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REMARK ON THE RESULT ON HOMOGENIZATION IN HYDRODYNAMICAL LUBRICATION BY G. BAYADA AND M. CHAMBAT (*)

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Résumé. — *Dans le travail récent* Homogenization of the Stokes system in a thin film flow with rapidly varying thickness, *M^2 AN,* 23 (1989), *pp.* 205-234, *G. Bayada and M. Chambat ont prouvé la convergence du processus de l'homogénéisation en faisant une hypothèse sur le comportement de la pression. En effet ils ont utilisé la convergence forte dans L^2 de la pression sans être capables de la prouver. Dans cette courte communication nous prouvons la convergence du processus de l'homogénéisation sans aucune conjecture, i.e. nous complétons leur étude mathématique.*

Abstract. — *In a recent paper entitled* Homogenization of the Stokes system in a thin film flow with rapidly varying thickness, *M^2 AN,* 23 (1989), *pp.* 205-234, *G. Bayada and M. Chambat have proved the convergence of the homogenization process under an additional conjecture on the behavior of the pressure. Actually, they used the strong L^2-convergence of the pressure, but were unable to prove it. In this short note we prove the convergence of the homogenization process without using any unproven conjecture, i.e. we make their mathematical study complete.*

In a recent paper [1] Bayada and Chambat have studied the asymptotic behavior of a viscous fluid flow in a narrow gap with mean thickness η whose surfaces are supposed to be rough, with a periodic roughness of wavelength ε, when the two small parameters ε and η tend to zero. Depending on $\lambda = \eta/\varepsilon$, various situations may occur. In that paper the particular case $\lambda =$ cst. was studied in detail, but the proof of convergence

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was left uncompleted. More precisely, in order to prove that the limit pressure P^* satisfies the equation

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{2} \frac{\partial}{\partial x_j} ([\alpha_j^i] P^*) \right) = 0 \tag{1}$$

with an appropriate boundary condition, the authors conjectured that $\varepsilon^2 P^\varepsilon \to P^*$ strongly in $L^2(\Omega)$ (see [1], Theorem 3.3, p. 219).

The goal of this paper is to prove Theorem 3.3 from [1], without making this conjecture. In [1] the authors used the strong convergence of $\{\varepsilon^2 P^\varepsilon\}$ in order to calculate

$$\lim_{\varepsilon \to 0} \sum_{j=1}^{2} \int_\Omega \varepsilon^2 P^\varepsilon \frac{\partial \phi}{\partial x_j} \alpha_j^{i\varepsilon} \, dx \, dz , \tag{2}$$

where

$$\alpha_j^{i,\varepsilon} \to \frac{[\alpha_j^i]}{Y_1 Y_2} \quad \text{in} \quad L^2(\Omega) \quad \text{weakly} \tag{3}$$

and $\phi \in C_0^\infty(\Omega)$.

We make more careful estimates of ∇P^ε and the operator R^ε, connected with a continuation of the pressure introduced originally by L. Tartar. These estimates allow us to calculate the limit (2), without having the strong convergence of $\varepsilon^2 P^\varepsilon$.

For a detailed formulation of the problem we refer to [1]. We start with the rescaled weak formulation ([1], p. 211). In order to define geometrical data, we follow the same reference. We suppose that ω is an open set in R^2 with a Lischitz boundary $\partial\omega$ and that ε is a small parameter related to the roughness wavelength scale. h is a smooth function, defined for y in R^2, periodic with period Y_i in $y_i (i = 1, 2)$. We set $Y = [0, Y_1] \times [0, Y_2]$. Next we define

$$h_{\min} = \min_{y \in Y} h(y) ; \quad h_{\min} > 0 ; \quad h_{\max} = \max_{y \in Y} h(y) ; \quad h^\varepsilon(x) = h\left(\frac{x}{\varepsilon} \right) .$$

Our rescaled equations hold in the domain

$$\Omega_\varepsilon = \{ (x, z) \in R^3, \quad x \in \omega, \quad 0 < z < h^\varepsilon(x) \} .$$

In addition, we have to introduce a fixed Ω involving Ω_ε :

$$\Omega = \{ (x, z) \in R^3, \quad x \in \omega, \quad 0 < z < h_{\max} \} .$$

The problem to be considered is a mixed weak formulation for the Stokes system :

Find $(u^\varepsilon, p^\varepsilon) \in H^1(\Omega_\varepsilon)^3 \times L^2(\Omega_\varepsilon)/R$ such that

$$(4a)\quad \sum_{i=1}^{3} \int_{\Omega_\varepsilon} \left(\sum_{j=1}^{2} \frac{\partial u_i^\varepsilon}{\partial x_j} \cdot \frac{\partial \phi_i}{\partial x_j} + \frac{1}{\lambda^2 \varepsilon^2} \cdot \frac{\partial u_i^\varepsilon}{\partial z} \cdot \frac{\partial \phi_i}{\partial z} \right) dx\, dz =$$

$$= \int_{\Omega_\varepsilon} p^\varepsilon \left(\sum_{i=1}^{2} \frac{\partial \phi_i}{\partial x_i} + \frac{1}{\lambda\varepsilon} \cdot \frac{\partial \phi_3}{\partial z} \right) dx\, dz\,, \quad \forall \phi \in H_0^1(\Omega_\varepsilon)^3\,,$$

$$(4b)\quad \int_{\Omega_\varepsilon} q \cdot \left(\sum_{i=1}^{2} \frac{\partial u_i^\varepsilon}{\partial x_i} + \frac{1}{\lambda\varepsilon} \cdot \frac{\partial u_3^\varepsilon}{\partial z} \right) dx\, dz = 0\,, \quad \forall q \in L^2(\Omega_\varepsilon)\,,$$

$$(4c)\quad u^\varepsilon \big|_{\partial\Omega_\varepsilon} = (K^\varepsilon, 0, 0)\big|_{\partial\Omega_\varepsilon} = (K, 0, 0)\,,$$

where K does not depend on ε and λ,

$$K^\varepsilon \in H^1(\Omega)^3, \quad \sum_{i=1}^{2} \frac{\partial K_i^\varepsilon}{\partial x_i} + \frac{1}{\lambda\varepsilon} \cdot \frac{\partial K_3^\varepsilon}{\partial z} = 0, \quad K\,|_\omega = s \in H^{1/2}(\omega)\,,$$

$$\|K^\varepsilon\|_\varepsilon^2 = \sum_{i=1}^{2} \left[\sum_{j=1}^{2} \left(\frac{\partial K_i^\varepsilon}{\partial x_i} \right)^2 + \frac{1}{\lambda^2 \varepsilon^2} \cdot \left(\frac{\partial K_i^\varepsilon}{\partial z} \right)^2 \right] \leqslant \frac{C}{\varepsilon^2}$$

and

$$t(x) = Y_1\, Y_2 \int_0^{h_{\min}(x)} K(x, z) \cos\,(n, x_1)\, dz\,.$$

Let us introduce

$$\Sigma_\varepsilon = \{(x, z) \in R^3, x \in \omega, z = h^\varepsilon(x)\}\,,$$

$$V_z = \left\{ v \in L^2(\Omega)^3, \ \frac{\partial v}{\partial z} \in L^2(\Omega)^3 \right\}\,,$$

$$V_2(\Omega) = \{ v \in H_0^1(\Omega)^3, \ \operatorname{div} v = 0 \}\,,$$

and extend $v \in H^1(\Omega_\varepsilon)$, $v = 0$ on Σ_ε, by zero to Ω.

In [1] G. Bayada and M. Chambat have proved the following :

THEOREM 1 ([1], p. 212) : *There exists u^* in V_z such that*

$$u^\varepsilon \rightharpoonup u^* \qquad L^2(\Omega)^3 \ \textit{weak}\,,$$

$$\frac{\partial u^\varepsilon}{\partial z} \rightharpoonup \frac{\partial u^*}{\partial z} \qquad L^2(\Omega)^3 \ \textit{weak}\,,$$

$$\varepsilon \frac{\partial u^\varepsilon}{\partial x_i} \rightharpoonup 0 \qquad L^2(\Omega)^3 \ \textit{weak}, \quad i = 1, 2\,.$$

Moreover, $u_3^* = 0$, $u^* = 0$ *on* Σ, $u^* = (s, 0, 0)$ *on* ω.

We now prove the following properties :

PROPOSITION 1 :

$$\left|\left\langle \frac{\partial p^{\varepsilon}}{\partial x_j}, \phi \right\rangle_{\Omega_\varepsilon}\right| \leq C \left\{ \frac{1}{\varepsilon} \sum_{i=1}^{2} \left\| \frac{\partial \phi}{\partial x_i} \right\|_{L^2(\Omega_\varepsilon)} + \frac{1}{\varepsilon^2} \left\| \frac{\partial \phi}{\partial z} \right\|_{L^2(\Omega_\varepsilon)} \right\},$$
$$\forall \phi \in H_0^1(\Omega_\varepsilon),\ j = 1, 2,$$

$$\left|\left\langle \frac{\partial p^{\varepsilon}}{\partial z}, \phi \right\rangle_{\Omega_\varepsilon}\right| \leq C \left\{ \sum_{i=1}^{2} \left\| \frac{\partial \phi}{\partial x_i} \right\|_{L^2(\Omega_\varepsilon)} + \frac{1}{\varepsilon} \left\| \frac{\partial \phi}{\partial z} \right\|_{L^2(\Omega_\varepsilon)} \right\},$$
$$\forall \phi \in H_0^1(\Omega_\varepsilon).$$

Proof: It is analogous to the proof of Proposition 3.1 in [1], p. 213-214.

Now, following [1], p. 214, we make a few more assumptions on the geometrical structure (see assumptions H1, H2 and H3 in [1], p. 214) and obtain the following result.

LEMMA 1 : *There exists an operator* $R^{\varepsilon}: H_0^1(\Omega)^3 \to H_0^1(\Omega_\varepsilon)^3$ *such that*

$$v \in H_0^1(\Omega_\varepsilon)^3 \Rightarrow R^{\varepsilon}(v) = v,$$

$$\operatorname{div} v = 0 \Rightarrow \operatorname{div} R^{\varepsilon}(v) = 0,$$

$$\left\| \frac{\partial}{\partial x_j} R^{\varepsilon} v \right\|_{L^2(\Omega_\varepsilon)^3} \leq C \left\{ \sum_{i=1}^{2} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^2(\Omega)^3} + \frac{1}{\varepsilon} \left\| \frac{\partial v}{\partial z} \right\|_{L^2(\Omega)^3} \right\},$$

$$\left\| \frac{\partial}{\partial z} R^{\varepsilon} v \right\|_{L^2(\Omega_\varepsilon)^3} \leq C \left\{ \varepsilon \sum_{i=1}^{2} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^2(\Omega)^3} + \left\| \frac{\partial v}{\partial z} \right\|_{L^2(\Omega)^3} \right\}.$$

Proof: The proof is similar to the proof of Lemma 3.2 in [1], p. 217, and therefore we do not repeat it here.

THEOREM 2 ([1], p. 218-219) : *There exists* $P^{\varepsilon} \in L^2(\Omega)$ *such that* ∇P^{ε} *is an extension of* ∇p^{ε}. *Moreover, there exists* $P^* \in L^2(\Omega)/R$ *such that a subsequence verifies*

$$\varepsilon^2 P^{\varepsilon} \rightharpoonup P^* \quad L^2(\Omega)/R \text{ weak},$$

$$\frac{\partial P^*}{\partial z} = 0.$$

Now we introduce some auxiliary periodic problems. They are defined on a basic cell B, given by

$$B = \{(y, z) \in R^3,\ \ y \in Y,\ \ 0 < z < h(y)\}.$$

Let

$$H^Y = \{\phi \in H^1(B)^3, \phi \text{ is } Y\text{-periodic in the } y_i \text{ variables}\},$$
$$H_0^Y = \{\phi \in H^Y, \phi(y, 0) = \phi(y, h(x, y)) = 0\}$$
$$a(\phi^1, \phi^2) = \sum_{i=1}^{3} \left[\sum_{j=1}^{2} \int_B \frac{\partial \phi_i^1}{\partial y_j} \frac{\partial \phi_i^2}{\partial y_j} + \frac{1}{\lambda^2} \int_B \frac{\partial \phi_i^1}{\partial z} \frac{\partial \phi_i^2}{\partial z} \right],$$
$$\operatorname{div}_\lambda \phi = \sum_{i=1}^{2} \frac{\partial \phi}{\partial y_i} + \frac{1}{\lambda} \frac{\partial \phi}{\partial z}.$$

Now we can define the problem (L^0).

Find $(\alpha^0, q^0) \in H^Y \times L^2(B)$ such that

$$a(\alpha^0, \phi) = \int_B q^0 \operatorname{div}_\lambda \phi, \quad \forall \phi \in H_0^Y,$$
$$\int_B \xi \operatorname{div}_\lambda \alpha^0 = 0, \quad \forall \xi \in L^2(B),$$
$$\alpha^0(y, h(x, y)) = 0, \quad \alpha^0(y, 0) = (s, 0, 0),$$

and the problem (L^i), $i = 1, 2$:

Find $(\alpha^i, q^i) \in H_0^Y \times L^2(B)$ such that

$$a(\alpha^i, \phi) = \int_B q^i \operatorname{div}_\lambda \phi - \int_B \phi_i, \quad \forall \phi \in H_0^Y,$$
$$\int_B \xi \operatorname{div}_\lambda \alpha^i = 0, \quad \forall \xi \in L^2(B).$$

We set

$$\alpha^{i,\varepsilon} = \alpha^i\left(\frac{x}{\varepsilon}, z\right), \quad q^{i,\varepsilon} = q^i\left(\frac{x}{\varepsilon}, z\right).$$

Then, after extending $\alpha^{i,\varepsilon}$ by zero to the whole Ω, we have

(5) $$\alpha^{i,\varepsilon} \rightharpoonup \frac{1}{Y_1 Y_2}[\alpha^i] = \frac{1}{Y_1 Y_2} \int_B \alpha^i(y, z)\, dy\, dz \quad \text{in} \quad L^2(\Omega)^3 \text{ weak}.$$

We also have

(6) $$\|\alpha^{i,\varepsilon}\|_{L^2(\Omega)^3} \leq C, \quad \left\| \frac{\partial \alpha^{i,\varepsilon}}{\partial z} \right\|_{L^2(\Omega)^3} \leq C.$$

(see [1], p. 219).

Now we come to the main result of this paper:

THEOREM 3: *We have*

(7) $$\lim_{\varepsilon \to 0} \sum_{j=1}^{2} \int_\Omega \varepsilon^2 P^\varepsilon \frac{\partial \phi}{\partial x_j} \alpha_j^{i\varepsilon} \, dx \, dz = $$
$$= \sum_{j=1}^{2} \int_\omega \frac{\partial \phi}{\partial x_j} P^*[\alpha_j^i] \, dx , \quad \forall \phi \in C_0^\infty(\Omega) .$$

Proof: Let us define $\omega^{i,\varepsilon}$ by setting

$$\omega^{i,\varepsilon} = \alpha^{i,\varepsilon} - \frac{1}{|\Omega|} \int_\Omega \alpha^{i,\varepsilon} .$$

Then we have

$$\omega^{i,\varepsilon} \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega)^3/R ,$$
$$\frac{\partial \omega^{i,\varepsilon}}{\partial z} \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega)^3 ,$$
$$\frac{1}{|\Omega|} \int_\Omega \alpha^{i,\varepsilon} \to \frac{1}{Y_1 Y_2} [\alpha^i] \quad \text{in } R .$$

Obviously, if we prove

$$\lim_{\varepsilon \to 0} \sum_{j=1}^{2} \int_\Omega \varepsilon^2 P^\varepsilon \frac{\partial \phi}{\partial x_j} \omega_j^{i,\varepsilon} \, dx \, dz = 0 ,$$

we easily obtain (7).

Let us define a sequence $\{(\psi_j^{i,\varepsilon}, \pi_j^{i,\varepsilon})\}$ as weak solutions to a sequence of nonhomogeneous Stokes problems

(8a) $$-\Delta \psi_j^{i,\varepsilon} + \nabla \pi_j^{i,\varepsilon} = 0 \quad \text{in } \Omega ,$$

(8b) $$\operatorname{div} \psi_j^{i,\varepsilon} = \omega_j^{i,\varepsilon} \quad \text{in } \Omega ,$$

(8c) $$\psi_j^{i,\varepsilon} = 0 \quad \text{on } \partial\Omega .$$

Because of Theorem 2.4, p. 31 from [2] the problem (8*a*)-(8*c*) is uniquely solvable in $H_0^1(\Omega)^3 \times L^2(\Omega)/R$ and $\psi_j^{i,\varepsilon} \rightharpoonup 0$ weakly in $H_0^1(\Omega)^3$. Therefore, there exists a subsequence which converges to zero weakly in $H_0^1(\Omega)^3$ and strongly in $L^2(\Omega)^3$. Furthermore, let us investigate the interior regularity for $\partial \psi_j^{i,\varepsilon}/\partial z$. For fixed $\xi \in C_0^\infty(\Omega)$, we have

(9a) $$-\Delta(\xi \psi_j^{i,\varepsilon}) + \nabla(\xi \pi_j^{i,\varepsilon}) = -2 \nabla \psi_j^{i,\varepsilon} \nabla \xi - \psi_j^{i,\varepsilon} \Delta \xi \quad \text{in } \Omega ,$$

(9b) $$\operatorname{div}(\xi \psi_j^{i,\varepsilon}) = \xi \omega_j^{i,\varepsilon} + \nabla \xi \psi_j^{i,\varepsilon} \quad \text{in } \Omega ,$$

(9c) $$\xi \psi_j^{i,\varepsilon} = 0 \quad \text{on } \partial\Omega .$$

Now, by using the same result from [2], we easily conclude that $\partial \psi_j^{l,\varepsilon}/\partial z \in H^1_{\mathrm{loc}}(\Omega)^3$ and (after passing to a subsequence)

$$\frac{\partial \phi}{\partial x_j} \cdot \frac{\partial \psi_j^{l,\varepsilon}}{\partial z} \to 0 \quad \text{strongly in } L^2(\Omega)^3 .$$

Consequently, we have

$$\left| \sum_{j=1}^{2} \int_\Omega \varepsilon^2 P^\varepsilon \frac{\partial \phi}{\partial x_j} \omega_j^{l,\varepsilon}\, dx\, dz \right|$$

$$\leqslant \left| \left\langle \varepsilon^2 \nabla P^\varepsilon, \frac{\partial \phi}{\partial x_j} \psi_j^{l,\varepsilon} \right\rangle_\Omega \right| + \left| \int_\Omega \varepsilon^2 P^\varepsilon \psi_j^{l,\varepsilon} \nabla \frac{\partial \phi}{\partial x_j} \right| .$$

We consider the first term on the right-hand side. By definition we have

$$\left| \left\langle \varepsilon^2 \nabla P^\varepsilon, \frac{\partial \phi}{\partial x_j} \psi_j^{l,\varepsilon} \right\rangle_\Omega \right| = \left| \left\langle \varepsilon^2 \nabla p^\varepsilon, R^\varepsilon \left(\frac{\partial \phi}{\partial x_j} \psi_j^{l,\varepsilon} \right) \right\rangle_{\Omega_\varepsilon} \right| .$$

After some straightforward calculations, by using Proposition 1 and Lemma 1 we conclude that

$$\left| \left\langle \varepsilon^2 \nabla p^\varepsilon, R^\varepsilon \left(\frac{\partial \phi}{\partial x_j} \psi_j^{l,\varepsilon} \right) \right\rangle_{\Omega_\varepsilon} \right| \leqslant$$

$$\leqslant C \left\{ \left\| \frac{\partial}{\partial z} \left(\frac{\partial \phi}{\partial x_j} \psi_j^{l,\varepsilon} \right) \right\|_{L^2(\Omega)^3} + \varepsilon \left\| \nabla \left(\frac{\partial \phi}{\partial x_j} \psi_j^{l,\varepsilon} \right) \right\|_{L^2(\Omega)^9} \right\} \to 0, \text{ as } \varepsilon \to 0 .$$

Obviously,

$$\left| \int_\Omega \varepsilon^2 P^\varepsilon \psi_j^{l,\varepsilon} \nabla \frac{\partial \phi}{\partial x_j} \right| \to 0 \text{ as } \varepsilon \to 0 .$$

Therefore

$$\lim_{\varepsilon \to 0} \sum_{j=1}^{2} \int_\Omega \varepsilon^2 P^\varepsilon \frac{\partial \phi}{\partial x_j} \omega_j^{l,\varepsilon}\, dx\, dz = 0 ,$$

and Theorem 3 is proved.

Now we define the proposed asymptotic equation

Find $p^{-2} \in H^1(\omega)$ such that

$$(10) \quad \sum_{i=1}^{2} \sum_{j=1}^{2} \int_\omega [\alpha_j^i] \frac{\partial p^{-2}}{\partial x_j} \frac{\partial \phi}{\partial x_i}\, dx + \sum_{i=1}^{2} \int_\omega [\alpha_i^0] \frac{\partial \phi}{\partial x_i}\, dx =$$

$$= \int_{\partial \omega} t \phi\, dx , \quad \forall \phi \in H^1(\omega) .$$

THEOREM 4 : *P^* satisfies (10).*

Proof: We follow the proof of Theorem 3.3. from [1], p. 219-222. Theorem 3 from this paper allow us to calculate the limit (7). The rest of the proof is exactly the same as ın [1].

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