Modélisation mathématique et analyse numérique

D. D. Bainov
A. B. Dishliev

Population dynamics control in regard to minimizing the time necessary for the regeneration of a biomass taken away from the population

Modélisation mathématique et analyse numérique, tome 24, nº 6 (1990), p. 681-691
<http://www.numdam.org/item?id=M2AN_1990__24_6_681_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol 24, n° 6, 1990, p 681 à 692)

POPULATION DYNAMICS CONTROL IN REGARD TO MINIMIZING THE TIME NECESSARY FOR THE REGENERATION OF A BIOMASS TAKEN AWAY FROM THE POPULATION (*)

D. D. BAINOV (1), A. B. DISHLIEV (2)

Communicated by R TEMAM

Abstract — *The dynamics of a population whose mathematical model is the equation of Verhulst is considered From the population by discrete outer effects (in the form of impulses) a certain quantity of biomass is taken away (supplemented) This process is described by means of differential equations with impulses at fixed moments The moments of impulse effect and the magnitudes of the impulses are determined so that the time of regeneration of the quantity of biomass taken away from the population be minimal*

Résume — *Contrôle de la dynamique d'une population par minimisation du temps nécessaire à la regenération d'une biomasse enlevee*

On considere la dynamique d'une population dont le modèle mathématique est l'équation de Verhulst Par des effets extérieurs discrets (sous forme d'impulsions) une certaine quantite de biomasse est ôtée (ou ajoutee) a la population Ce phenomène est decrit par des equations differentielles, avec impulsions à des instants fixes Les instants des effets impulsionnels et les amplitudes des impulsions sont déterminés de façon que le temps de régenération de la quantite de biomasse ôtee à la population soit minimal

I. INTRODUCTION

In the paper the equation of Verhulst is investigated :

$$\frac{dN}{dt} = \frac{\mu}{K} N(K - N), \tag{1}$$

The present investigation is supported by the Ministry of Culture, Science and Education of People's Republic of Bulgaria under Grant 61

(*) Received in July 1989

(1) Academy of Medicine, Sofia, Bulgaria

(2) Higher Institute of Chemical Technology, Sofia, Bulgaria

which describes the dynamics of many populations. In (1) $N = N(t) > 0$ denotes the biomass of the population at the moment $t \geqslant 0$, $K > 0$ is the capacity of the environment and $\mu > 0$ is the difference between the birth-rate and the death-rate. We assume that at the initial moment $t = 0$ we have

$$N(0) = N_0 , \tag{2}$$

where
$$0 < N_0 < \frac{K}{2} .$$

A frequently occurring situation is that, when the population is subject to outer effects, anthropogeneous ones included. We shall consider the case when the outer effects are discrete in time and are expressed in abrupt (instantaneous) taking away or supplementing of certain quantities of biomass. The differential equations with impulses are an adequate mathematical model of such dynamical processes [1]-[4]. Consider the impulsive analogue of the equation of Verhulst :

$$\frac{dN}{dt} = \frac{\mu}{K} N(K - N) , \quad t \neq \tau_i , \tag{3}$$

$$\Delta N(t)\,|_{t = \tau_i} = N(\tau_i + 0) - N(\tau_i) = -I_i , \quad i = 1, 2, \dots , \tag{4}$$

where $\tau_1, \tau_2, \dots$ $(0 < \tau_1 < \tau_2 < \dots)$ are the moments at which the discrete outer effects are realized, also called moments of impulse effect ; I_i, $i = 1, 2, \dots$ are the quantities of biomass taken away $(I_i > 0)$ or supplemented $(I_i < 0)$ respectively at the moments τ_i, $i = 1, 2, \dots$ We shall consider the important in regards to the applications in the biotechnologies case of taking away of biomass, i.e. $I_i \geqslant 0$, $i = 1, 2, \dots$

The solution of problem (3), (4), (2) is a piecewise continuous function with points of discontinuity of the first kind $\tau_1, \tau_2, \dots$ at which it is continuous from the left. We shall note that for $\tau_i < t \leqslant \tau_{i+1}$, $i = 1, 2, \dots$, the solution of problem (3), (4), (2) coincide with the solution of equation (1) with initial condition

$$N(\tau_i + 0) = N(\tau_i) - I_i . \tag{5}$$

Since the solutions of problems (1), (5) for $t > \tau_i$, $i = 1, 2, \dots$, are monotone increasing functions, then in a certain time after the moment τ_i, the population will regenerate its biomass which it had at the moment τ_i (just before the taking away of the biomass). This time we shall call time of regeneration of the biomass taken away at the moment τ_i and denote by $\Delta\tau_i$. It is clear that if $\tau_{i+1} - \tau_i = \Delta\tau_i$, then $N(\tau_i + \Delta\tau_i) = N(\tau_i)$.

Let a certain quantity of biomass be planned to be taken away from the population by several consecutive discrete outer effects. We shall denote this quantity by I. It is natural to assume that the quantities of biomass taken away at a single moment are bounded below, i.e.

$$I_0 \leq I_i\,, \quad i = 1, 2, \dots,$$

where $0 < I_0 < K$.

The aim of the present paper is to find the moments $\tau_1, \tau_2, \dots$, of impulse effect at which biomass is taken away, as well as the quantities I_1, $I_2, \dots$ of biomass taken away so that the sum of the times of regeneration of the quantities of biomass taken away be minimal.

II. PRELIMINARY NOTES

In the form of lemmas we shall give some inequalities which are useful for the further considerations.

LEMMA 1 : *If* $0 < I_1, I_2 < K$, *then*

$$\frac{(K+I_1)(K+I_2)}{(K-I_1)(K-I_2)} \geq \left(\frac{K+(I_1+I_2)/2}{K-(I_1+I_2)/2}\right)^2.$$

The proof of Lemma 1 is trivial.

LEMMA 2 : *If* $I_i > 0$, $i = 1, 2, \dots, n$ *and* $I_1 + I_2 + \cdots + I_n < K$, *then*

$$\frac{K+I_1+I_2+\cdots+I_n}{K-I_1-I_2-\cdots-I_n} \geq \frac{(K+I_1)(K+I_2)\dots(K+I_n)}{(K-I_1)(K-I_2)\dots(K-I_n)}.$$

Lemma 2 is proved by induction. Note that the equality is achieved for $n = 1$.

LEMMA 3 : *If* $0 < I < K$ *and* $1 \leq m < n$, *then*

$$\left(\frac{K+I/m}{K-I/m}\right)^m > \left(\frac{K+I/n}{K-I/n}\right)^n. \tag{6}$$

Proof : Set $a = \frac{I}{K}$. It is clear that $0 < a < 1$. Then inequality (6) becomes

$$\left(\frac{m+a}{m-a}\right)^m > \left(\frac{n+a}{n-a}\right)^n.$$

The above inequality will hold if we establish that the function

$$\varphi(X) = X \ln\frac{X+a}{X-a}, \quad X \geq 1,$$

is strictly monotone decreasing, i.e. that the inequality

$$\varphi'(X) = \ln \frac{X+a}{X-a} - \frac{2\,aX}{X^2 - a^2} < 0\,, \quad X \geqslant 1$$

is valid. For this purpose we establish that

$$\varphi''(X) = \frac{4\,a^3}{(X^2 - a^2)^2} > 0\,, \quad X \geqslant 1\,,$$

which implies that the function φ' is strictly monotone increasing for $X \geqslant 1$. Moreover,

$$\lim_{X \to \infty} \varphi'(X) \quad \lim_{X \to \infty} \left[\ln \frac{X+a}{X-a} - \frac{2\,aX}{X^2 - a^2} \right] = 0\,.$$

Hence $\varphi'(X) < 0$ for $X \geqslant 1$, i.e. the function φ is strictly monotone decreasing. Thus Lemma 3 is proved.

LEMMA 4 [5] : *Let $f : R \to R$ and for any two points I_1, $I_2 \in R$ we have*

$$f(I_1) + f(I_2) \geqslant 2\,f\left(\frac{I_1 + I_2}{2} \right).$$

Then for any $n \geqslant 2$ we have

$$f(I_1) + f(I_2) + \cdots + f(I_n) \geqslant nf\left(\frac{I_1 + I_2 + \cdots + I_n}{n} \right),$$

where $I_1, I_2, \ldots, I_n \in R$.

The proof of this well known assertion can be found for instance in [5].

We shall make some remarks :

Equation (1) is Bernoulli's, hence its solution can be found in quadratures. The solution of the initial value problem (1), (2) is given by the formula

$$N(t) = \frac{K}{\left(\frac{K}{N_0} - 1 \right) e^{-\mu t} + 1}\,, \quad t \geqslant 0\,. \tag{7}$$

The inflection point of the graph of the function $N = N(t)$ has coordinates $\left(\frac{1}{\mu} \ln \left(\frac{K}{N_0} - 1 \right), \frac{K}{2} \right)$.

Since the function (7) is strictly monotone increasing for $t \geqslant 0$, then it has an inverse function :

$$t = t(N) = \frac{1}{\mu} \ln \frac{\frac{K}{N_0} - 1}{\frac{K}{N} - 1}, \quad 0 < N < K. \tag{8}$$

From (8) we obtain immediately that for $0 < I < K$ and $0 < N < K - I$,

$$\psi(N) = t(N + I) - t(N) = \frac{1}{\mu} \ln \frac{\frac{K}{N} - 1}{\frac{K}{N + I} - 1}. \tag{9}$$

In particular, for $N = \frac{K - I}{2}$,

$$\psi\left(\frac{K - I}{2}\right) = \frac{2}{\mu} \ln \frac{K + I}{K - I}. \tag{10}$$

III. MAIN RESULTS

THEOREM 1: *If* $0 \leq I < K$ *and* $0 < N < K - I$, *then* $\psi\left(\frac{K - I}{2}\right) \leq \psi(N)$, *i.e.*

$$t\left(\frac{K + I}{2}\right) - t\left(\frac{K - I}{2}\right) \leq t(N + I) - t(N). \tag{11}$$

Proof: From (9) it follows that the function $\psi = \psi(N)$ achieves its minimum at $N = \frac{K - I}{2}$, which immediately implies (11).

Remark 1: The above theorem allows us to claim that the time of regeneration of the quantity of biomass I_i taken away at the moment τ_i is minimal if the moments of impulse effect τ_i satisfy the equality

$$\frac{N(\tau_i) + N(\tau_i + 0)}{2} = N(\tau_i) - \frac{I_i}{2} = \frac{K}{2},$$

i.e. if

$$N(\tau_i) = \frac{K + I_i}{2}, \quad i = 1, 2, \ldots$$

The impulse effects I_i, $i = 1, 2, \ldots$, satisfying the last equalities will be called centered.

THEOREM 2 : *If* $I_i > 0$, $i = 1, 2, \ldots, n$ *and* $I_1 + I_2 + \cdots + I_n = I < K$, *then for any* N, $0 < N < K - I$, *we have*

$$t(N+I) - t(N) \geqslant \left[t\left(\frac{K+I_1}{2}\right) - t\left(\frac{K-I_1}{2}\right)\right] + \\ + \left[t\left(\frac{K+I_2}{2}\right) - t\left(\frac{K-I_2}{2}\right)\right] + \cdots + \left[t\left(\frac{K+I_n}{2}\right) - t\left(\frac{K-I_n}{2}\right)\right]. \tag{12}$$

Proof : For the left-hand side of (12), by Theorem 1, we have

$$t(N+I) - t(N) \geqslant t\left(\frac{K+I}{2}\right) - t\left(\frac{K-I}{2}\right).$$

From the above inequality by means of (10) it is seen that

$$t(N+I) - t(N) \geqslant \frac{2}{\mu} \ln \frac{K+I}{K-I}, \quad 0 < N < K - I. \tag{13}$$

For the right-hand side of (12) we obtain

$$\left[t\left(\frac{K+I_1}{2}\right) - t\left(\frac{K-I_1}{2}\right)\right] + \cdots + \left[t\left(\frac{K+I_n}{2}\right) - t\left(\frac{K-I_n}{2}\right)\right] = \\ = \frac{2}{\mu} \ln \frac{K+I_1}{K-I_1} + \cdots + \frac{2}{\mu} \ln \frac{K+I_n}{K-I_n} \\ = \frac{2}{\mu} \ln \frac{(K+I_1)(K+I_2) \ldots (K+I_n)}{(K-I_1)(K-I_2) \ldots (K-I_n)}. \tag{14}$$

In view of Lemma 2 we establish that

$$\frac{K+I}{K-I} \geqslant \frac{(K+I_1)(K+I_2) \ldots (K+I_n)}{(K-I_1)(K-I_2) \ldots (K-I_n)},$$

whence it follows that

$$\frac{2}{\mu} \ln \frac{K+I}{K-I} \geqslant \frac{2}{\mu} \ln \frac{(K+I_1)(K+I_2) \ldots (K+I_n)}{(K-I_1)(K-I_2) \ldots (K-I_n)}.$$

From the last inequality, in view of (13) and (14), we get (12).

Remark 2 : Theorem 2 shows that the time of regeneration of the biomass I taken away by an arbitrary outer effect is not smaller than the sum of the times of regeneration of the same biomass taken away by several centered impulse effects.

THEOREM 3 : *If* $0 < I_1, I_2 < K$, *then*

$$\left[t\left(\frac{K+I_1}{2}\right)-t\left(\frac{K-I_1}{2}\right)\right]+\left[t\left(\frac{K+I_2}{2}\right)-t\left(\frac{K-I_2}{2}\right)\right]\geqslant$$
$$\geqslant 2\left[t\left(\frac{K+(I_1+I_2)/2}{2}\right)-t\left(\frac{K-(I_1+I_2)/2}{2}\right)\right]. \quad (15)$$

Proof: By Lemma 1 we have

$$\frac{(K+I_1)(K+I_2)}{(K-I_1)(K-I_2)}\geqslant\left(\frac{K+(I_1+I_2)/2}{K-(I_1+I_2)/2}\right)^2 .$$

We take the logarithm of both sides of the last inequality and obtain

$$\frac{2}{\mu}\ln\frac{K+I_1}{K-I_1}+\frac{2}{\mu}\ln\frac{K+I_2}{K-I_2}\geqslant 2\,\frac{2}{\mu}\ln\frac{K+(I_1+I_2)/2}{K-(I_1+I_2)/2},$$

whence, by (9) and (10), we get to (15).

Remark 3 : Theorem 3 allows us to claim that the time of regeneration of the biomass of quantity I_1+I_2 taken away by two centered outer effects of magnitude respectively I_1 and I_2 is not smaller than the time of regeneration of the same biomass by two centered outer effects of magnitude I_1' and I_2' where $I_1'=I_2'=\dfrac{I_1+I_2}{2}$.

THEOREM 4 : *If* $0<I_0\leqslant I_i<K$, $0<N_i<K-I_i$ *for* $i=1,2,\ldots,m$ *and if, moreover,* $nI_0\leqslant I<(n+1)I_0$ *where* $I=I_1+I_2+\cdots+I_m$ *(n is a positive integer), then*

$$[t(N_1+I_1)-t(N_1)]+[t(N_2+I_2)-t(N_2)]+\cdots+$$
$$+[t(N_m+I_m)-t(N_m)]\geqslant n\left[t\left(\frac{K+I/n}{2}\right)-t\left(\frac{K-I/n}{2}\right)\right]. \quad (16)$$

Proof: From Theorem 1 it follows that

$$[t(N_1+I_1)-t(N_1)]+\cdots+[t(N_m+I_m)-t(N_m)]\geqslant$$
$$\geqslant\left[t\left(\frac{K+I_1}{2}\right)-t\left(\frac{K-I_1}{2}\right)\right]+\cdots+\left[t\left(\frac{K+I_m}{2}\right)-t\left(\frac{K+I_m}{2}\right)\right]. \quad (17)$$

Consider the function $f:(0,K)\to R^+$ defined by means of the equality

$$f(X)=\psi\left(\frac{K-X}{2}\right)=t\left(\frac{K+X}{2}\right)-t\left(\frac{K-X}{2}\right),\quad 0<X<K .$$

By Theorem 3 we have

$$f(I_1) + f(I_2) \geqslant 2 f \left(\frac{I_1 + I_2}{2} \right),$$

whence, in view of Lemma 4, we conclude that

$$f(I_1) + f(I_2) + \cdots + f(I_m) \geqslant mf \left(\frac{I_1 + I_2 + \cdots + I_m}{m} \right) = mf \left(\frac{I}{m} \right).$$

From the above inequality, (17) and the definition of the function f we obtain the estimate

$$[t(N_1 + I_1) - t(N_1)] + \cdots + [t(N_m + I_m) - t(N_m)] \geqslant \geqslant m \left[t \left(\frac{K + I/m}{2} \right) - t \left(\frac{K - I/m}{2} \right) \right]. \quad (18)$$

In view of (10) we obtain that

$$m \left[t \left(\frac{K + I/m}{2} \right) - t \left(\frac{K - I/m}{2} \right) \right] = \frac{2}{\mu} \ln \left(\frac{K + I/m}{K - I/m} \right)^m.$$

It is easy to see that $m \leqslant n$. Then from Lemma 3 we conclude that

$$\left(\frac{K + I/m}{K - I/m} \right)^m \geqslant \left(\frac{K + I/n}{K - I/n} \right)^n,$$

whence it follows that

$$\frac{2}{\mu} \ln \left(\frac{K + I/m}{K - I/m} \right)^m \geqslant \frac{2}{\mu} \ln \left(\frac{K + I/n}{K - I/n} \right)^n.$$

From the last inequality, (18) and (19) we obtain (16).

Remark 4 : Theorem 4 will be given the following interpretation. Let a quantity I be taken away from the biomass of the population investigated in the form of discrete outer effects. Moreover, let the quantity of biomass which can be taken away at a single moment, be bounded below by the positive constant I_0. Let n be the greatest integer in $\frac{I}{I_0}$. Then the minimal time of regeneration of the quantity I taken away from the biomass of the population considered is realized by n centered discrete outer effects of magnitude $\frac{I}{n}$.

IV. MODEL OF THE OPTIMAL REGIME OF OUTER EFFECTS

Optimal here (as well as throughout the paper) is understood in the sense of the quickest regeneration of the quantity of biomass taken away from the population. The assumptions here coincide with the conditions of Theorem 4. Let us formulate them :

1) $0 < I_0 \leq I_i < K, \ i = 1, 2, \dots$;
2) $I_1 + I_2 + \cdots = I, \ I_0 < I$;
3) $nI_0 \leq I < (n+1) I_0$, where n is a positive integer ;
4) $0 < N_0 < \frac{K}{2}$.

Then by Theorem 4 the optimal regime of the impulse effects on the dynamics of the population is described by the following initial value problem for a equation with impulses :

$$\frac{dN}{dt} = \frac{\mu}{K} N(K-N), \quad t \neq \tau_i, \tag{20}$$

$$\Delta N(t)|_{t=\tau_i} = \frac{-I}{n}, \quad i = 1, 2, \dots, n, \tag{21}$$

$$N(0) = N_0, \tag{22}$$

where the moments of impulse effect $\tau_1, \tau_2, \dots, \tau_n$ are determined in the following way. We find τ_1 by the formula

$$\tau_1 = t\left(\frac{K + I/n}{2}\right) = \frac{1}{\mu} \ln \frac{(K - N_0)(nK + I)}{N_0(nK - I)}.$$

Moreover, we have

$$\tau_{i+1} = \tau_i + \Delta\tau_i = \tau_i + \Delta\tau, \quad i = 1, 2, \dots, n,$$

where

$$\Delta\tau = t\left(\frac{K + I/n}{2}\right) - t\left(\frac{K - I/n}{2}\right) = \frac{1}{\mu} \ln \left(\frac{nK + I}{nK - I}\right)^2.$$

Hence

$$\tau_i = \frac{1}{\mu} \ln \left[\frac{K - N_0}{N_0} \left(\frac{nK + I}{nK - I}\right)^{2i+1}\right], \quad i = 1, 2, \dots, n.$$

Finally we shall note that under the restrictions 1-4 imposed above the minimal time of regeneration of the biomass I taken away from a population whose dynamics is described by the equation of Verhulst is

$$n \, \Delta\tau = \frac{2n}{\mu} \ln \frac{nK + I}{nK - I} ,$$

where n is the greatest integer in $\frac{I}{I_0}$ and I_0 is the minimal quantity of biomass which can be taken away at a single moment.

Thus, for instance, for $\mu = 0.03$; $K = 100$; $N_0 = 15$; $I_0 = 15$ and $I = 50$ the graph of the optimal solution of problem (20), (21), (22) is shown on figure 1.

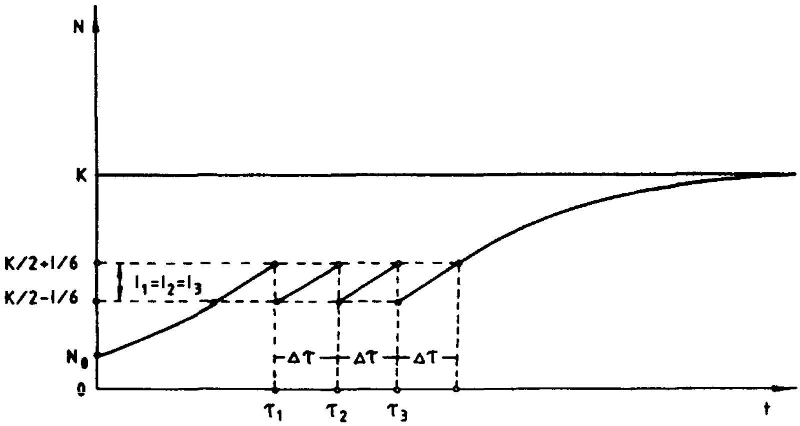

Figure 1.

For these initial data we have: $n = 3$; $\tau_1 \approx 69.034$; $\tau_2 \approx 91.466$; $\tau_3 \approx 113.897$; $I_1 = I_2 = I_3 = I/n \approx 16.67$ and the minimal time of regeneration of the biomass taken away is $3 \, \Delta\tau \approx 67.294$.

The authors express their gratitude to Prof. Bl. Sendov for the statement of the problem considered.

REFERENCES

[1] A. B. DISHLIEV, D. D. BAINOV (1988), Continuous dependence of the solution of a system of differential equations with impulses on the initial condition and a parameter in the presence of beating, Int. J. Systems sci., Vol. 19, No. 5, 669-682.

[2] A. B. DISHLIEV, D. D. BAINOV (1988), Continuous dependence of the solution of a system of differential equations with impulses on the impulse hypersurfaces, J. of Math. Analysis and Applications, Vol. 135, No. 2, 369-389.

[3] V. LAKSHMIKANTHAM, LIU XINZHI (1989), On quasi stability for impulsive differential systems, J. of Nonlinear Analysis (to appear).

[4] V. D. MIL'MAN, A. D. MYSHKIS (1960), On the stability of motion in the presence of impulses, Sib. Math. J., Vol. 1, No. 2, 233-237 (in Russian).

[5] I. P. NATANSON (1957), Theory of the Functions of a Real Variable, State Edition of Technico-Theoretical Literature, Second Ed., 581 p. (in Russian).