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Mixed finite element methods for quasilinear second order elliptic problems : the $p$-version

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MIXED FINITE ELEMENT METHODS
FOR QUASILINEAR SECOND ORDER ELLIPTIC PROBLEMS:
THE $p$-VERSION (*)

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Abstract — The $p$-version of the finite element method is analyzed for quasilinear second order elliptic problems in mixed weak form. Approximation properties of the Raviart-Thomas projection are demonstrated and $L^2$-error bounds for the three relevant variables in the mixed method are derived.

Résumé — Nous analysons la version-$p$ de la méthode d'éléments finis mixtes pour des problèmes quasilinéaires elliptiques du second ordre en forme faible mixte. Nous démontrons des propriétés d'approximation de la projection de Raviart-Thomas et on dérive des bornes de l'erreur dans $L^2(\Omega)$ pour les trois variables d'intérêt dans la méthode mixte.

I. INTRODUCTION

We consider here the numerical solution of the following boundary-value problem:

\[
\begin{cases}
\mathcal{D}(u) = - \nabla \cdot (a(u) \nabla u + b(u)) + c(u) = 0 & \text{in } \Omega, \\
u = - g & \text{on } \partial \Omega,
\end{cases}
\]  

(1.1)

where $\Omega$ is a convex polygon with boundary $\partial \Omega$, $\nabla w$ denotes the gradient of the scalar function $w$ and $\nabla \cdot v$ and $\text{div } v$ denote the divergence of the vector.

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function $v$. We shall assume that for $r \geq 2$ and for each $g \in H^{r-1/2}(\partial \Omega)$ there exists a unique isolated solution $u \in H^r(\Omega)$ of (1.1) (that is, a solution not situated at a bifurcation point). Note that Sobolev’s embedding theorem implies then that $u \in W^{r-1-\varepsilon, \infty}(\Omega)$, $\varepsilon > 0$, $\varepsilon \ll 1$, which will be needed throughout the paper.

We shall also assume that the coefficients $a : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$, $b : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}^2$ and $c : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ are twice continuously differentiable with bounded derivatives through second order, and that $a(x, q) \equiv a > 0$. The variable $x$ will be omitted as explicit argument of all functions, except when necessary to avoid ambiguity.

For $1 \leq s \leq \infty$ and $k$ any nonnegative integer, we let

$$W^{k,s}(\Omega) = \{ f \in L^s(\Omega) : D^\alpha f \in L^s(\Omega), \ |\alpha| \leq k \}$$

denote the Sobolev space endowed with its standard norm

$$\| f \|_{k,s,\alpha} = \left( \sum_{|\alpha| \leq k} \| D^\alpha f \|_{L^s(\Omega)}^{s} \right)^{1/s}, \ s \leq \infty ,$$

$$\| f \|_{k,\infty,\alpha} = \max_{|\alpha| \leq k} \| D^\alpha f \|_{L^\infty(\Omega)} .$$

The subscript $\Omega$ in the norms will be omitted. Let $H^k(\Omega) = W^{k,2}(\Omega)$ with norm $\| \cdot \|_k = \| \cdot \|_{k,2}$. In particular, the notation $\| \cdot \|_0$ will mean $\| \cdot \|_{L^2(\Omega)}$ or $\| \cdot \|_{L^2(\Omega)^2}$. For $0 \leq r < \infty$ let $W^{r,s}(\Omega)$, $W^{r,s}(\partial \Omega)$, $H^r(\Omega)$, and $H^r(\partial \Omega)$ denote the fractional order Sobolev spaces with norms $\| \cdot \|_{r,s,\alpha}$, $\| \cdot \|_{r,s,\partial \Omega}$, $\| \cdot \|_{r,\alpha}$ and $\| \cdot \|_{r,\partial \Omega}$, respectively, defined by interpolation [7].

We shall denote by $(\cdot, \cdot)$ the Hilbert inner product in either $L^2(\Omega)$ or $L^2(\Omega)^2$ and by $\langle \cdot, \cdot \rangle$ the $L^2$-inner product on the boundary of $\Omega$. The same notation will be used to indicate the dualities between $W^{r,s}(\Omega)$ and $W^{r,s}(\Omega)'$ and $H^r(\partial \Omega)$ and $H^{-s}(\partial \Omega)$, respectively. Throughout the paper, $C$, $Q$, and $K$ will denote generic positive constants which need not have the same value in all their occurrences.

Let

$$\mathcal{V} = H(\text{div} ; \Omega) = \{ v \in L^2(\Omega)^2 : \text{div} \ v \in L^2(\Omega) \} ,$$

normed by

$$\| v \|^2_{H(\text{div} ; \Omega)} = \| v \|_0^2 + \| \text{div} \ v \|_0^2 ,$$

and

$$W = L^2(\Omega) .$$
The mixed finite element method we shall consider seeks simultaneous approximations of the solution of (1.1), $u$, and of the flux

$$z = - a(u) \nabla u - b(u).$$

The mixed weak formulation of (1.1) consists of finding $(z, u) \in V \times W$ such that

$$\begin{cases} (\alpha(u) z, \psi) - (u, \nabla \psi) + (\beta(u), \psi) = \langle g, \psi \cdot n \rangle, & \psi \in V, \\ (\nabla z, w) + (c(u), w) = 0, & w \in W, \end{cases}$$

(1.3)

where we have set

$$\alpha(u) = 1/a(u), \quad \beta(u) = \alpha(u) b(u),$$

and $n$ is the outward unit normal vector on $\partial \Omega$. Our mixed finite element method is a discrete form of (1.3).

Let $\mathcal{C}$ be a decomposition of $\Omega$ by parallelograms which will be the «elements» $E$ and let $\mathcal{P}_{p,q}(E) = \{\text{polynomials } f(x,y) \text{ on } E, \text{ of degree } \leq p \text{ in } x \text{ and degree } \leq q \text{ in } y\}$, $\mathcal{P}_p(E) = \{\text{polynomials of degree } \leq p \text{ on } E\}$; next define, for each element $E$,

$$V^p(E) = \mathcal{P}_{p+1,p}(E) \times \mathcal{P}_{p,p+1}(E),$$

and let

$$V^p \times W^p \subset V \times W$$

be the Raviart-Thomas-Nedelec space of index $p \geq 0$ associated with this decomposition [3, 5], given by

$$\begin{cases} V^p = \left( \prod_{E \in \mathcal{C}} V^p(E) \right) \cap \{ f : \Omega^2 \to \mathbb{R} \mid f \cdot n_E \text{ on } E \cap E', E, E' \in \mathcal{C} \} \\ W^p = \prod_{E \in \mathcal{C}} \mathcal{P}_p(E), \end{cases}$$

where $n_E$ denotes the outward unit normal vector along $\partial E$, $E \in \mathcal{C}$. It is known [3, 5] that $\text{div } V^p \subset W^p$, a property we shall exploit later.

We seek $(z^p, u^p) \in V^p \times W^p$ so that

$$\begin{cases} (\alpha(u^p) z^p, \psi) - (u^p, \nabla \psi) + (\beta(u^p), \psi) = \langle g, \psi \cdot n \rangle, & \psi \in V^p, \\ (\nabla z^p, w) + (c(u^p), w) = 0, & w \in W^p. \end{cases}$$

(1.5)

Equations (1.5) define the $p$-version of the mixed finite element approximation for (1.3). This version is based on using a fixed mesh and increasing
the degree of the finite elements (as opposed to the $h$-version that keeps the degree fixed and refines the mesh). The $p$-version has been analyzed for the linearized version of (1.1) in terms of the standard variational form in [1] and in terms of the mixed variational form in [6]. In this paper, we extend the results obtained in [6] for the linear problem to the quasilinear case. We also obtain an improved version of lemma 3.1 of [6] by reducing the regularity assumed there. We restrict our attention to the mixed method, the corresponding generalization for the standard finite element method is more straightforward.

Milner [3] described the $h$-version of this method for the same problem, demonstrated the unique solvability (for small $h$) of the nonlinear algebraic system (1.5) and derived error estimates in $L^2(\Omega)$, $2 \leq s \leq +\infty$, for the error in $u$, and in $H(div; \Omega)$ for the error in $\varepsilon$. The assumption there was that the solution of (1.1) was in the space $H^{2+\epsilon}(\Omega)$. In contrast, for the present paper we shall need an extra half derivative, that is, $u \in H^{5/2+\epsilon}(\Omega)$.

We shall follow very closely the analysis of [3]. In order to do so we shall use the $L^2$-projection onto $W_p$, $P^p : L^2 \to W_p$, given by

$$(P^p w - w, \chi) = 0, \quad \chi \in W_p, \quad w \in W,$$  

for which the following approximation properties follow by repeating the arguments of [4] in two dimensions and using interpolation from the cases $s = 2$ and $s = +\infty$:

$$\left\| P^p w - w \right\|_{0,s} \leq Q_p^{-m + 3/2 - 3/s} \| w \|_m, \quad s \geq 2, \quad 3/2 - 3/s \leq m,$$  

if $w \in H^m(\Omega)$. We shall also use the Raviart-Thomas projection of $V$ onto $V_p$, $\pi^p : V \to V_p$, [5] for which we shall demonstrate in Section 2 the following approximation property:

$$\left\| \pi^p u - u \right\|_0 \leq Q_p^{1/2 - r} \left\| u \right\|_r, \quad r > 1/2, \quad u \in H^r(\Omega)^2 \cap V.$$

Our proof of (1.8) improves upon the one presented in [6], which imposed extra regularity on $u$ by requiring that $r > 1$. In contrast, the condition $r > 1/2$ is optimal (see remark 2.1). We also obtain estimates for the approximation properties of $\pi_p$ in the $W^{0,s}(\Omega)$-norm.

We shall find very useful the following inverse-type inequalities, the two dimensional form of the ones found in [4]:

$$\left\| \chi \right\|_{0,s} \leq Q_p^{4r - 4ls} \left\| \chi \right\|_{0,r}, \quad 1 \leq r \leq s \leq +\infty,$$

$$\chi \in L^s(\Omega) \cap W_p$$  

(1.9)

The plan of the paper is as follows: in Section 2 we demonstrate (1.8), in Section 3 we prove that, for $p$ sufficiently large, (1.5) is uniquely solvable.
and its solution \((\varepsilon, u^p)\) converges to \((\varepsilon, u)\) in \(V \cap L^2 + \varepsilon(\Omega)^2 \times L^{(2 + 4 \varepsilon)\varepsilon(\Omega)}\) for any fixed \(\varepsilon, 0 \leqslant \varepsilon \leqslant 1\), and in Section 4 we establish the rate of convergence of the approximation to the exact solution.

II. THE APPROXIMATION PROPERTIES OF \(\pi^p\)

We recall that \(\pi^p \psi\) is given locally (on every element \(E\)) by the following relations (2.1) and (2.2) (see [5]):

\[
\left\langle [\pi^p \psi - \psi], \varphi \right\rangle_{S_i} = 0, \quad \varphi \in \mathcal{P}_p, \quad (2.1)
\]

where \(\left\langle \ldots, \cdot \right\rangle_{S_i}, 1 \leqslant i \leqslant 4\), denotes the line integral along each side \(S_i\) of the element \(E\) and \(\mathcal{P}_p\) is the set of all polynomials in one variable of degree less than or equal to \(p\),

\[
(\pi^p \psi - \psi, \psi)_{E} = 0, \quad \psi \in \mathcal{V}^p(E), \quad (2.2)
\]

where \(\left\langle \ldots, \cdot \right\rangle_{E}\) denotes the standard \(L^2(E)\)-inner product.

Now, let \(R = [-1, 1] \times [-1, 1]\) and let \(\{P_i\}_{i \geq 0}\) denote the \(L^2([-1, 1])\)-complete orthogonal Legendre polynomials. For any \(\psi \in H(\text{div}; R)\), let

\[
\psi(x, y) = \left[ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} P_i(x) P_j(y), \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{i,j} P_i(x) P_j(y) \right], \quad (2.3)
\]

and let

\[
\pi^p \psi(x, y) = \left[ \sum_{i=0}^{p+1} \sum_{j=0}^{p} \tilde{a}_{i,j} P_i(x) P_j(y), \sum_{i=0}^{p} \sum_{j=0}^{p+1} \tilde{b}_{i,j} P_i(x) P_j(y) \right]. \quad (2.4)
\]

It follows from (2.2)-(2.4) that

\[
\begin{aligned}
a_{i,j} &= \tilde{a}_{i,j}, & 0 \leqslant i \leqslant p - 1, & 0 \leqslant j \leqslant p, \\
b_{kl} &= \tilde{b}_{kl}, & 0 \leqslant k \leqslant p, & 0 \leqslant l \leqslant p - 1.
\end{aligned} \quad (2.5)
\]

Next, we see that (2.1), (2.3)-(2.5) imply that

\[
\begin{aligned}
\sum_{i=p}^{p+1} \tilde{a}_{i,j} P_i(\pm 1) &= \sum_{i=p}^{\infty} a_{i,j} P_i(\pm 1), & 0 \leqslant j \leqslant p, \\
\sum_{j=p}^{p+1} \tilde{b}_{j,i} P_j(\pm 1) &= \sum_{j=p}^{\infty} b_{i,j} P_i(\pm 1), & 0 \leqslant i \leqslant p.
\end{aligned} \quad (2.6)
\]
Since $P_t(-1) = (-1)^t$ and $P_t(1) = 1$, (2.6) implies that
\[
\bar{a}_{pq} = \sum_{i=0}^{\infty} a_{2i+p,j}, \quad \bar{a}_{p+1,j} = \sum_{i=0}^{\infty} a_{2i+p-1,j}, \quad 0 \leq j \leq p,
\]
\[
\bar{b}_{ip} = \sum_{j=0}^{\infty} b_{i,p+2j}, \quad \bar{b}_{i,p+1} = \sum_{j=0}^{\infty} b_{i,p+1+2j}, \quad 0 \leq i \leq p.
\]

**Proposition 2.1:** Let $\psi \in V$ and let $\pi^p \psi$ be its Raviart-Thomas projection in $V^p$ given by (2.1)-(2.2). Then, if $\psi \in H^r(\Omega)^2$, we have
\[
\|\psi - \pi^p \psi\|_0 \leq Q p^{1/2-r} \|\psi\|_r, \quad r > 1/2,
\]
where $Q > 0$ is a constant independent of $p$ and $\psi$ but depending on $r$.

**Proof:** We first assume that $\Omega = R$ and that the decomposition consists of just one element. Then, $\psi \in V$ and $\pi^p \psi \in V^p$ can be given, respectively, by (2.3) and (2.4).

The following relation is a trivial consequence of well known properties of the Legendre polynomials,
\[
\|\psi - \pi^p \psi\|_0^2 = \sum_{i=p+1}^{p+1} \sum_{j=0}^{p} \frac{4(a_{i,j} - \bar{a}_{i,j})^2}{(2i+1)(2j+1)} + \sum_{i=0}^{p+1} \sum_{j=0}^{p+1} \frac{4(b_{i,j} - \bar{b}_{i,j})^2}{(2i+1)(2j+1)} + \sum_{i=0}^{p+1} \sum_{j=0}^{p+1} \frac{4}{(2i+1)(2j+1)}
\]
\[
\quad + \sum_{i=p+2}^{p+1} \sum_{j=0}^{p+1} \frac{4a_{i,j}}{(2i+1)(2j+1)} + \sum_{i=p+2}^{p+1} \sum_{j=0}^{p+2} \frac{4b_{i,j}}{(2i+1)(2j+1)}
\]
\[
\quad + \sum_{i=p+2}^{p+1} \sum_{j=p+1}^{p+1} \frac{4a_{i,j}}{(2i+1)(2j+1)} + \sum_{i=p+1}^{p+2} \sum_{j=p+2}^{p+2} \frac{4b_{i,j}}{(2i+1)(2j+1)}
\]
\[
= I + II + \cdots + VIII.
\]

Note that III-VIII can be bounded as follows:
\[
III + V + VII \leq Q p^{-2(r-1)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{a_{i,j}^2(1 + i^2 + j^2)^r}{(2i+1)(2j+1)}, \quad r > 0,
\]
while
\[
IV + VI + VIII \leq Q p^{-2(r-1)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{b_{i,j}^2(1 + i^2 + j^2)^r}{(2i+1)(2j+1)}, \quad r > 0,
\]
which implies that (see [6])
\[
III + \cdots + VIII \leq Q p^{-2(r-1)} \|\psi\|_r^2, \quad r > 0.
\]
On the other hand, it follows from (2.7) that
\[
I = \frac{4}{2p + 1} \sum_{j=0}^{p} \left( a_{p,j} - \sum_{j=0}^{\infty} a_{2i+p,j} \right)^2 + \frac{4}{2p + 3} \sum_{j=0}^{p} \left( a_{p+1,j} - \sum_{j=0}^{\infty} a_{2i+p+1,j} \right)^2
\]
and
\[
II = \frac{4}{2p + 1} \sum_{i=0}^{p} \left( \sum_{j=0}^{\infty} a_{2i+p,j} \right)^2 + \frac{4}{2p + 3} \sum_{i=0}^{p} \left( \sum_{j=0}^{\infty} a_{2i+p+1,j} \right)^2
\]

Next observe that bounding the series \( \sum_{k=p+1}^{\infty} (c + k^2)^{-s} (1 + 2k) \) using the integral method for \( \int_{p}^{\infty} (c + t^2)^{-s} (1 + 2t) \) \( dt \approx \frac{K}{s-1} p^{2-2s} \) \( (s > 1) \), and using the Cauchy-Schwarz inequality, we see that, for \( s \) bounded away from 1,
\[
\left( \sum_{i=1}^{\infty} a_{2i+p,j} \right)^2 \leq \sum_{i=1}^{\infty} \frac{a_{2i+p,j}^2}{4i + 2p + 1} \left[ 1 + (2i + p)^2 + j^2 \right]^s \sum_{i=1}^{\infty} [1 + (2i + p)^2 + j^2]^{-s} \times \left( 1 + 4i + 2p \right)
\]
\[
\leq \sum_{i=0}^{\infty} \frac{a_{2i,j}^2}{2i + 1} \left( 1 + i^2 + j^2 \right)^y \sum_{i=p+2}^{\infty} \left( 1 + i^2 + j^2 \right)^{-s} (1 + 2i)
\]
\[
\leq Qp^{2-2s} \sum_{i=0}^{\infty} \frac{a_{2i,j}^2 (1 + i^2 + j^2)^y}{2i + 1}, \quad (2.12)
\]
with exactly the same final bound holding for \( \left( \sum_{t=1}^{\infty} a_{2t+p+1,j} \right)^2 \). It follows from (2.10) and (2.12) that, for \( s \) bounded away from 1,

\[
I \leq Q p^{1-2s} \sum_{j=0}^{\infty} \sum_{t=0}^{\infty} \frac{a_{t,j}^2 (1 + i^2 + j^2)^s}{(2i+1)(2j+1)}.
\]  

(2.13)

In an entirely analogous way (replacing \( a_{i,j} \) by \( b_{i,j} \) and reversing the roles of \( i \) and \( j \)) we deduce from (2.11) that, for \( s \) bounded away from 1,

\[
II \leq Q p^{1-2s} \sum_{j=0}^{\infty} \sum_{t=0}^{\infty} \frac{b_{t,j}^2 (1 + i^2 + j^2)^s}{(2i+1)(2j+1)}.
\]  

(2.14)

Combining (2.13) and (2.14) results, for \( s \) bounded away from 1, in

\[
I + II \leq Q p^{1-2s} \| \bar{v} \|^2_s .
\]  

(2.15)

Next note that

\[
\bar{v} \cdot \bar{v} \mid_{\partial R} =
\]

\[
v_1(\pm 1, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} P_i(\pm 1) P_j(y), \quad -1 \leq y \leq 1 ,
\]

\[
v_2(x, \pm 1) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{i,j} P_i(y) P_j(\pm 1), \quad -1 \leq x \leq 1 .
\]  

(2.16)

The trace theorem (Sobolev’s embedding theorem) implies that \( v_1, v_2 \in L^2(\partial R) \) for \( s > 1/2 \). Consequently, since \( P_i(1) = 1 \) and \( P_i(-1) = (-1)^i \), we see from (2.16) that

\[
\| v_1(\pm 1, . ) \|^2_{0, \partial \Omega} = 2 \sum_{j=0}^{\infty} \left[ \sum_{i=0}^{\infty} (\pm 1)^i a_{i,j} \right]^2 \frac{2}{2j+1} < \infty ,
\]  

(2.17)

\[
\| v_2(., \pm 1) \|^2_{0, \partial \Omega} = 2 \sum_{j=0}^{\infty} \left[ \sum_{i=0}^{\infty} (\pm 1)^i b_{i,j} \right]^2 \frac{2}{2i+1} < \infty .
\]  

Let now \( \bar{v} \in H^{1/2+\varepsilon}(\Omega) \). We shall prove that

\[
\| \pi^p \bar{v} - \bar{v} \|_0 \leq Q p^{-\varepsilon} \| \bar{v} \|_{1/2+\varepsilon} .
\]  

(2.18)
In view of (2.8) and (2.9) it is sufficient to prove that $I, II < Qp^{-2} \|\psi\|_{1/2+\epsilon}^2$.

It follows from (2.10) that

$$I \leq 4 p^{-1} \sum_{j=0}^p (2j + 1)^{-1} \left[ \left( \sum_{i=1}^\infty a_{2i+p,j} \right)^2 + \left( \sum_{i=1}^\infty a_{2i+p+1,j} \right)^2 \right]$$

$$= 2 p^{-1} \sum_{j=0}^p (2j + 1)^{-1} \left[ \left( \sum_{i=0}^\infty a_{i,j} \right)^2 + \left( \sum_{i=0}^\infty (-1)^i a_{i,j} \right)^2 \right]$$

$$= 2 p^{-1} \sum_{j=0}^p (2j + 1)^{-1} \left[ \left( \sum_{i=0}^p a_{i,j} - \sum_{i=0}^{p+1} a_{i,j} \right)^2 + \left( \sum_{i=0}^\infty (-1)^i a_{i,j} \right)^2 \right]$$

$$\leq 4 p^{-1} \sum_{j=0}^p (2j + 1)^{-1} \left[ \left( \sum_{i=0}^p a_{i,j} \right)^2 + \left( \sum_{i=0}^\infty (-1)^i a_{i,j} \right)^2 + \left( \sum_{i=0}^{p+1} a_{i,j} \right)^2 + \left( \sum_{i=0}^\infty (-1)^i a_{i,j} \right)^2 \right]$$

$$\leq 4 p^{-1} \left( \|v_1(1, \cdot)\|_{0, \partial\Omega}^2 + \|v_1(-1, \cdot)\|_{0, \partial\Omega}^2 \right) + 4 p^{-1} \sum_{j=0}^p (2j + 1)^{-1} \left[ \left( \sum_{i=0}^{p+1} a_{i,j} \right)^2 + \left( \sum_{i=0}^\infty (-1)^i a_{i,j} \right)^2 \right]. \quad (2.19)$$

Note that the next to last term on the right hand side of (2.19) can be bounded, using the integral method for

$$\int_0^{p+1} (2i + 1)(1 + i^2 + j^2)^{-1/2-\epsilon} = O(p^{1-2\epsilon})$$

as $p \to \infty$, as follows:

$$\sum_{j=0}^p (2j + 1)^{-1} \left( \sum_{i=0}^{p+1} a_{i,j} \right)^2 \leq \sum_{j=0}^p (2j + 1)^{-1} \sum_{i=0}^{p+1} \frac{a_{i,j}^2 (1 + i^2 + j^2)^{1/2+\epsilon}}{2i + 1}$$

$$\times \sum_{i=0}^{p+1} (2i + 1)(1 + i^2 + j^2)^{-1/2-\epsilon}$$

$$\leq Q \|\psi\|_{1/2+\epsilon}^2 p^{1-2\epsilon}, \quad (2.20)$$

with an identical bound holding for the last term of (2.19). Combining (2.19) and (2.20) and using Sobolev’s embedding theorem yields

$$I \leq Qp^{-2\epsilon} \|\psi\|_{1/2+\epsilon}^2. \quad (2.21)$$
In an entirely analogous fashion we can see that
\[ \Pi \lesssim Qp^{-2} \| v \|_{1/2 + \epsilon}^2, \]
which together with (2.21) yields (2.18). Using interpolation [7], it follows from (2.15) and (2.18) that, for \( s \) bounded away from 1/2,
\[ I + \Pi \lesssim Qp^{1 - 2s} \| v \|_s^2, \]
which together with (2.8) and (2.9) concludes the proof for the case \( \Omega = R \). For the case when \( \Omega \) is a disjoint union of parallelograms the result follows on each element by using affine mappings onto \( R \). The proposition then follows by summing over all the elements (see [6] for details).

Remark 2.1 : This result differs from the one known for the \( h \)-version of the finite element method [3, (1.5)],
\[ \| v - \pi_h v \|_0 \lesssim Kh' \| v \|_r, \quad r > 1/2. \] (2.22)
The constraint \( r > 1/2 \) (or \( r \geq 1/2 + \epsilon \)) stems from the fact that, according to the trace theorem, this is the minimal requirement to ensure that \( v \) has a trace on the boundary which is an \( L^2 \)-function (not just a distribution). In [6] the corresponding result required an additional half derivative on \( \Omega \) \( r > 1 \). In contrast, proposition 2.1 assumes the minimum regularity necessary. It is possible, however, that the bound still holds with the exponent of \( p \) replaced by \( -r \), as suggested by (2.22).

Corollary 2.1 : For \( s \geq 2, r > \max \{ 1/2 ; 3/2 - 3/s \} \),
\[ \| v - \pi^p v \|_{0, s} \lesssim Qp^{5/2 - r - 4/s} \| v \|_r. \] (2.23)

Proof : Let \( P^p v \) be the \( L^2 \)-projection \( P^p \times P^p : V \rightarrow V^p \). Then the following analogue of (1.7) holds :
\[ \| P^p v - v \|_{0, s} \lesssim Qp^{-r + 3/2 - 3/s} \| v \|_r, \quad s \geq 2, \quad 3/2 - 3/s \lesssim r. \] (2.24)
Also,
\[ \| \pi^p v - v \|_{0, s} \lesssim \| P^p v - v \|_{0, s} + \| \pi^p v - P^p v \|_{0, s}. \] (2.25)
The second term in this expression may be bounded using the inverse inequality (1.9) as follows :
\[ \| \pi^p v - P^p v \|_{0, s} \lesssim Qp^{2 - 4/s} \| \pi^p v - P^p v \|_0 \]
\[ \lesssim Qp^{2 - 4/s} (\| P^p v - v \|_0 + \| \pi^p v - v \|_0). \] (2.25)
Combining (2.23)-(2.25) and using proposition 2.1, we obtain the corollary.
III. SOLVABILITY OF THE DISCRETE PROBLEM

Following [3] we introduce, for $\rho \in W^p$, the notation

$$\alpha (\rho ) - \alpha (u) = - \tilde{\alpha}_u(\rho )(u - \rho ) = - \alpha_u(u)(u - \rho ) + \tilde{\alpha}_{uu}(\rho )(u - \rho )^2 ,$$

(3.1)

where

$$\tilde{\alpha}_u(\rho ) = \int_0^1 \alpha_u(\rho + t[u - \rho ]) \, dt ,$$

and

$$\tilde{\alpha}_{uu}(\rho ) = \int_0^1 (1 - t) \alpha_{uu}(u + t[\rho - u]) \, dt ,$$

are bounded functions in $\tilde{\Omega}$. Similarly, we write

$$\beta (\rho ) - \beta (u) = - \tilde{\beta}_u(\rho )(u - \rho ) = - \beta_u(u)(u - \rho ) + \tilde{\beta}_{uu}(\rho )(u - \rho )^2 ,$$

(3.2)

and

$$c(\rho ) - c(u) = - \tilde{c}_u(\rho )(u - \rho ) = - c_u(u)(u - \rho ) + \tilde{c}_{uu}(\rho )(u - \rho )^2 ,$$

(3.3)

where $\tilde{\beta}_u(\rho ), \tilde{\beta}_{uu}(\rho ), \tilde{c}_u(\rho ),$ and $\tilde{c}_{uu}(\rho )$ are bounded functions in $\tilde{\Omega}$. Also, let

$$\Gamma = \alpha_u(u) z + \beta_u(u) , \quad \gamma = c_u(u) .$$

(3.4)

With the notation of (3.1)-(3.4), the following error equations follow from (1.3) and (1.5), [3]:

$$(\alpha(u)[\pi^p z - \pi^p], \psi) - (\text{div} \, \psi, P^p u - u^p) + ([P^p u - u^p]) \Gamma, \psi) =$$

$$= (q(w^p, \pi^p), \psi) , \quad \psi \in V^p ,$$

$$(\text{div} \, [\pi^p z - \pi^p], w) + (\gamma [P^p u - u^p], w) = (\eta(w^p), w) , \quad w \in W^p ,$$

(3.5)

where

$$q(u^p, \pi^p) = \alpha(u)[\pi^p z - \pi^p] + [P^p u - u] \Gamma +$$

$$+ (u - u^p)^2 \left[ \tilde{\alpha}_{uu}(u^p) z + \tilde{\beta}_{uu}(u^p) \right] + \tilde{\alpha}_u(u^p)(u - u^p)(z - \pi^p) ,$$

(3.6)
and

\[ \eta(u^p) = \gamma [P^p u - u] + \bar{c}_{uu}(u^p)(u - u^p)^2. \]  

(3.7)

Just as in [3], we let

\[ \Phi : \mathcal{V}^p \times \mathcal{W}^p \to \mathcal{V}^p \times \mathcal{W}^p \]

be given by \( \Phi((\mu, \rho)) = (\lambda, \kappa), (\lambda, \kappa) \) being the (unique) solution of the system

\[
\begin{align*}
(a(\mu)[\pi^p(z - \lambda)], v) - (\text{div} v, P^p u - \kappa) + ([P^p u - \kappa] \Gamma, v) &= (q(\rho, \mu), v), \quad v \in \mathcal{V}^p, \\
(\text{div} [\pi^p(z - \lambda)], w) + (\gamma [P^p u - \kappa], w) &= (\eta(\rho), w), \quad w \in \mathcal{W}^p,
\end{align*}
\]

(3.8)

where \( q(\rho, \mu) \) and \( \eta(\rho) \) are given by (3.6) and (3.7), respectively, replacing \( u^p \) by \( \rho \) and \( \bar{z}^p \) by \( \mu \). The unique solvability of this (linear) system follows, for \( p \) sufficiently large, from [2], since the left hand side of (3.8) corresponds to the mixed method for the operator \( M : H^2(\Omega) \cap H^1_0(\Omega) \to \mathcal{L}^2(\Omega) \) given by

\[ Mw = -\nabla \cdot (a(u)\nabla w + a(u) w \Gamma) + \gamma w, \]

which has a bounded inverse. In fact, note that (1.2), (1.4) and (3.4) give

\[
\begin{align*}
Mw &= -\nabla \cdot [a(u)\nabla w + a(u) w (\alpha(u) z + \beta(u))] + c_u(u) w \\
&= -\nabla \cdot \left[ a(u)\nabla w + a(u) w \left( -\frac{a_u(u)}{a^2(u)} (-a(u) \nabla u) + \alpha(\mu) b_u(u) \right) \right] + \frac{\alpha(u) b_u(u)}{a^2(u)} + c_u(u) w \\
&= -\nabla \cdot [a(u)\nabla w + (a_u(u) \nabla u + b_u(u)) w] + c_u(u) w,
\end{align*}
\]

which shows that \( M \) is the linearization of the operator \( \partial \) in (1.1) about the function \( u \), and, thus, it has a bounded inverse since we have assumed that (1.1) admits unique isolated solutions.

The solvability of (1.5) is now equivalent to showing that \( \Phi \) has a fixed point. This will follow from the Brouwer fixed point theorem if we show that \( \Phi \) maps a ball of \( \mathcal{V}^p \times \mathcal{W}^p \) into itself. We shall need the following technical result, a \( p \)-version of lemma 2.1 of [3]. Let \( \epsilon > 0 \) be fixed for the rest of the paper, \( \epsilon \ll 1. \)
LEMMA 3.1: Let $2 \leq \theta \leq 4 - \varepsilon$. Let $\omega \in V$, $q \in L^2(\Omega)^2$, and $\eta \in L^2(\Omega)$. If $\tau \in W^p$ satisfies
\[
\begin{align*}
(\alpha(u) \omega, \psi) - (\operatorname{div} \psi, \tau) + (\tau \Gamma', \psi) &= (q, \psi), \quad \psi \in V^p, \\
(\operatorname{div} \omega, w) + (\gamma \tau, \omega) &= (\eta, w), \quad w \in W^p,
\end{align*}
\]
then, there exists a constant $C = C(\theta, u, \alpha, \Gamma, \gamma, \Omega, \varepsilon)$ such that, for $p$ sufficiently large, depending upon $\varepsilon$,
\[
\|\tau\|_{0, \theta} \leq C [p^{1/2 - 2/\theta} \|\omega\|_0 + p^{-1 - 2/\theta} \|\operatorname{div} \omega\|_0 + \|q\|_0 + \|\eta\|_0].
\]

Proof: We follow the proof of lemma 2.1 of [3]. Let $\theta' = \theta/(\theta - 1)$ be the conjugate exponent of $\theta$. For $\psi \in L^{\theta'}(\Omega)$ let $\phi \in W^{2, \theta'}(\Omega)$ be the (unique) solution of $M^* \phi = \psi$ in $\Omega$, $\psi = 0$ on $\partial \Omega$, where $M^*$ is the formal adjoint of $M$. It follows that $\|\phi\|_{2, \theta'} \leq Q \|\psi\|_{0, \theta'}$. We then have [3],
\[
(\tau, \psi) = (q, \alpha(u) \nabla \phi) + (q, \pi^p a(u) \nabla \phi - a(u) \nabla \phi) + \\
+ (\operatorname{div} \omega + \gamma \tau, \phi - P^p \phi) \\
+ (\alpha(u) \omega + \tau \Gamma', a(u) \nabla \phi - \pi^p a(u) \nabla \phi) + (\eta, \phi) + (\eta, P^p \phi - \phi).
\]

(3.9)

Note that Sobolev’s embedding theorem implies that
\[
(q, \alpha(u) \nabla \phi) \leq C \|q\|_0 \|\phi\|_1 \leq C \|q\|_0 \|\phi\|_{2, \theta'}.
\]

(3.10)

Next, (1.8) and Sobolev’s embedding theorem imply that
\[
(q - \alpha(u) \omega, \pi^p a(u) \nabla \phi - a(u) \nabla \phi) \leq \\
\leq C \left(\|q\|_0 + \|\omega\|_0\right) p^{1/2 - 2/\theta} \|\nabla \phi\|_{2/\theta} \\
\leq C \left(\|q\|_0 + \|\omega\|_0\right) p^{1/2 - 2/\theta} \|\phi\|_{2, \theta'} ,
\]

(3.11)

and that
\[
(\tau \Gamma', a(u) \nabla \phi - \pi^p a(u) \nabla \phi) \leq C \|\tau\|_{0, \theta} \|a(u) \nabla \phi - \pi^p a(u) \nabla \phi\|_0 \\
\leq C \|\tau\|_{0, \theta} p^{-\varepsilon/8} \|\phi\|_{2, \theta'}.
\]

(3.12)

On the other hand, (1.7) and Sobolev’s embedding theorem lead to
\[
(\operatorname{div} \omega, \phi - P^p \phi) \leq K \|\operatorname{div} \omega\|_0 p^{-1 - 2/\theta} \|\phi\|_{2, \theta'},
\]

(3.13)

\[
(\gamma \tau, \phi - P^p \phi) \leq K \|\tau\|_{0, \theta} p^{-1 - 2/\theta} \|\phi\|_{2, \theta'},
\]

(3.14)

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and

$$(\eta, \phi) + (\eta, P^P \phi - \phi) \leq K \| \eta \|_0 \| \phi \|_0 \leq K \| \eta \|_0 \| \phi \|_{2, \theta}.$$

(3.15)

Collecting (3.9)-(3.15) we see that

$$(\tau, \psi) \leq K \| \psi \|_{0, \theta} \| P^{1/2-2\theta} \| \omega \|_0 + P^{-1-2\theta} \| \text{div} \omega \|_0 +$$

$$+ P^{-1/8} \| \tau \|_{0, \theta} + \| q_\theta \|_0 + \| \eta \|_0 \},$$

which, for $p$ sufficiently large, yields the desired estimate.

Now let $\mathcal{Y}^p = \mathcal{V}^p$ with the stronger norm $\| \psi \|_{\mathcal{Y}^p} = \| \psi \|_{0,2+\varepsilon} + \| \text{div} \psi \|_0$

and let $\mathcal{W}^p = W^p$ with the stronger norm $\| w \|_{\mathcal{W}^p} = \| w \|_{0,2}$, where $t = \frac{4 + 2\varepsilon}{\varepsilon}$. We can prove now the existence of a solution of (1.5).

**Theorem 3.1:** For $\delta > 0$ sufficiently small (dependent on $p$) and for $p$ sufficiently large, $\Phi$ maps a ball of radius $\delta$ centered at $(\pi^p z, P^p u)$ of $\mathcal{Y}^p \times \mathcal{W}^p$ into itself.

**Proof:** Note that $1/t + 1/(2 + \varepsilon) = 1/2$. Let

$$\| \pi^p z - \mu \|_{\mathcal{Y}^p} \leq \delta \quad \text{and} \quad \| P^p u - \rho \|_{\mathcal{W}^p} \leq \delta < 1.$$

Let us use lemma 3.1 on (3.8) with $\tau = P^p u - \kappa$, $\omega = \pi^p z - \frac{1}{\delta}$, $q = q(\rho, \mu)$, $\eta = \eta(\rho)$, and $\theta = 4 - \varepsilon$. Observe that (1.7)-(1.9) and corollary 2.1 imply that, for $r > 1/2$, $m = r + 1$,

$$\| q(\rho, \mu) \|_0 + \| \eta(\rho) \|_0 \leq 2 [p^{1/2-r} \| z \|_r + P^{-m} \| u \|_m + \| u - \rho \|_{0, 1/4}^2 +$$

$$+ \| u - \rho \|_{0, 1} \| z - \mu \|_{0, 2+\varepsilon}]$$

$$
\leq 2 [p^{1/2-r} \| u \|_{r+1} + (\| u - P^p u \|_{0, 4} + \| P^p u - \rho \|_{0, 4})^2 +$$

$$+ (\| u - P^p u \|_{0, 1} + \| P^p u - \rho \|_{0, 1}) \times$$

$$\times (\| z - \pi^p z \|_{0, 2+\varepsilon} + \| \pi^p z - \mu \|_{0, 2+\varepsilon})]$$

$$\leq 2 [p^{1/2-r} \| u \|_{r+1} + (p^{-m+3/4} \| u \|_{m} + \delta)^2 +$$

$$+ (p^{5/2-r-4/(2+\varepsilon)} \| u \|_{r+1} + \delta)(p^{-m+3/2-3/4} \| u \|_{m} + \delta)]$$

$$\leq 2 (\delta^2 + p^{1/2-r} \| u \|_{r+1}).$$

(3.16)
where $\mathcal{Q}$ depends on $\|u\|_m$. Therefore,

$$
\|P^p u - \kappa\|_{0,4-\varepsilon} \leq \mathcal{Q} \left[ p^{-\varepsilon/8} \|\pi^p z - \lambda\|_0 + p^{-1-2/(4-\varepsilon)} \times \right.
\left. \|\text{div} (\pi^p z - \lambda)\|_0 + \delta^2 + p^{1/2-r} \right]. \quad (3.17)
$$

On the other hand, taking $v = \pi^p z - \lambda$ and $w = P^p u - \kappa$ in (3.8), we see that

$$
\|\pi^p z - \lambda\|_0 \leq \mathcal{Q} \left[ \|P^p u - \kappa\|_0 + \|q\|_0 + \|\eta\|_0 \right], \quad (3.18)
$$

and, taking $w = \text{div} (\pi^p z - \lambda)$ in the second equation of (3.8) results in

$$
\|\text{div} (\pi^p z - \lambda)\|_0 \leq \mathcal{Q} \left[ \|P^p u - \kappa\|_0 + \|q\|_0 + \|\eta\|_0 \right]. \quad (3.19)
$$

Combining (3.17)-(3.19) yields the relation

$$
\|P^p u - \kappa\|_{0,4-\varepsilon} \leq \mathcal{Q} \left[ p^{-\varepsilon/8} \|P^p u - \kappa\|_0 + \delta^2 + p^{1/2-r} \right],
$$

which, for $p$ sufficiently large and $r = 5/2$, implies that

$$
\|P^p u - \kappa\|_{0,4-\varepsilon} \leq \mathcal{Q} [\delta^2 + p^{-2}], \quad (3.20)
$$

where the constant $\mathcal{Q}$ depends on $\|u\|_{\gamma_2}$. Combining (3.20) with (1.9) we see that

$$
\|P^p u - \kappa\|_{0,4-\varepsilon} \leq \mathcal{Q} \left[ p^{4-\varepsilon} - \varepsilon + 2 \|P^p u - \kappa\|_{0,4-\varepsilon} \right.
\left. \leq \mathcal{Q} (p^{1-\varepsilon/4} \delta^2 + p^{1-\varepsilon/4}) \right], \quad (3.21)
$$

while (1.9), (3.18), (3.16), and (3.20) imply that

$$
\|\pi^p z - \lambda\|_{0,2+\varepsilon} \leq \mathcal{Q} \left[ p^{2\varepsilon/(2+\varepsilon)} \|\pi^p z - \lambda\|_0 \right.
\left. \leq \mathcal{Q} (p^\varepsilon \delta^2 + p^{-2+\varepsilon}) \right]. \quad (3.22)
$$

Combining (3.19) and (3.22) yields

$$
\|\pi^p z - \lambda\|_{\gamma_p} \leq \mathcal{Q} (p^\varepsilon \delta^2 + p^{-2+\varepsilon}). \quad (3.23)
$$

We can now combine (3.21) and (3.23) in the bound

$$
\|P^p u - \kappa\|_{\gamma_p} + \|\pi^p z - \lambda\|_{\gamma_p} \leq \mathcal{Q}_1 (p^{1-\varepsilon/4} \delta^2 + p^{1-\varepsilon/4}). \quad (3.24)
$$

We want to choose $p$ and $\delta$ so that $\mathcal{Q}_1 p^{1-\varepsilon/4} \delta^2 \leq \frac{\delta}{2}$ and $\mathcal{Q}_1 p^{-1-\varepsilon/4} \leq \frac{\delta}{2}$.
Let $p \geq (2 \mathcal{D}_1)^{4/5}$, so that $I = \left[ 2 \mathcal{D}_1 p^{-1 - \varepsilon/4}, \frac{15/4 - 1}{2 \mathcal{D}_1} \right]$ is not empty. Then, for $\delta \in I$, (3.24) implies that

$$\| P^p u - \kappa \|_{\mathcal{W}^p} \leq \delta \quad \text{and} \quad \| \pi^p z - \lambda \|_{\chi^p} \leq \delta,$$

as we needed.

\textbf{Remark 3.1} : Note that the choice $\delta = 2 \mathcal{D}_1 p^{-1 - \varepsilon/4}$ in theorem 3.1 shows (using (1.7) and (1.8)) not only that (1.5) is solvable but also that, for $p \to \infty$, the solution of (1.5), $(z^p, u^p)$, differs from $(z, u)$ in the $\chi^p \times \mathcal{W}^p$ norm by $O(p^{-1 - \varepsilon/4})$ at most. We shall need this observation in order to arrive at the correct error estimates.

\section{4. THE $L^2$-ERROR BOUNDS}

Just as in [3], using (3.1)-(3.3) we now rewrite (3.5) in the form

\begin{equation}
\begin{cases}
(\alpha(u) \xi, \psi) - (\text{div} \psi, \tau) + (\tau \tilde{F}, \psi) = (q, \psi), & \psi \in \mathcal{V}^p, \\
(\text{div} \xi, w) + (\tilde{\gamma} \tau, w) = (\eta, w), & w \in \mathcal{W}^p,
\end{cases}
\end{equation}

where $\xi = z - z^p$, $\tau = P^p u - u^p$, $\tilde{F} = \tilde{\alpha}_u(u^p) z^p + \tilde{\beta}_u(u^p)$, $\tilde{\gamma} = \tilde{c}_u(u^p)$, $q = (P^p u - u) \tilde{F}$, and $\eta = (P^p u - u) \tilde{\gamma}$. Note that the left hand side of (4.1) corresponds to the mixed method for the operator $N : H^2(\Omega) \to L^2(\Omega)$ given by

$$N w = - \nabla \cdot (a(u) \nabla w + a(u) w \tilde{F}) + \tilde{\gamma} w.$$ 

Therefore, if we show that its formal adjoint, $N^*$, has a bounded inverse $L^2 \to H^2(\Omega) \cap H_0^1(\Omega)$, then lemma 3.1 would apply to (4.1) without any change in the proof. Since we know that $M^*$ has a bounded inverse, all we need to do is to check that the operator norm of $M^* - N^*$ can be made arbitrarily small by taking $p$ large enough.

\textbf{Lemma 4.1} : \textit{There exists a positive integer $p_0$ such that, for all $p \geq p_0$, $N^*$ has a bounded inverse $L^2(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega)$. ($N^*$ depends on $p$ through $\tilde{\gamma}$ and $\tilde{F}$).}

\textbf{Proof} : Just as in [3], we have

$$\left( M^* - N^* \right) \chi = a(u) \left( [\tilde{\alpha}_{uu} z + \tilde{\beta}_{uu}] (u - u^p) + \tilde{\alpha}_u(u^p) (z - z^p) \right) \times
$$

$$\times \nabla \chi + \tilde{c}_{uu} (u - u^p) \chi, \quad \chi \in L^2(\Omega),$$

where $\alpha_{uu}$ and $\beta_{uu}$ are the entries of the matrices $\alpha(u)$ and $\beta(u)$, respectively.
where \( \bar{\alpha}_{uu} = \frac{\alpha_u(u) - \bar{\alpha}_u(u^p)}{u - u^p} \) and \( \bar{\beta}_{uu} \), and \( \bar{c}_{uu} \), defined by analogous relations, are bounded functions in \( \bar{\Omega} \). It follows from remark 3.1 and Sobolev’s embedding theorem that

\[
\| (M^* - N^*) z \|_0 \leq K \left[ \| z \|_0 \infty \| u - u^p \|_0 \| \nabla x \|_0 2 + \varepsilon \right. \\
+ \left. \| z - z^p \|_0 2 + \varepsilon \| \nabla x \|_0 \ v + \| u - u^p \|_0 \| x \|_0 \infty \right] \\
\leq K \left( \| \nabla x \|_1 + \| x \|_1 + \varepsilon \right) p^{-1 - \varepsilon/4} \\
= K p^{-1} \| x \|_2 ,
\]

as needed.

To conclude, we establish the rate of convergence of \((z^p, u^p)\) to \((z, u)\).

**Theorem 4.1:** Assume that the solution \( u \) of (1.1) is in \( H^{7/2}(\Omega) \). There is a positive constant \( Q \), independent of \( p \) but dependent on \( |M| 7/2 + 2 \varepsilon \), such that, for \( p \) sufficiently large and \( m = 7/2 \),

\[
1) \quad \| u - u^p \|_0 \leq Q p^{1 - m} \| u \|_m , \\
11) \quad \| z - z^p \|_0 \leq Q p^{3/2 - m} \| u \|_m , \\
11i) \quad \| \text{div} (z - z^p) \|_0 \leq Q p^{2 - m} \| u \|_m .
\]

**Proof** In view of remark 3.1 and lemma 4.1, we can use lemma 3.1 on (4.1) with \( \theta = 2 \). Thus,

\[
\| \tau \|_0 \leq C \left[ p^{-1/2} \| z \|_0 + p^{-2} \| \text{div} z \|_0 + \| q \|_0 + \| \eta \|_0 \right] \quad (4.2)
\]

Note that remark 3.1 together with (1.7) lead to the following estimate for \( r \geq 0, \ m \geq 3/2, \)

\[
\| z \|_0 + \| \eta \|_0 = \| (P^p u - u) \bar{f} \| + \| (P^p u - u) \bar{v} \| \leq \\
\leq K \left( \| P^p u - u \|_0 + \| (P^p u - u) \bar{z}^p \|_0 \right) \\
\leq K \left( p^{-r} \| u \|_r (1 + \| z \|_0 \infty ) + \| z - z^p \|_0 2 + \varepsilon \| P^p u - u \|_0 \right. \ v \\
\leq \left. K \left( p^{-r} \| u \|_r + p^{-1 - \varepsilon/4} p^{3/2 - 3 \varepsilon/2 (2 + \varepsilon)} - m \| u \|_m \right) \right) \\
\leq K p^{-m - \varepsilon} \| u \|_m .
\]
Combining (4.2), (4.3), (3.18), (3.19), (1.7) and (1.8) yields,
\[ \| \tau \|_0 \leq C \left[ p^{1/2} \left( \| \zeta - \pi^p \zeta \|_0 + \| \pi^p \zeta - \zeta' \|_0 \right) + p^{-2} \left( \| \text{div} \ z - P^p \text{div} \ z \|_0 \right) \right] \]
\[ + \left( \| \text{div} \ (\pi^p \zeta - \zeta') \|_0 \right) + p^{1-m-\epsilon} \| u \|_m \] \]
\[ \leq C \left[ p^{1/2} \| \tau \|_0 + p^{1/2} - p^{1/2} p^{1/2} \right] \| u \|_{r+1} + p^{-2} p^{-2} \| u \|_{s+2} + \]
\[ + p^{-1/2} p^{1/2} + p^{1-m-\epsilon} \| u \|_m \right] , \ r > 1/2 , \ s \geq 0 , \ m > 3/2 , \]
which, for \( p \) sufficiently large, leads to
\[ \| \tau \|_0 \leq C p^{1-m} \| u \|_m , \ m \geq 2 , \quad (4.4) \]
where the constant \( C \) depends on \( \| u \|_{7/2} \). The first part of the theorem is an immediate consequence of (1.7) and (4.4). On the other hand, it follows from (1.8), (3.18), (4.3) and (4.4), that
\[ \| \zeta - \zeta' \|_0 \leq \| \zeta - \pi^p \zeta \|_0 + \| \pi^p \zeta - \zeta' \|_0 \]
\[ \leq C \left[ p^{3/2-\epsilon} \| u \|_m + p^{1-m} \| u \|_m \right] , \]
which proves the second part of the theorem.

Finally, we deduce from (3.19), (1.7), (4.3) and (4.4) that
\[ \| \text{div} \ (\zeta - \zeta') \|_0 \leq \| \text{div} \ z - P^p \text{div} \ z \|_0 + \| \text{div} \ (\pi^p \zeta - \zeta') \|_0 \]
\[ \leq C \left[ p^{2-\epsilon} \| u \|_m + p^{1-m} \| u \|_m \right] , \]
which gives iii).

Remark 4.1 : The estimate for the error in \( \zeta \) is the best we could hope for in view of (1.8). The estimate for the error in \( \text{div} \ z \) is optimal in rate and regularity, while the one for \( u \) is probably not sharp in view of (1.7).

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