+} E_\varepsilon(\alpha^*_{\varepsilon H'}, \mu^*_{\varepsilon h'}) \leq \liminf_{h' \to 0^+} E_\varepsilon(\bar{\alpha}_{H'}, \bar{\mu}_{h'}) = E_\varepsilon(\bar{\alpha}, \bar{\mu}) . \tag{3.14}$$

As $(\bar{\alpha}, \bar{\mu}) \in U_{\text{ad}} \times Q^f_{\text{ad}}(\bar{\alpha})$ is an arbitrary element, (3.12) and (3.14) yield the assertion of Proposition. □

Remark 3.3 : The numerical realization of $(\hat{\mathbf{P}})_{\varepsilon h}$ has two great advantages :

1) the state problem is still solved on the *fixed* domain $\hat{\Omega}$ with the *fixed* triangulation $\mathscr{T}_h$;

2) the stiffness matrix of the state problem $(\hat{\mathscr{P}}(\alpha_H, \mu_h))_h$ *does not change* during the calculations. The right-hand side of the corresponding linear system is changed, only. This is of a great importance. Using the method of factorization, one can solve very efficiently the discrete state problem.

4. SENSITIVITY ANALYSIS AND NUMERICAL REALIZATION OF $(\hat{\mathrm{P}})_{\varepsilon h}$

Let $h > 0$ and $H > 0$ be fixed. The state problem $(\hat{\mathscr{P}}(\alpha_H, \mu_h))_h$ expressed in the matrix form reads as follows :

$$A\mathbf{u}(\boldsymbol{\alpha}, \boldsymbol{\mu}) = F(\boldsymbol{\alpha}, \boldsymbol{\mu}), \tag{4.1}$$

where A is a symmetric, positive definite matrix (stiffness matrix) and $F(\boldsymbol{\alpha}, \boldsymbol{\mu})$ is a right-hand side vector given by

$$F(\boldsymbol{\alpha}, \boldsymbol{\mu}) = F_1(\boldsymbol{\alpha}) + F_2(\boldsymbol{\alpha}, \boldsymbol{\mu}),$$

$$(F_1(\boldsymbol{\alpha}))_i = \int_{\Omega_h(\alpha_H)} f\varphi_i \,\mathrm{d}x,$$

$$(F_2(\boldsymbol{\alpha}, \boldsymbol{\mu}))_i = \int_{\Xi_h(\alpha_H)} \left(w_{h0}\,\varphi_i + w_{h1}\frac{\partial \varphi_i}{\partial x_1} + w_{h2}\frac{\partial \varphi_i}{\partial x_2} \right) \mathrm{d}x,$$

$$i = 1, \dots, \dim \hat{V}_h,$$

with φ_i being the (Courant) basis functions of $\hat{V}_h$. Hence, the vector F depends and the matrix A *does not* depend on the vector of discrete design variables $(\boldsymbol{\alpha}, \boldsymbol{\mu}) \in \mathbb{R}^{N+3N_1+4}$,

$$\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \dots, \alpha_N), \quad \alpha_i = \alpha_H(a_i), \quad i = 0, 1, \dots, N, \tag{4.2}$$

$$\boldsymbol{\mu} = (w_{00}, w_{01}, \dots, w_{0N_1} w_{10}, w_{11}, \dots, w_{1N_1}, w_{20}, w_{21}, \dots, w_{1N_1}),$$

$$w_{0i} = w_{h0}|_{T_i}, \quad w_{1i} = w_{h1}|_{T_i}, \quad w_{2i} = w_{h2}|_{T_i},$$

$$T_i \in \mathcal{T}_h, \quad i = 1, \dots, N_1, \quad \hat{\Omega} = \bigcup_{i=1}^{N_1} T_i,$$

for $(\alpha_H, \mu_h) \in U^H_{\mathrm{ad}} \times Q^f_{\mathrm{ad},\,h}(\alpha_H)$, $\mu_h = (f, w_{h0}, w_{h1}, w_{h2})$.

Now it is important to notice that the mapping

$$\boldsymbol{\alpha} \mapsto \mathbf{u}(\boldsymbol{\alpha}, \boldsymbol{\mu}) \tag{4.3}$$

is not continuously differentiable, in general. This is readily seen from the following simple example.

Example 4 1 Consider a one-dimensional boundary-value problem

$$\begin{cases} \text{Find } \hat{u}(v) \in \hat{V} = H_0^1((0, 1)) \text{ such that} \\ \int_0^1 \hat{u}(v)' \varphi' \, dx = \int_0^\alpha f\varphi \, dx + \int_\alpha^1 v\varphi \, dx \quad \forall \varphi \in \hat{V} \end{cases} \tag{4 4}$$

where $\alpha \in (0, 1)$, f, $v \in L^2((0, 1))$, f fixed and v variable Further, consider a partition of $[0, 1]$ The discretization of (4 4) by piecewise linear elements leads to an algebraic system

$$A\mathbf{u}(\alpha, w) = F(\alpha, w),$$

where A is the well-known three-diagonal stiffness matrix and

$$(F(\alpha, w))_i = \int_0^\alpha f\varphi_i \, dx + \int_\alpha^1 w\varphi_i \, dx$$

It is obvious that if w is a piecewise constant approximation of v on (0, 1), then the function

$$\alpha \mapsto \int_\alpha^1 w\varphi_i \, dx$$

is not continuously differentiable at points of discontinuity of w but only directionally differentiable Thus one can not expect the differentiability of the mapping (4 3) □

Analogously, the mapping $\boldsymbol{\alpha} \mapsto F(\boldsymbol{\alpha}, \boldsymbol{\mu})$ is not differentiable in two-dimensional situation (and neither the mapping (4 3)), in general However, the mapping $\boldsymbol{\alpha} \mapsto F(\boldsymbol{\alpha}, \boldsymbol{\mu})$ is locally Lipschitz continuous and we are able to employ methods of nondifferentiable optimization

Remark 4 1 The nondifferentiably of (4 3) is caused by the type of finite elements used When we use C^1 elements for discretization of $(\hat{\mathscr{P}}(\alpha, v))$ and C^0 elements for approximation of functions from $Q(\alpha)$, the problem remains differentiable but becomes very complicated and very large □

Now let us assume that the objective functional J is quadratic, for example Therefore, its discretization is given by a quadratic function, determined by a symmetric, positive definite matrix $A_0 = A_0(\boldsymbol{\alpha})$ (for the sake of simplicity, we assume that the linear and absolute term are equal to zero) Again, the mapping $\boldsymbol{\alpha} \mapsto A_0(\boldsymbol{\alpha})$ is generally nondifferentiable Let $M = M(\alpha)$ be a matrix realizing the penalty integral (the exact definition will be given below). The matrix form $\mathbf{E}_\varepsilon(\boldsymbol{\alpha}, \boldsymbol{\mu})$ of the functional $E_\varepsilon(\alpha_H, \mu_h)$ is given by

$$\mathbf{E}_\varepsilon(\boldsymbol{\alpha}, \boldsymbol{\mu}) = \frac{1}{2}(\mathbf{u}, A_0 \mathbf{u}) + \frac{1}{2\varepsilon}(\mathbf{u}, M\mathbf{u}), \tag{4.5}$$

where $\mathbf{u}$ solves (4.1), the symbol (,) stands for the scalar product in corresponding vector spaces and the problem $(\hat{\mathbf{P}})_{\varepsilon h}$ reads now as follows :

$$\begin{cases} \text{Find } (\boldsymbol{\alpha}^*, \boldsymbol{\mu}^*) \in \mathbf{U} \times \mathbf{Q}(\boldsymbol{\alpha}^*) \text{ such that} \\ \mathbf{E}_\varepsilon(\boldsymbol{\alpha}^*, \boldsymbol{\mu}^*) \leq \mathbf{E}_\varepsilon(\boldsymbol{\alpha}, \boldsymbol{\mu}) \quad \forall (\boldsymbol{\alpha}, \boldsymbol{\mu}) \in \mathbf{U} \times \mathbf{Q}(\mathbf{a}), \end{cases} \tag{4.6}$$

where $\mathbf{U} \times \mathbf{Q}(\boldsymbol{\alpha})$ is a subset of $\mathbb{R}^{N+3N_1+4}$, isometrically isomorphic with $U_{\text{ad}}^H \times Q_{\text{ad},h}^f(\alpha_H)$ by (4.2).

The mapping $(\boldsymbol{\alpha}, \boldsymbol{\mu}) \mapsto F(\boldsymbol{\alpha}, \boldsymbol{\mu})$ is locally Lipschitz continuous over $\mathbf{U} \times \mathbf{Q}(\boldsymbol{\alpha})$ and directionally differentiable, in general. Therefore, there is a chance for a successfull implementation of some nondifferentiable optimization (NDO) method for the solution of (4.6). Such methods require computation of at least one vector (subgradient) $\xi \in \partial \mathbf{E}_\varepsilon(\boldsymbol{\alpha}, \boldsymbol{\mu})$ of the generalized gradient of $\mathbf{E}_\varepsilon$, at any point $(\boldsymbol{\alpha}, \boldsymbol{\mu})$. Here and in what follows, $\partial G(x_0)$ denotes the generalized gradient of G at x_0 (see e.g. [3]) and $\nabla G(x_0)$ denotes the gradient of G at x_0. As usual, $\nabla_{x_i} G(x_1, \ldots, x_n)$ means the gradient of G with respect to x_i-variable.

In our case, the subgradient ξ can be computed by Proposition 1.3 from [6] : If $\mathbf{E}_\varepsilon$ is continuously differentiable as a function of $\mathbf{u}$, A is a regular matrix and the mapping $(\boldsymbol{\alpha}, \boldsymbol{\mu}) \mapsto F(\boldsymbol{\alpha}, \boldsymbol{\mu})$ is locally lipschitz continuous (all these assumptions are fulfilled) then $\mathbf{E}_\varepsilon$ is locally Lipschitz continuous as the implicit function of $(\boldsymbol{\alpha}, \boldsymbol{\mu})$ and

$$(\partial_{\boldsymbol{\alpha}} F(\boldsymbol{\alpha}, \boldsymbol{\mu}), \mathbf{p}) + \frac{1}{2}(\mathbf{u}, \partial_{\boldsymbol{\alpha}} A_0(\boldsymbol{\alpha})\mathbf{u}) + \frac{1}{\varepsilon}(\partial_{\boldsymbol{\alpha}} F(\boldsymbol{\alpha}, \boldsymbol{\mu}), \mathbf{q}) + \\ + \frac{1}{2\varepsilon}(\mathbf{u}, \nabla_{\boldsymbol{\alpha}} M(\boldsymbol{\alpha})\mathbf{u}) \subset \partial_{\boldsymbol{\alpha}} \mathbf{E}_\varepsilon(\boldsymbol{\alpha}, \boldsymbol{\mu}). \tag{4.7}$$

$$(\nabla_{\boldsymbol{\mu}} F(\boldsymbol{\alpha}, \boldsymbol{\mu}), \mathbf{p}) + \frac{1}{\varepsilon}(\nabla_{\boldsymbol{\mu}} F(\boldsymbol{\alpha}, \boldsymbol{\mu}), \mathbf{q}) \in \partial_{\boldsymbol{\mu}} \mathbf{E}_\varepsilon(\boldsymbol{\alpha}, \boldsymbol{\mu}), \tag{4.8}$$

where $\mathbf{p}$, $\mathbf{q}$ solve the adjoint equations

$$A\mathbf{p} = A_0 \mathbf{u}, \qquad A\mathbf{q} = M\mathbf{u}. \tag{4.9}$$

If we assume that A_0 does not depend on $\boldsymbol{\alpha}$ (as in [9]), the formula (4.7) becomes more simple :

$$(\partial_{\boldsymbol{\alpha}} F(\boldsymbol{\alpha}, \boldsymbol{\mu}), \mathbf{p}) + \frac{1}{\varepsilon}(\partial_{\boldsymbol{\alpha}} F(\boldsymbol{\alpha}, \boldsymbol{\mu}), \mathbf{q}) + \\ + \frac{1}{2\varepsilon}(\mathbf{u}, \nabla_{\boldsymbol{\alpha}} M(\boldsymbol{\alpha})\mathbf{u}) \subset \partial_{\boldsymbol{\alpha}} \mathbf{E}_\varepsilon(\boldsymbol{\alpha}, \boldsymbol{\mu}), \tag{4.7a}$$

Thus, computing of a subgradient $\xi \in \partial \mathbf{E}_\varepsilon$ requires

(i) solving of the adjoint equations (4 9) with the matrix A which is independent of $(\boldsymbol{\alpha}, \boldsymbol{\mu})$ and which has been factorized at the beginning of the computation once for all ,

(ii) computation of a subgradient from $\partial_{\boldsymbol{\alpha}} F(\boldsymbol{\alpha}, \boldsymbol{\mu})$ and the gradients $\nabla_{\boldsymbol{\mu}} F(\boldsymbol{\alpha}, \boldsymbol{\mu})$ and $\nabla_{\boldsymbol{\alpha}} M(\boldsymbol{\alpha})$

The latter will be the subject of the rest of this section

To this end, let us assume that the triangulation $\mathcal{T}$ is constructed as follows first we divide $\hat{\Omega}$ into rectangles

$$R_{ij} = [ih, (i+1)h] \times [jk, (j+1)k], \quad h, \quad k > 0$$

and then each R_{ij} will be divided into two triangles We shall also assume that nodes of the partition D_H are given by $a_j = jk$, $j = 0, \ldots, N$, i e , nodes of $\mathcal{T}$ and vertices of $\alpha_H \in U_{\text{ad}}^H$ lie on the same lines, parallel with x_1-axe The triangulation $\mathcal{T}$ can be splitted into two parts

(i) $\mathcal{T}_0$ containing triangles $T_i \subset \overline{\Omega(\alpha_H)}$ and $T_i \subset \overline{\Xi(\alpha_H)}$,

(ii) $\mathcal{T}_{\alpha_H}$ containing triangles « cutted » by α_H, i e , lying partly in $\overline{\Omega(\alpha_H)}$ and partly in $\overline{\Xi(\alpha_h)}$

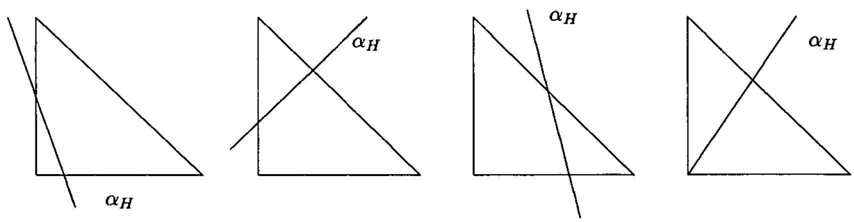

Figure 4 1

Some examples of triangles from $\mathcal{T}_{\alpha_H}$ are depicted on figure 4 1

For a particular triangle $T_i \in \mathcal{T}_{\alpha_H}$, see figure 4 2, we shall show how one can compute the element right-hand side vector F^i, the element matrix M^i and corresponding gradients

The element right-hand side vector is given as the sum of integrals of the type

$$\int_{T_i^k} w\varphi_j \, dx , \tag{4 10}$$

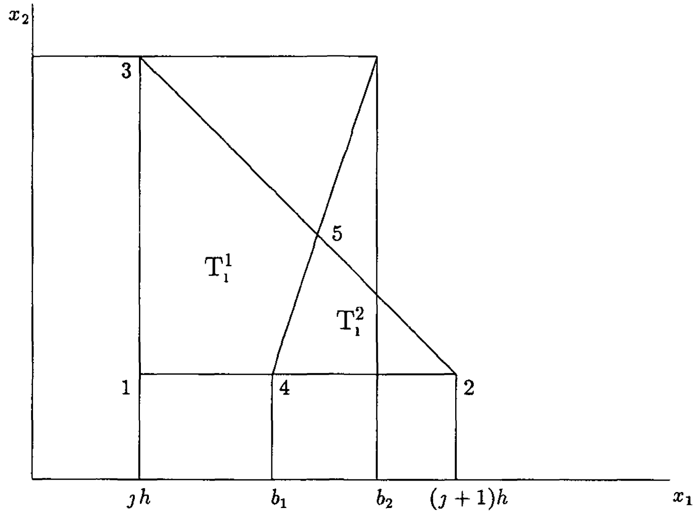

Figure 4.2.

or

$$\int_{T_\iota^k} w \frac{\partial \varphi_j}{\partial x_l} dx, \quad k = 1, 2, \quad j = 1, 2, 3, \quad l = 1, 2, \qquad (4.10)'$$

where

$$T_\iota^1 = T_\iota \cap \overline{\Omega(\alpha_H)}, \quad T_\iota^2 = T_\iota \cap \overline{\Xi(\alpha_H)}$$

and φ_j are the (linear) basis functions corresponding to nodes 1, 2, 3 (or their derivatives — this case is trivial). Suppose that w is constant on T_ι. Then for $\varphi \equiv \varphi_j$, j fixed, the integral (4.10) can be computed either as

$$w_\iota \text{ meas } (T_\iota^1) \frac{\varphi^1 + \varphi^3 + \varphi^4 + \varphi^5}{4} \quad \text{for } k = 1$$

or as

$$w_\iota \text{ meas } (T_\iota^2) \frac{\varphi^2 + \varphi^4 + \varphi^5}{3} \quad \text{for } k = 2,$$

where φ^j are values of φ at points 1, 2, 3, 4, 5, the cartesian coordinates of which are $(x_1^{(j)}, x_2^{(j)})$. The values φ^j, vol (T_ι^k) depend on b_1, b_2, e.g.

$$\text{meas } (T_\iota^2) = \frac{1}{2} \frac{((j+1)h - b_1)^2 h}{(b_2 - b_1 + h)}$$

(here we suppose that $h = k$, for simplicity) ; other formulae are similar. Once one has these formulae, the computation of corresponding gradients will be obvious.

The element matrix M^ι is defined as a mass matrix of the segment $[x_2^{(4)}, x_2^{(5)}]$, i.e.,

$$M^\iota = \frac{x_2^{(5)} - x_2^{(4)}}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

the « element » penalty part of $\mathbf{E}_\varepsilon$ as

$$(u_4, u_5)\, M^\iota (u_4, u_5)^T . \tag{4.11}$$

As u_4, u_5 depends on u_1, u_2, u_3 and b_1, b_2 and $x_2^{(4)}$, $x_2^{(5)}$ depends on b_1, b_2, (4.10) can be written as

$$(u_1, u_2, u_3)\, \tilde{M}^\iota (u_1, u_2, u_3)^T$$

with $\tilde{M}^\iota$ depending on b_1, b_2. Once we have the formulae for the elements of $\tilde{M}^\iota$, we shall be able to compute the corresponding gradients easily.

5. NUMERICAL EXAMPLE

In this section we shall demonstrate the capability of the presented method on a simple one-dimensional model example introduced in Example 4.1. For the sake of simplicity, we shall use linear, nondifferentiable cost functional and penalty term. In fact, this approach is not quite correct from the theoretical point of wiev, because the inclusion (4.7) no longer hold. Nevertheless, in this simple example, the method as well as the used NDO algorithm work very satisfactorily.

Example 5.1 : Let the state problem be given as in Example 4.1 with $f = 0.1$. Its solution $\hat{u}(v)$ is approximated by means of piecewise linear functions over an equidistant partition D_h of [0, 1] and the variable right-hand side by piecewise constant functions over the same partition D_h. Let $\mathbf{U} = [0.55, 0.75]$, $\mathbf{Q} = \{w \in \mathbb{R}^N \mid -1 \leq w_\iota \leq 1\}$, $N = 1/h$. The discretized cost functional reads as

$$\mathbf{E} \equiv |u_c - \psi_c| + |u_\alpha| ,$$

where $\psi = -0.05\,(x - 0.31)^2 + 0.004805$ is the exact solution of (4.4) for $\hat{V} = H_0^1((0, 0.62))$, u_c is the c-th component of the vector $\mathbf{u}$, i.e., the value of the approximate solution of (4.4) at a node $x^{(c)} \in D_h$, lying in interval

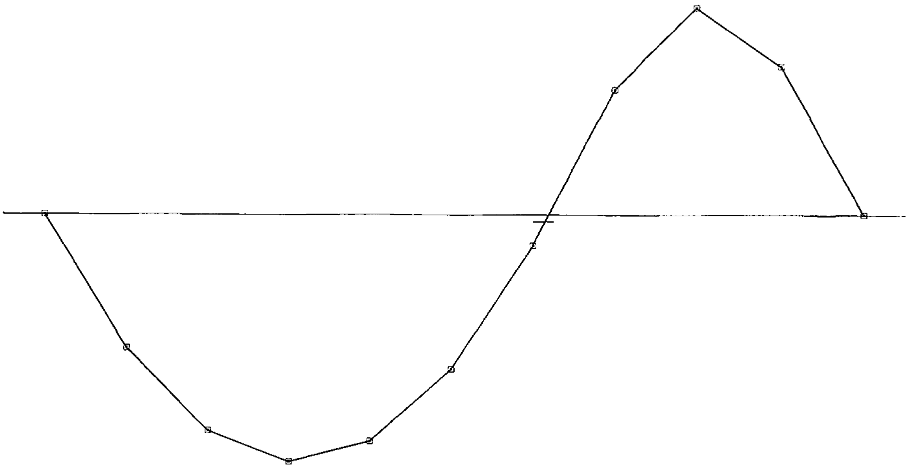

Figure 5.1.

[0, 0.62], $\psi_c = \psi(x^{(c)})$ and u_α is obtained by linear interpolation of **u** from the nodes neighbouring to α. Thus we can expect the optimal value of α to be $\alpha_{\text{opt}} = 0.62$ and the optimal value of **E** to be $\mathbf{E}_{\text{opt}} = 0$. We have computed this example for two values of the discretization parameter $h = 1/10$ and $h = 1/100$ by the NDO code BT [8]. The values of α_{opt}, $\mathbf{E}_{\text{opt}}$ are given in Table 1. Figures 5.1 and 5.2 show the final values of the design variables α, the « additional » right-hand side ω, as well as the solutions of the state problems.

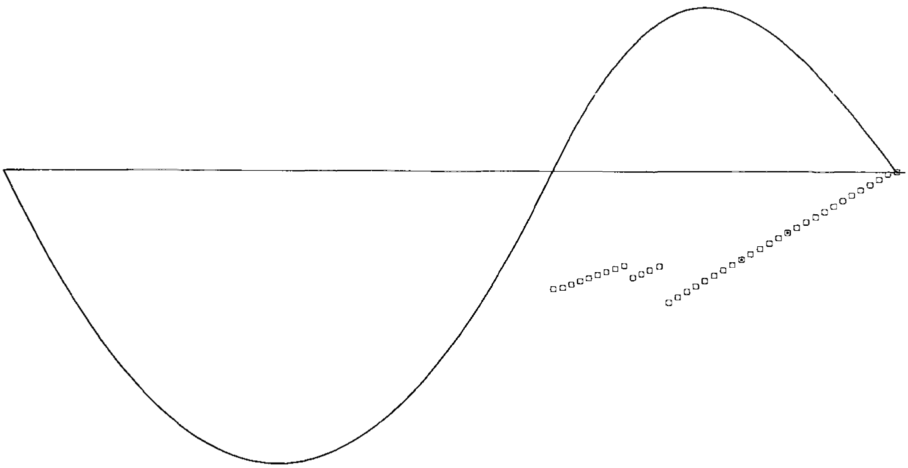

Figure 5.2.

Table 1

h	c	α_{opt}	$\mathbf{E}_{\text{opt}}$
1/10	4	0 619836	$0\,4 \times 10^{-6}$
1/100	31	0 619952	$0\,1 \times 10^{-5}$

APPENDIX

Notation will be the same as before Let $(\alpha, v) \in U_{\text{ad}} \times Q(\alpha)$ In Section 2 we have introduced the following problem

$$(\hat{\mathscr{P}}(\alpha, v)) \quad \begin{cases} \text{Find } \hat{u} \equiv \hat{u}(v) \in \hat{V} \text{ such that} \\ (\nabla \hat{u}, \nabla \varphi)_{\Omega} = [v, \varphi]_{\alpha} \quad \forall \varphi \in \hat{V} \end{cases}$$

The aim of this Appendix is to prove

PROPOSITION A 1 *For any $\alpha \in U_{\text{ad}}$, there exists $\bar{v} \in Q(\alpha)$ such that $\hat{u}(\bar{v})|_{\Omega(\alpha)}$ is the solution of $(\mathscr{P}(\alpha))$, i e , the solution of the homogeneous Dirichlet boundary value problem on $\Omega(\alpha)$*

Proof Let $u_1 \in V_1(\alpha)$ be the solution of $(\mathscr{P}(\alpha))$

$$(\nabla u_1, \nabla \varphi)_{\Omega(\alpha)} = (f, \varphi)_{\Omega(\alpha)} \quad \forall \varphi \in V_1(\alpha) \tag{A 1}$$

As $f \in L^2(\Omega(\alpha))$, the normal derivative $\dfrac{\partial u_1}{\partial n} \in H^{-1/2}(\Gamma(\alpha))$ and the following Green's formula holds

$$(\nabla u_1, \nabla \varphi)_{\Omega(\alpha)} = (f, \varphi)_{\Omega(\alpha)} + \left[\frac{\partial u_1}{\partial n}, \varphi\right]$$

for any $\varphi \in V(\alpha) = \{y \in H^1(\Omega(\alpha)) | y = 0 \text{ on } \partial\Omega(\alpha) \backslash \Gamma(\alpha)\}$ The symbol $[\ ,\ ]$ stands for the corresponding duality pairing Let $g = -\dfrac{\partial u_1}{\partial n}$ on $\Gamma(\alpha)$, i e ,

$$[g, \varphi] \equiv (f, \varphi)_{\Omega(\alpha)} - (\nabla u_1, \nabla \varphi)_{\Omega(\alpha)} \quad \forall \varphi \in V(\alpha) \tag{A 2}$$

Let $u_2 \in H_0^1(\Xi(\alpha))$ be fixed and $\varphi \in V_2(\alpha)$ The family $\{\Xi(\alpha), \alpha \in U_{\text{ad}}\}$ has a uniform extension property, i e , there exists a continuous extension mapping π from $\Xi(\alpha)$ on $\Omega(\alpha)$, the norm of which does not depend on $\alpha \in U_{\text{ad}}$ It is easy to see that the formula

$$\varphi \mapsto (\nabla u_2, \nabla \varphi)_{\Xi(\alpha)} - [g, \pi\varphi], \quad \varphi \in V_2(\alpha)$$

defines the linear, continuous functional on $V_2(\alpha)$, i.e., there exists $\bar{v}_2 \in V_2'(\alpha)$ such that

$$\langle \bar{v}_2, \varphi \rangle_\alpha \equiv (\nabla u_2, \nabla \varphi)_{\Xi(\alpha)} - [g, \pi \varphi] . \tag{A.3}$$

Moreover,

$$\|\bar{v}_2\|_{*, \alpha} \leq c(\|u_2\|_{V_2(\alpha)} + \|f\|_{\hat{\Omega}}) \quad (\|f\|_{\hat{\Omega}} \equiv \|f\|_{L^2(\hat{\Omega})}) \tag{A.4}$$

with a constant $c > 0$, which does not depend on α. From (A.2) and (A.3) we see that

$$(\nabla u_1, \nabla(\pi \varphi))_{\Omega(\alpha)} + (\nabla u_2, \nabla \varphi)_{\Xi(\alpha)} = (f, \pi \varphi)_{\Omega(\alpha)} + \langle \bar{v}_2, \varphi \rangle_\alpha \tag{A.5}$$

holds for any $\varphi \in V_2(\alpha)$. As $[g, \varphi] = 0$ for any $\varphi \in H_0^1(\Omega(\alpha))$, (A.5) holds for any $\varphi \in \hat{V}$. Let

$$\hat{u} = \begin{cases} u_1 & \text{on } \Omega(\alpha) \\ u_2 & \text{on } \Xi(\alpha) . \end{cases}$$

Then (A.5) is equivalent to

$$(\nabla \hat{u}, \nabla \varphi)_{\hat{\Omega}} = [\bar{v}, \varphi]_\alpha \quad \forall \varphi \in \hat{V} , \tag{A.6}$$

where

$$[\bar{v}, \varphi]_\alpha = (f, \varphi)_{\Omega(\alpha)} + \langle \bar{v}_2, \varphi \rangle_\alpha , \quad \varphi \in \hat{V} . \qquad \square$$

Remark A.1 : In fact, we proved more, namely : for any $u \in V_1(\alpha)$ being the solution of $(\mathscr{P}(\alpha))$, there exists $\bar{v} \in Q(\alpha)$ such that the solution $\hat{u}(\bar{v})$ of $(\hat{\mathscr{P}}(\alpha, \bar{v}))$ coincides with u on $\Omega(\alpha)$.

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