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AN ITERATIVE CLOUGH-TOCHER INTERPOLANT

by G. FARIN [1] and P. KASHYAP [1]

Abstract. — *We present a method for scattered data interpolation that is based on the classical Clough-Tocher element in Bézier formulation. It uses a modified form of the standard element in an iterative way.*

Résumé. — Un interpolant itératif du type « Clough-Tocher ». *Nous présentons une méthode pour l'interpolation de données dispersées basée sur l'élément classique de Clough-Tocher sous sa forme de Bézier. Cette méthode utilise une modification itérative de la formulation standard.*

1. INTRODUCTION

Clough-Tocher interpolants were invented as a tool for the finite element method [5], but for some years they have been used in the field of CAGD (Computer Aided Geometric Design) in the area of scattered data interpolation [1], [7], [12].

A Clough-Tocher interpolant produces a C^1 piecewise polynomial surface, defined over a triangulation of scattered data sites. Previously the original interpolant was modified, [7], to increase the smoothness of the overall interpolant by using the available degrees of freedom. In this paper we use an iterative scheme to further increase the smoothness of the overall interpolant.

The iterative scheme is then compared to the other methods by using an interrogation technique which simulates reflection lines [7].

It is assumed that the reader is familiar with the theory of Bézier triangular patches, as outlined in [8], [6]. An n-th degree Bernstein-Bézier triangular patch is of the form

$$b^n(\mathbf{u}) = \sum_{|\mathbf{i}| = n} b_{\mathbf{i}} B_{\mathbf{i}}^n(\mathbf{u}) \tag{1}$$

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where the Bernstein polynomials $B_{\mathbf{i}}^n(\mathbf{u})$ are defined by

$$B_{\mathbf{i}}^n(\mathbf{u}) = \binom{n}{\mathbf{i}} u^i v^j w^k ; \quad |\mathbf{i}| = n$$

the $b_{\mathbf{i}}$ are Bézier ordinates which form the control net of the triangular patch ; $\mathbf{u} = (u, v, w)$ are the barycentric coordinates of the domain triangle, and $\mathbf{i} = (i, j, k)$ is a multi-index with $|\mathbf{i}| = i + j + k$.

Now consider two adjacent domain triangles and Bézier nets defined on each of them ; we want to find the conditions which the nets must satisfy in order for the surface patches defined by the two nets be C^1 or C^2.

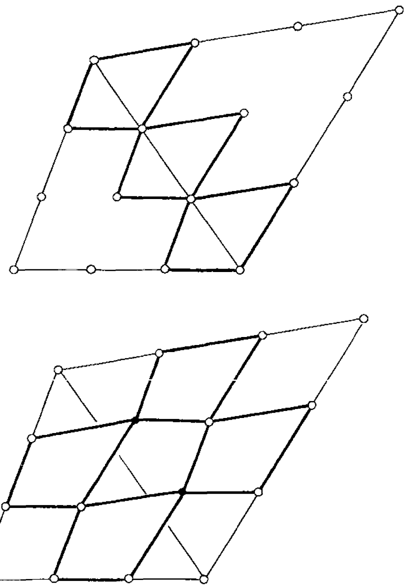

Figure 1. — Continuity conditions of Bézier triangular patches : top, C^1 continuity, bottom, additional C^2 continuity.

For the C^1 case figure 1 (top) illustrates the result : the condition for C^1 continuity of the interpolant is that the emphasized pairs of triangles should be coplanar.

For the C^2 case : the emphasized pairs of triangles in figure 1 (bottom) are constructed to be coplanar. Each of the extension points (marked points) as found from the two triangles would have a different z-value. The condition for C^2 continuity is that these two values be identical. These extension points are analogous to the extension points used to define C^2 conditions for spline curves in Bézier form, see [4] or [6].

2. THE CLOUGH-TOCHER INTERPOLANT

Given the z-values and gradients over a set of triangulated data points, we want a « good » (in terms of continuity) piecewise cubic interpolant over the data set. As a first approximation, the 9-parameter piecewise cubic Bernstein-Bézier interpolant [8] can be used. Its 9 boundary ordinates (all $b_{i,j,k}$ except $b_{1,1,1}$) are determined from the data by univariate cubic Hermite interpolation, the remaining ordinate is given by

$$b_{1,1,1} = \frac{1}{4}(b_{2,0,1} + b_{1,0,2} + b_{0,2,1} + b_{0,1,2} + b_{2,1,0} + b_{1,2,0}) - \\ - \frac{1}{6}(b_{3,0,0} + b_{0,3,0} + b_{0,0,3}) .$$

This choice of $b_{1,1,1}$ ensures quadratic precision.

This interpolant is only C^0 in general, and needs to be modified if one desires an overall C^1 smoothness. This can be done by splitting each triangle in the given triangulation into three minitriangles (for an algorithm see [8]). This subdivided domain now has enough degrees of freedom (twelve per subdivided triangle instead of ten before subdivision) to allow for C^1 continuity of the overall interpolant.

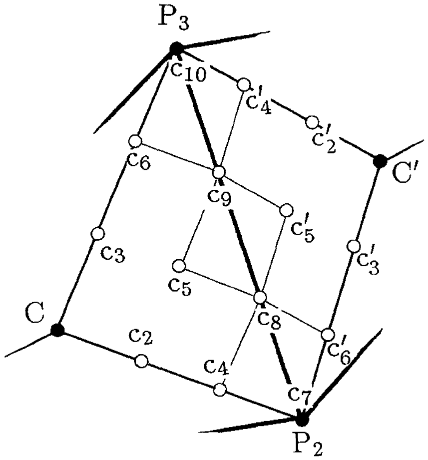

Figure 2. — Cross boundary derivatives : the Bézier ordinates that are involved in the C^1 and C^2 conditions.

In figure 2, let C, P_2, P_3 and C', P_3, P_2 be the vertices of two adjacent minitriangles (coming from two different macrotriangles). Expressing center points C and C' in terms of barycentric coordinates of opposite triangles C', P_3, P_2 and C, P_2, P_3 respectively :

$$C = u' C' + v' P_3 + w' P_2$$

and

$$C' = uC + vP_2 + wP_3 .$$

The C^1 conditions for adjacent Bézier triangles are fulfilled by the subtriangle pair formed by c_6, c_9, c_{10} and $c_{4'}$, c_{10}, c_9 and pair c_4, c_7, c_8 and $c_{6'}$, c_8, c_7 (same tangent plane). So the only condition for a C^1 patch is that the middle pair of subtriangles should be coplanar i.e.,

$$c_5' = uc_5 + vc_8 + wc_9 . \tag{2}$$

This can be achieved by following the given scheme : chose a direction $\mathbf{l}$ (the components ℓ_1, ℓ_2, ℓ_3 are its barycentric representation in terms of triangle C, P_2, P_3), not parallel to the triangle edge $P_2 P_3$. Then the directional derivative of mini-cubic $\mathcal{P}_1$ defined over the minitriangle C, P_2, P_3 is a univariate quadratic Bézier polynomial with Bézier ordinates

$$3(\ell_1 c_6 + \ell_2 c_9 + \ell_3 c_{10}) , \quad 3(\ell_1 c_5 + \ell_2 c_8 + \ell_3 c_9) ,$$
$$3(\ell_1 c_4 + \ell_2 c_7 + \ell_3 c_8) .$$

We can fix the unknown c_5 by choosing a linear variation for the directional derivative. This choice can be expressed as

$$(\ell_1 c_6 + \ell_2 c_9 + \ell_3 c_{10}) - 2(\ell_1 c_5 + \ell_2 c_8 + \ell_3 c_9) + (\ell_1 c_4 + \ell_2 c_7 + \ell_3 c_8) = 0 , \tag{3}$$

see [2]. The unknown c_5' may be found from (2), or by an analogous procedure for the other mini cubic $\mathcal{P}_2$ over the minitriangle C', P_3, P_2. It is imperative the $\mathbf{l}$ denotes the same direction both in $\mathcal{P}_1$ and $\mathcal{P}_2$. One way of doing this is by choosing $\mathbf{l}$ to be perpendicular to edge P_2, P_3, although it makes the interpolant affinely variant (perpendicular lines, in general do not map to perpendicular lines in an affine transformation).

After finding all three center points of the macro triangle, we can compute the rest of the interior points by applying the C^1 conditions four times.

3. SMOOTHING THE INTERPOLANT

The C^1 condition (2) has two unknowns, one of which can be fixed by chosing the linear cross boundary derivative condition (3). This appears to be quite arbitrary and so in this section we describe conditions which would improve the smoothness of the surface [7]. In section 3, we observed that C^1 cubics over split triangles enjoy an extra degree of freedom (this is manifested in (2) which has two unknowns in one equation). We can improve upon condition (3) by trying to achieve

$$uc_3 + vc_5 + wc_6 = u' c_2' + v' c_4' + w' c_5' \tag{4}$$

and

$$uc_2 + vc_4 + wc_5 = u' c_3' + v' c_5' + w' c_6' , \tag{5}$$

thereby hopefully minimizing the jump in the second derivative across the boundary.

So we have a constrained minimization problem : minimize the sum of errors in (4), (5) constrained by (2). We use the standard Lagrange multiplier method to obtain

$$c_5' = (us_1 + ua_{12} + u^2 s_2 + r_3 a_{11})/D$$

where

$$s_1 = 2(vr_1 + wr_2) , \quad s_2 = -2(w' r_1 + v' r_2)$$

and

$$\begin{aligned} r_1 &= u' c_2' + v' c_4' - uc_3 - wc_6 \\ r_2 &= u' c_3' + w' c_6' - uc_2 - vc_4 \\ r_3 &= vc_8 + wc_9 \\ a_{11} &= 2(v^2 + w^2) \\ a_{12} &= -2(vw' + wv') \\ a_{22} &= 2(w'^2 + v'^2) \end{aligned}$$

and the denominator

$$D = 2\, ua_{12} + u^2 a_{22} + a_{11} .$$

The unknown c_5 then can be found from (2). After fixing the center points of the macro triangle, the inner points are recomputed using the C^1 conditions.

4. ITERATIVE IMPROVEMENT

In the previous section, the C^2 error across the macro triangle was reduced by adjusting the center points c_5 and c_5' ; based on the same idea, we now present an iterative scheme which further improves the smoothness of the overall interpolant by changing the inner points. These points can be found either by :

a) Form the control net over the macro triangle by using the 9-parameter interpolant, then subdivide it at the centroid into three subnets (use de-Casteljau algorithm). The inner points are Bézier ordinates neighboring the centroid. This interpolant is C^{∞} over the macro triangle but only C^0 across them.

b) Find the center points using the linearized cross boundary derivative condition (2) ; the inner points are then found by using the C^1 conditions in the macro triangle. The resulting global interpolant is C^2 at the centroid of the macro triangle and C^1 all over.

Neither of these two schemes produces the « best » C^2 global interpolant. So we see that minimization of C^2 error (section 5) could yield smoother surfaces if we could start with « better » inner points.

The iterative idea is based on the fact that after application of the C^2 error minimization scheme, the C^2 error is reduced, hence the new inner points are « better » than the old ones. Now using these « better » inner points and once again applying the minimization scheme we can reduce the C^2 errors further.

The iterative improvement weakens the locality of the standard or of the modified Clough-Tocher schemes : suppose just one data value were nonzero. Using the standard Clough-Tocher scheme, all triangular patches sharing the corresponding data site would be nonzero. Using the modified method, all neighboring patches would be nonzero as well. With the iterative method, one further layer of triangles is added per iteration. This gradual loss of locality may explain the apparent shape improvement of the resulting surfaces ; the less local a scheme is, the more potential exists for « ironing out » shape imperfections.

5. REFLECTION LINES

In order to judge the performance of different interpolation schemes, one may print out errors relative to known test functions, one may compare perspective views, or one may inspect contour plots. We have found that a third method is far more powerful : this is the use of reflection lines. The idea comes from the automotive industry. Here designers judge the aesthetic appearance of a car body by placing it under parallel fluorescent light bulbs. These reflect in the car surface, and instead of judging the car body directly, one judges how the light sources reflect. Tiny imperfections are detectable with this method. We use this method of quality inspection to evaluate our new interpolants. For literature on reflection lines etc., see [8], [3], [9], [10].

As shown in [8], the problem of finding reflection lines of surfaces of the form $z = f(x, y)$ amounts to contouring a directional derivative of that

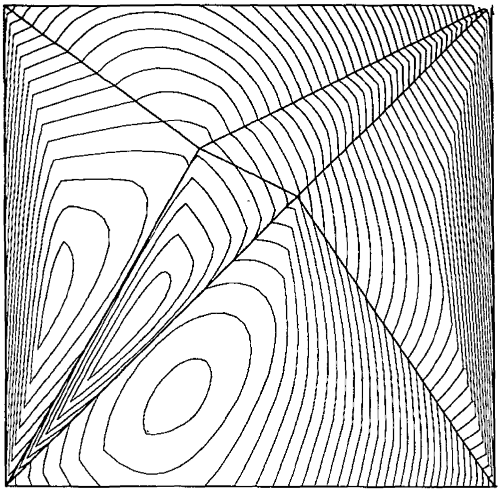

Figure 3. — Reflection lines : the original Clough-Tocher interpolant with linearized cross-boundary derivatives.

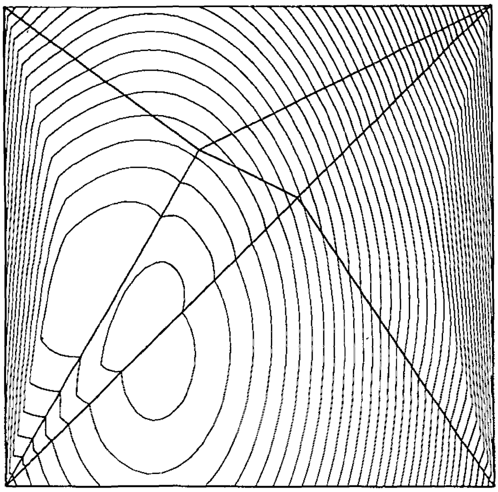

Figure 4. — Reflection lines : the modified Clough-Tocher interpolant with no iterations.

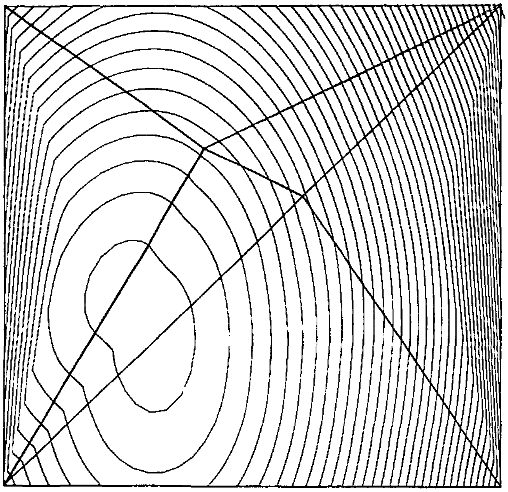

Figure 5. — Reflection lines : the iterated Clough-Tocher interpolant after ten iterations.

surface. Since we are using triangular cubic patches, its directional derivatives are quadratic patches, the contours of which are conic sections [11].

The test function that we used is :

$$f(x, y) = (x - 0.3)^3 + x(y - 0.3)^2 - 0.1\, x$$

with exact gradients. The data sites consist of the corners of the unit square with two additional points (0.4, 0.7) and (0.6, 0.6).

In figures 3, 4, 5, we can see that the boundaries of the minitriangles are much more visible in the reflection lines of the older methods, and in general the new scheme produces visually more pleasing (smoother) reflection line patterns.

We are currently experimenting with a scheme that would, by suitably adjusting control points within a macro-triangle, produce surfaces with the promise of cubic precision. If successful, this research will be reported elsewhere.

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