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An upwind finite element method for singularly perturbed elliptic problems and local estimates in the $L^\infty$-norm

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AN UPWIND FINITE ELEMENT METHOD
FOR SINGULARLY PERTURBED ELLIPTIC PROBLEMS
AND LOCAL ESTIMATES IN THE $L^\infty$-NORM (*)

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Abstract. — We consider a finite element method for solving singularly perturbed second order elliptic problems in 2D domains. We give a strategy for proving local $L^\infty$-error estimates for a class of upwind-FEM and use this strategy to derive such local estimates for a special finite element method.

Résumé. — On étudie une méthode d'éléments finis adaptée à la résolution de problèmes elliptiques du second ordre avec perturbations singulières, posés sur des domaines bidimensionnels. On décrit une méthode permettant d'établir des estimations $L^\infty$ locales de l'erreur pour une classe de méthodes « upwind », et on en déduit des estimations locales pour une méthode d'éléments finis particulière.

1. INTRODUCTION

We consider the problem

$$- \varepsilon \Delta u + b(x) \nabla u + c(x)u = f(x) \quad \text{in} \quad \Omega \subset \mathbb{R}^2$$

$$u = 0 \quad \text{on} \quad \Gamma = \partial \Omega$$

with the small parameter $0 < \varepsilon \ll 1$. The qualitative behaviour of the solution is characterized by the existence of boundary layers (narrow regions where the norms $\|u\|_{k,p}$, $k \geq 1$, are not bounded independently of $\varepsilon$) which influence the properties of discretization methods. So the

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application of standard finite element methods results in undesired oscillations in the numerical solution $u_h$ unless the discretization parameter is very small. These oscillations can spread over a region much larger than the boundary layer. A strong dependence on $\varepsilon$ also is to be seen in the classical error estimates which for piecewise linear discrete solutions can be written in the form

$$\sqrt{\varepsilon} |u - u_h|_{1,2} + \|u - u_h\|_{0,2} \leq Ch \|u\|_{2,2}$$

where generally $\|u\|_{2,2} = O\left(\varepsilon^{-\frac{3}{2}}\right)$.

($\|\cdot\|_{k,p,G}$ is the norm in $W^{k,p}(G)$. If $G = \Omega$ we omit the index $G$.)

It is obvious to try to improve the estimates in subdomains of where boundary layers are excluded (local estimates).

One objective of local estimates is with their help to ensure that the numerical method «recognizes» boundary layers. Some literature on the subject of local estimates can be found in [9] (using asymptotic means), [3], [4], [9], [10] (using cut-off techniques, [10] for $\varepsilon = O(1)$).

In this article we derive local estimates for a special upwind FEM in the $L^\infty$-norm using both cut-off functions similar to [4] and a method proposed already in 1973 by Ciarlet-Raviart [1] and generalized by Tabata [8] for proving global $L^\infty$-estimates. It is known that the method of Ciarlet-Raviart even for global estimates often doesn't give the optimal rate of convergence with respect to $h$. However, due to the difficulties of the dominating convection term and especially due to the fact that in our special upwind FEM the approximation error is only of order $h$ it is not clear whether other methods of proof can yield better estimates.

We call our special FEM «hybrid upwind-FEM» due to the fact (contrary to some other usual applications of the word «hybrid») that in the discretization both FEM- and FDM-ideas are used. It is only one representative of a class of FEM which all have the same main objective, i.e. preserving inverse-monotony (and by that such meaningful physical properties as non-negativity of the solutions and often $L^\infty$-a priori estimates and $L^\infty$-stability) for the discrete problem. A good survey of these methods and an extensive bibliography can be found in [2]. Additionally a rather easy to construct but nevertheless very interesting nonconforming variant is described in [5].

In this article we describe our hybrid upwind FEM and note some properties (Section 2), then we give a more general principle to get local $L^\infty$-error estimates (theorem 1, Section 3) and in section 4 we use theorem 1 to obtain $L^\infty$-error estimates for the hybrid upwind FEM. Finally, in Section 5, a numerical example is given.
2. THE HYBRID UPWIND FEM

The weak formulation of our problem is:

Find $u \in V_0 = H^1_0(\Omega)$ s.t.

$$\left\{ \begin{array}{c}
\langle Lu, v \rangle = l(u, v) = \varepsilon (\nabla u, \nabla v) + (b \nabla u, v) + (cu, v) = \langle f, v \rangle \quad (P) \\
\forall v \in V_0 
\end{array} \right.$$ 

Let $b, c$ be sufficiently smooth (we will need $b, c \in C^{0+1}(\bar{\Omega})$ and sometimes, for (2.3), $b \in C^{1+1}(\bar{\Omega})$).

Let be a bounded polygonal domain divided into triangles the angles of which are less equal $\frac{\pi}{2}$. Let the triangulation $\{T_h\}$ be regular in the usual sense and $h$ denote the maximal diameter of all triangles.

We consider also a dual decomposition of $\Omega$ which can be constructed in the following way.

To each node $P_i$ corresponds a dual polygon $D_i$ bounded by parts $\Gamma_{ij}$ of the mid-perpendiculars of the adjoining triangles.

Furthermore we use the notations:

$\Lambda_i$ : set of indices of the nodes adjoining to $P_i$

$n_{ij}$ : outer normal vector to $D_i$.

Let $V_h = \left\{ v_h \in C(\bar{\Omega}) \mid v_h|_T \in P_1(T) \right\}$, $V_{0h} = \left\{ v_h \in V_h \mid v_h|_{\Gamma} = 0 \right\}$ denote spaces of piecewise linear functions.
In contrast to standard FEM we approximate $(b \nabla u_h, v_h), (c u_h, v_h)$ by

$$b_h(u_h, v_h) = \sum_i v_h(P_i) \sum_{j \in \Lambda_i} \beta_{ij} (\lambda_{ij} - 1)(u_h(P_i) - u_h(P_j)),$$

$$c_h(u_h, v_h) = \sum_i \text{meas } D_i u_h(P_i) v_h(P_i)$$

where $\beta_{ij}$ is an approximation of $\int_{\Gamma_{ij}} b n_i \, d\Gamma_{ij}$ and $\lambda_{ij} = \frac{1}{2} (1 + \text{sgn } \beta_{ij})$.

The discretization of the convection term is motivated by the splitting

$$(b \nabla u_h, v_h) = (\text{div } (u_h b), v_h) - (u_h \text{ div } b, v_h)$$

and the transformations (using integration by parts)

$$(\text{div } (u_h b), v_h) = \sum_i \int_{D_i} \text{div } (u_h b) \, v_h \, dx \approx \sum_i v_h(P_i) \int_{D_i} \text{div } (u_h b) \, dx$$

$$\approx \sum_i v_h(P_i) \sum_{j \in \Lambda_i} \int_{\Gamma_{ij}} b n_{ij} u_h \, d_{ij}$$

$$\approx \sum_i v_h(P_i) \sum_{j \in \Lambda_i} \beta_{ij} (\lambda_{ij} u_h(P_i) + (1 - \lambda_{ij}) u_h(P_j))$$

and

$$(u_h \text{ div } b, v_h) = \sum_i \int_{D_i} v_h u_h \text{ div } b \, dx \approx$$

$$\approx \sum_i v_h(P_i) u_h(P_i) \int_{D_i} \text{div } b \, dx \approx \sum_i v_h(P_i) \sum_{j \in \Lambda_i} \beta_{ij} u_h(P_i).$$

We will need $\beta_{ij}$ such that

$$\beta_{ij} + \beta_{ji} = 0 \quad \text{if } \quad \Gamma_{ij} \cap \partial \Omega = \emptyset$$

(2.1)

and

$$\left| \beta_{ij} - \int_{\Gamma_{ij}} b n_i \, d\Gamma_{ij} \right| \leq C h^2.$$ (2.2)

Sometimes instead of (2.2) we demand

$$\left| \beta_{ij} - \int_{\Gamma_{ij}} b n_i \, d\Gamma_{ij} \right| \leq C h^3.$$ (2.3)

If $\Gamma_{ij} \cap \partial \Omega = \emptyset$ then (2.1) and (2.2) can be obtained (with $C = O(\| b \|_{C^{0+1}(\Omega)})$) e.g. for $\beta_{ij} = b(P_{ij}) n_i$ meas $\Gamma_{ij}$ where $P_{ij}$ is the
midpoint of \( \overline{P_i P_j} \); (2.3) can be obtained (with \( C = O(\|b\|_{C^{1+1}(\Omega)}) \)) for \( \beta_{ij} = b(\hat{P}_{ij}) n_i \) meas \( \Gamma_{ij} \) where \( \hat{P}_{ij} \) is the midpoint of \( \Gamma_{ij} \).

Now our discrete problem is

Find \( u_h \in V_{0h} \) s.t.

\[
l_h(u_h, v_h) = \varepsilon (\nabla u_h, \nabla v_h) + b_h(u_h, v_h) + c_h(u_h, v_h) = \langle f, v_h \rangle \quad \forall v_h \in V_{0h}
\]

Taking into consideration that \((P_h)\) corresponds to a linear system of equations

\[
L_h z_h = (\varepsilon A_h + B_h + C_h) z_h = f_h
\]

our discretization ensures that \( C_h \) is a diagonal matrix and that (due to the fact that we have no obtuse-angled triangles and due to the special choice of the \( \lambda_{ij} \)) the off-diagonal elements of \( A_h \) and \( B_h \) are non-positive.

Let us note the following essential facts about \((P)\) and \((P_h)\) (for proofs see [6]).

Let

(H1) \quad \begin{align*}
\forall \theta \in C_0^\infty, \quad & c(x) \equiv c_0 \equiv 0 \\
\text{or} \quad & c(x) \equiv \frac{1}{2} \text{div} \, b(x) \equiv \alpha_0 \equiv 0
\end{align*}

be fulfilled. Then the continuous problem \((P)\) admits a unique solution and \( L \) is inverse-monotone, i.e., from \( Lu \leq Lv \) and \( \sup (u - v, 0) \in H_0^1(\Omega) \) it follows \( u \leq v \). Moreover, (H1) with \( c_0 > 0 \) yields \( L^\infty \)-stability

\[
\|u - v\|_{0,\infty} \leq C \|Lu - Lv\|_{0,\infty}
\]

with a constant \( C \) independent on \( \varepsilon \) and from (H2) we obtain the \( V_0 \)-ellipticity of the bilinearform \( \langle L, \cdot, \cdot \rangle \).

Are similar properties valid for the discrete problem?

Under (H1) the hybrid upwind FEM preserves the properties of inverse-monotony, unique solvability and \( L^\infty \)-stability for all \( h \).

This is the main objective of the method and it is based on the fact that, contrary to standard FEM, the matrix \( L_h \) becomes an \( M \)-matrix. Furthermore, under (H2) with \( \alpha_0 > 0 \) the discrete bilinear-form is \( V_0h \)-elliptic for all \( h \leq h_0 \) independent of \( \varepsilon \) provided that (2.3) is fulfilled.

Under (H1), (H2) with \( \alpha_0 > 0 \) and (2.3) we get the estimate

\[
\|I_h u - u_h\|_{0,\infty} \leq C(\kappa) h^\kappa \varepsilon^{-\frac{1}{2}} \|u\|_{2,p} \quad \text{with} \quad \kappa \to 1 \quad \text{for} \quad p \to \infty
\]
where $I_h$ denotes the interpolation operator.

For triangulations of parallelogramm type (see below, def. 1) we obtain

$$\|I_h u - u_h\|_{0,\infty} \leq C(\kappa) h^k \|u\|_{2,p}.$$  

Unfortunately, the norm $\|u\|_{2,p}$ is not bounded uniformly with respect to $\varepsilon$ since boundary layers appear.

### 3. THE STRATEGY OF DERIVING THE LOCAL $L^\infty$-ERROR ESTIMATES

The central result of this section is theorem 1 where the local estimate for the hybrid upwind FEM is given in some more general formulation to make clearer the main principle of deriving the estimates.

First some notations.

For $z_h \in V_h$, $\delta \in R$

let $z_{h,\delta} = \sum z_{i,\delta} \varphi_i$ where $z_{i,\delta} = \max (0, z_h(P_i) - \delta)$

and $\varphi_i \in V_h$, $\varphi_i(P_j) = \delta_{ij}$.

For $\Psi \in C(\bar{\Omega})$, $\Psi \geq 0$, $p \geq 1$ we define the seminorms

$$|z|_{k,p,\Psi} = \left( \sum_{|\alpha| = k} \|\sqrt{\Psi} \, D^\alpha z\|_{0,p}^\frac{1}{p} \right)^\frac{1}{p}, \quad p < \infty$$

$$||z||_{k,p,\Psi} = \left( \sum_{|\alpha| = k} |z|_{l,p,\Psi}^p \right)^\frac{1}{p}, \quad p < \infty$$

$$|z|_{k,\infty,\Psi} = \max_{|\alpha| = k} \left( \text{ess sup}_{\Omega} |\sqrt{\Psi} \, D^\alpha z| \right)$$

$$||z||_{k,\infty,\Psi} = \max_{l = 0(1) k} |z|_{l,\infty,\Psi}$$

$$\|z\|_{\Psi} = \varepsilon^{1/2} |z|_{1,2,\Psi} + |z|_{0,2,\Psi}$$

and the norm

$$\|z\|_{\varepsilon} = \varepsilon^\frac{1}{2} |z|_{1,2} + \|z\|_{0,2}.$$

In the following let $I_h$ be an operator $C(\bar{\Omega}) \to V_h$, and for all $p \geq 1$ we define $p'$ by $\frac{1}{p} + \frac{1}{p'} = 1$.

Let us note some lemmata needed below.
LEMMA 1: For all \( z \in W^{k,p}(\Omega) \), \( k \geq 0 \), \( 2 \leq p \leq \infty \) is
\[
|z|_{k,p',\psi} \leq |z|_{k,2,\psi} \left( \frac{1}{2} - \frac{1}{p} \right). 
\]

Proof: It is sufficient to derive the result for \( k = 0 \).

The proof is a simple consequence of Hölder's inequality.
We have (for \( p > 2 \))
\[
|z|_{0,p',\psi} = \left( \int_{\text{supp}(\psi)} |\sqrt{\psi} z|^{p'} \cdot 1 \, dx \right)^{\frac{1}{p'}} 
\leq \left( \int_{\Omega} |\sqrt{\psi} z|^{p'} \cdot \frac{1}{2} \cdot \left( \int_{\text{supp}(\psi)} 1 \cdot dx \right)^{\frac{2}{p'}} \cdot \frac{1}{p'} \right)^{\frac{2}{p'}} 
\leq |z|_{0,2,\psi} \left( \frac{1}{2} - \frac{1}{p} \right). \]

LEMMA 2: For all \( 1 \leq q < \infty \) there is a constant \( C_1 < \infty \) s.t.
for all \( \beta, \delta \in \mathbb{R}^1 \) with \( \beta > \delta \) and for all \( z_h \in V_h \) it holds:
\[
C_1 \|z_{h,\delta}\|_{0,q} \geq (\beta - \delta) \left( \text{meas supp } z_{h,\delta} \right)^q. 
\]

Proof: See [1]. \( \square \)

LEMMA 3: For all \( 2 < q < \infty \) there is a constant \( C_2 < \infty \) s.t.
for all \( z_h \in V_{0h} \) it holds:
\[
\|z_h\|_{0,q} \leq C_2 \min \left( h^{\frac{q}{2} - 1}, \epsilon^{\frac{q}{2} - \frac{1}{2}} \right) \|z_h\|_{\varepsilon}. 
\]

Proof: From Sobolev’s imbedding theorem there follows the estimate
\[
\|z_h\|_{0,q} \leq C \|z_h\|_{1,2} \leq C \epsilon^{-\frac{1}{2}} \|z_h\|_{\varepsilon}. 
\]

The estimate \( \|z_h\|_{0,q} \leq C h^{\frac{q}{2} - 1} \|z_h\|_{0,2} \) can be derived from
\[
\|z_h\|_{0,q} \leq C \left( \sum_i |z_h(P_i)|^q \text{ meas } D_i \right)^{\frac{1}{q}} \leq C h^{\frac{q}{2}} \left( \sum_i |z_h(P_i)|^q \right)^{\frac{1}{q}} 
\]
and
\[
\|z_h\|_{0,2} \leq C \left( \sum_i |z_h(P_i)|^2 \text{ meas } D_i \right)^{\frac{1}{2}} \geq C h \left( \sum_i |z_h(P_i)|^2 \right)^{\frac{1}{2}}. \]

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The following generalization of a lemma due to Stampacchia is the basis of our estimates.

**Lemma 4:** Let $\zeta$ be a real-valued function that is defined on the measurable set $M \subset \mathbb{R}^1$, monotone non-increasing and non-negative.

Let $\infty > \zeta_0 \equiv \zeta(\delta) \ \forall \delta \in M$, $\delta_0 = \inf \{\delta | \delta \in M\}$.

Let there exist constants $C_3, q, \nu$ with $0 < q, C_3 < \infty, 1 < \nu < \infty$ s.t. for all $\beta, \delta \in M$ with $\beta > \delta$ the inequality

$$
\zeta(\beta) \leq \frac{C_3}{(\beta - \delta)^q} (\zeta(\delta))^\nu
$$

(3.1)

holds.

Then there exists a constant $C_4 < \infty$ (depending only on $q, \nu, \zeta_0$) s.t. $\zeta(\gamma) = 0$ for all $\gamma \in M$ with $\text{meas}(M \cap (\delta_0, \gamma)) \geq C_4 C_3^q$.

**Proof:** Let $\gamma \in M$ arbitrary with $\zeta(\gamma) = \eta > 0$.

For all $l \geq 1$ we carry out the following construction.

For given $\delta_1 \in M$ let

$$
\alpha_l = 2^q C_3^q (\zeta(\delta_l))^{\nu-1},
$$

$$
\delta_{l+1}^* = \inf \{\delta \in M | \delta \geq \delta_l + \alpha_l\},
$$

$$
\delta_{l+1} \in M \ \text{s.t.} \ \delta_{l+1}^* \leq \delta_{l+1} \leq \delta_{l+1}^* + \vartheta \cdot 2^{-(l+1)}
$$

with a certain positive constant $\vartheta$.

$\delta_1 \in M$ is chosen s.t. $\delta_1 \leq \delta_0 + \frac{\vartheta}{2}$.

Then it holds

$$
\zeta(\delta_{l+1}) \leq \frac{C_3}{2 C_3 (\zeta(\delta_l))^{\nu-1}} (\zeta(\delta_l))^\nu = \frac{1}{2} \zeta(\delta_l) \leq \cdots \leq \frac{1}{2^l} \zeta(\delta_1) \leq \frac{1}{2^l} \zeta_0.
$$

Hence there is a $k = k(\eta) < \infty$ with $\zeta(\delta_k) < \eta$.

Furthermore, we have

$$
\text{meas}(M \cap (\delta_0, \delta_k)) \leq \sum_{j=1}^{k-1} (\alpha_j + \vartheta \cdot 2^{-(j+1)}) + \frac{\vartheta}{2}
$$

$$
\leq \sum_{j=1}^{k-1} \frac{1}{2^q C_3^q} \left( \frac{1}{2^{j-1}} \zeta_0 \right)^{\nu-1} + \vartheta \sum_{j=1}^{k} \left( \frac{1}{2} \right)^j
$$

$$
\leq (2 C_3)^q (\zeta_0)^{\nu-1} q \left( 1 - \left( \frac{1}{2} \right)^{\nu-1} q (k-1) \right) \left( 1 - \left( \frac{1}{2} \right)^q \right)^{-1} + \vartheta.
$$
Thus for sufficiently small $\delta$ it holds
\[
\text{meas } (M \cap (\delta_0, \delta_k)) < C_3^q \xi_0^q 2^q \left( 2^{q-1} \frac{1}{\xi_0} \right)^{-1}.
\]

Hence, choosing $C_4 = 2^q \left( 2^{q-1} \frac{1}{\xi_0} \right)^{-1}$, we obtain
\[
\text{meas } (M \cap (\delta_0, \delta_k)) < C_4 C_3^q.
\]

By virtue of $\zeta(\delta_k) < \eta = \zeta(\gamma)$ and the monotony of $\zeta$ it follows $\gamma < \delta_k$ which proves the assertion. \qed

The main idea in deriving the local estimates of the error $w_h$ will be proving the assumptions of lemma 4 for $\zeta(\delta) = \text{meas } (\text{supp } w_{h, \delta} \cap \Omega')$ for a certain $\Omega' \subset \Omega$ with $\delta_0 = 0$ and small $C_3$ because then on $\Omega'$ the estimate $w_h \leq C_4 C_3^q$ follows. Now the general formulation of our local estimates.

**Theorem 1**: Let $\mu_{\max} = C_5 |\ln h|$ with a sufficiently large $C_5 < \infty$ (for exact definition of $C_5$ see lemma 5).
Assume the existence of a set of "cut-off functions" $\Psi_\mu$, $\mu = 1(1) \mu_{\max}$, and of domains $\Omega_\mu$, $\mu = 0(1) \mu_{\max}$, s.t.

\[
\Omega \supset \Omega_0 \supset \Omega_1 \supset \cdots \supset \Omega_{\mu_{\max}}
\]

\[
\Psi_\mu = 0 \quad \text{on} \quad \Omega \setminus \Omega_{\mu - 1}
\]

\[
\Psi_\mu \geq 1 \quad \text{on} \quad \Omega_\mu
\]

\[
\Psi_{\mu_{\max}} \equiv 0,
\]

$\Psi_\mu \in C(\overline{\Omega})$.

For a certain bilinearform $l_h(\cdot, \cdot)$ on $V_h \times V_h$ assume for all $z_h \in V_{0h}$, $\mu = 1(1) \mu_{\max}$

(H3) $l_h(z_{h, \delta}, I_h(\Psi_\mu z_{h, \delta})) \leq l_h(z_h, I_h(\Psi_\mu z_{h, \delta}))$

(H4) ($\Psi_\mu$-weighted pseudo-ellipticity).
There are constants $C_6 \leq \infty$ and $0 \leq K_1 = K_1(\varepsilon, h) \leq \infty$ s.t.

\[
\| z_h \|_{\Psi_\mu} \leq C_6 l_h(z_h, I_h(\Psi_\mu z_h)) + K^2 \| z_h \|_{0,2, \Omega_{\mu - 1}}^2.
\]

(H5) (error inequality).
For a certain element $w_h \in V_{0h}$ ($w_h$ is the element we want to estimate, e.g. $w_h = u_h - I_h u$) and a certain $2 < p \leq \infty$ there exist $K_i = K_i(w_h, \varepsilon, h) \leq \infty$, $i = 2, 3, 4$, s.t.

\[
| l_h(w_h, I_h(\Psi_\mu z_h)) | \leq K_2 \| z_h \|_{1,p', \Psi_\mu} + K_3 \| z_h \|_{0,p', \Psi_\mu} + K_4 \| z_h \|_{0,p', \Omega_{\mu - 1}}.
\]
Under these assumptions there holds
\[ \| w_h \|_{0, \infty, \Omega_{\mu_{\max}}} \leq C \left| \ln h \right| \min \left( h^{\frac{2}{q} - 1}, \varepsilon^{-\frac{1}{2}} \right) \times \]
\[ \times \left( C_6 \left( \frac{1}{4} K_2 + K_3 + \kappa K_4 \right) + \left( \frac{1}{\kappa} + K_1 \right) \| w_h \|_{0, \infty, \Omega_0} \right) \]  
(3.2)

\( q \) is a constant with \( q \geq \frac{p}{p - 2} \), \( C \) is proportional to \( C_1 C_2 C_4 C_5 \).
\( C_1 \) and \( C_2 \) are derived from lemmata 2 and 3, \( C_4 \) is derived from lemma 4
for \( v = q \left( \frac{1}{2} - \frac{1}{p} \right) \) and \( \xi_0 = \text{meas} \Omega_0, \kappa \) is an arbitrary positive constant.

For an application of (3.2) the assumptions of theorem 1 with very small \( K_1, K_2, K_3, K_4 \) and \( \text{meas} (\Omega_{\mu - 1} \setminus \Omega_{\mu}) \) are to be shown.

Proof of theorem 1: Let \( \delta \geq 0 \) arbitrary, let \( E(\delta) = \text{supp} w_{h, \delta} \).
Using one after another (H4), (H3), (H5), lemma 1 and the inequality
\[ ab \leq \kappa a^2 + \frac{1}{4 \kappa} b^2 \] which holds for all \( \kappa > 0 \) we obtain
\[ \| w_{h, \delta} \|_{\Psi_{\mu}}^2 \leq C_6 l_h (w_{h, \delta}, l_h (\Psi_{\mu} w_{h, \delta})) + K_2^2 \| w_{h, \delta} \|_{0, 2, \Omega_{\mu - 1}}^2 \]
\[ \leq C_6 l_h (w_{h, \delta}, l_h (\Psi_{\mu} w_{h, \delta})) + K_1^2 \| w_{h, \delta} \|_{0, 2, \Omega_{\mu - 1}}^2 \]
\[ \leq C_6 (K_2 \| w_{h, \delta} \|_{1, 2}, \Psi_{\mu} + K_3 \| w_{h, \delta} \|_{0, 2}, \Psi_{\mu} + K_4 \| w_{h, \delta} \|_{0, 2, \Omega_{\mu - 1}}) \]
\[ + K_2^2 \| w_{h, \delta} \|_{0, 2, \Omega_{\mu - 1}}^2 \]
\[ \leq C_6^2 \left( \text{meas} (E(\delta) \cap \Omega_{\mu - 1}) \right)^{\frac{1}{p} - \frac{1}{p}} (K_2 \| w_{h, \delta} \|_{1, 2, \Psi_{\mu}} + K_3 \| w_{h, \delta} \|_{0, 2}, \Psi_{\mu}) \]
\[ + K_4 \| w_{h, \delta} \|_{0, 2, \Omega_{\mu - 1}} + K_2^2 \| w_{h, \delta} \|_{0, 2, \Omega_{\mu - 1}} \]
\[ \leq C_6^2 \left( \text{meas} (E(\delta) \cap \Omega_{\mu - 1}) \right)^{\frac{1}{p} - \frac{1}{p}} \left( \kappa_1 K_2^2 + \kappa_2 K_3^2 + \frac{1}{2} \kappa^2 K_4^2 \right) \]
\[ + \frac{1}{4} \left( \frac{1}{\kappa_1} \left( \| w_{h, \delta} \|_{1, 2, \Psi_{\mu}}^2 + \| w_{h, \delta} \|_{0, 2, \Psi_{\mu}}^2 \right) \right) \]
\[ + \left( \frac{1}{2 \kappa^2} + K_1^2 \right) \| w_{h, \delta} \|_{0, 2, \Omega_{\mu - 1}}^2 \]

The choice \( \kappa_1 = \frac{1}{2 \epsilon}, \kappa_2 = \frac{1}{2} \) yields
\[ \| w_{h, \delta} \|_{\Psi_{\mu}}^2 \leq C_6 \left( \text{meas} (E(\delta) \cap \Omega_{\mu}) \right)^{\frac{1}{p} - \frac{1}{p}} \left( \epsilon^{-\frac{1}{2}} K_2 + K_3 + \kappa K_4 \right) + \]
\[ + \left( \frac{1}{\kappa} + K_1 \sqrt{2} \right) \| w_{h, \delta} \|_{0, 2, \Omega_{\mu - 1}}^2 \].
From
\[ \| w_{h, \delta} \|_{0, 2, \Omega_{\mu} - 1} \leq C \| w_{h, \delta} \|_{0, \infty, \Omega_{\mu} - 1} (\text{meas } (E(\delta) \cap \Omega_{\mu} - 1))^{\frac{1}{2}} \]
\[ \leq C_7 \| w_{h, \delta} \|_{0, \infty, \Omega_{\mu} - 1} (\text{meas } (E(\delta) \cap \Omega_{\mu} - 1))^{\frac{1}{2} - \frac{1}{p}} \]

\((C_7 \text{ depends on meas } \Omega)\) there follows
\[ \| w_{h, \delta} \|_{\psi_\mu} \leq C_8 (\text{meas } (E(\delta) \cap \Omega_{\mu} - 1))^{\frac{1}{2} - \frac{1}{p}} \]
with \( C_8 = C_6 \left( e^{-\frac{1}{2}} K_2 + K_3 + \kappa K_4 \right) + C_7 \left( \frac{1}{\kappa} + K_1 \sqrt{2} \right) \| w_{h, \delta} \|_{0, \infty, \Omega_{\mu} - 1} \).

By virtue of lemma 2 and lemma 3 we get for all \( \beta > \delta \) and \( q > 2 \)
\[ (\text{meas } (E(\delta) \cap \Omega_{\mu}))^q \leq C_1 \frac{1}{\beta - \delta} \| w_{h, \delta} \|_{0, q, \Omega_{\mu}} \]
\[ \leq \frac{C_1 C_2}{\beta - \delta} \min \left( h^{q - 1}, \varepsilon^{-\frac{1}{2}} \right) \| w_{h, \delta} \|_{0, \Omega_{\mu}} \]
\[ \leq \frac{C_1 C_2}{\beta - \delta} \min \left( h^{q - 1}, \varepsilon^{-\frac{1}{2}} \right) \| w_{h, \delta} \|_{\psi_\mu} \]
\[ \leq \frac{C_9}{\beta - \delta} (\text{meas } (E(\delta) \cap \Omega_{\mu} - 1))^{\frac{1}{2} - \frac{1}{p}} \]  
\[(3.3)\]

with \( C_9 = C_1 C_2 C_8 \min \left( h^{q - 1}, \varepsilon^{-\frac{1}{2}} \right) \).

Now the only problem for an application of lemma 4 with \( \mu = \text{meas } (E(\delta) \cap \Omega_{\mu}) \) lies in the fact that in (3.3) on the right hand side it stands \( \Omega_{\mu} - 1 \) instead of \( \Omega_{\mu} \). To overcome this difficulty we use the following lemma :

**Lemma 5**:
Let \( z_h \in V_h, E(\delta) = \text{supp } z_{h, \delta}, \)
Let \( C_5 = \max \{ h \ln h \}^{-1} \mu_{\max} \) be chosen such that
\[ (\text{meas } \Omega_0) h^{C_3 \ln 2} \leq \min \{ \text{meas } T | T \in \mathcal{T}_h \}. \]

(By virtue of the assumed regularity of the family of triangulations we can find such a \( C_5 < \infty \) independently of \( h \).)
Then we have for all \( \delta \geq 0 \):
If not
(i) there is a \( \mu \in \{ 1, \ldots, \mu_{\max} \} \) s.t.
\[ \text{meas } (E(\delta) \cap \Omega_{\mu}) \geq \frac{1}{2} \text{meas } (E(\delta) \cap \Omega_{\mu} - 1) \]
then
(ii) \( z_{h, \delta} = 0 \) on \( \Omega_{\mu_{\text{max}}} \).

Proof of lemma 5: If not (i) then

\[
\text{meas } \Omega_0 \geq \text{meas } (E(\delta) \cap \Omega_0) > 2 \text{meas } (E(\delta) \cap \Omega_1) > \cdots >
\]

\[
> 2^{\mu_{\text{max}}} \text{meas } (E(\delta) \cap \Omega_{\mu_{\text{max}}}),
\]

hence

\[
\text{meas } (E(\delta) \cap \Omega_{\mu_{\text{max}}}) < 2^{-C_{\delta} \ln h} \text{meas } \Omega_0 = h^{C_{\delta} \ln 2} \text{meas } \Omega_0,
\]

hence

\[
\text{meas } (E(\delta) \cap \Omega_{\mu_{\text{max}}}) = 0.
\]

Now let us turn back to the proof of theorem 1:

Let \( v = q \left( \frac{1}{2} - \frac{1}{p} \right) \) (by virtue of \( q > \frac{p}{p-2} \) is \( v > 1 \)), \( \xi_0 = \text{meas } \Omega_0 \) and let \( C_4 \) be the constant from lemma 4 for these \( q, v \) and \( \xi_0 \).

Let

\[
C_9' = C_9 2^{\frac{1}{2} - \frac{1}{p}}, \quad C_{10} = C_4 C_9'.
\]

If for any \( \delta \leq C_{10}(\mu_{\text{max}} + 1) \) the case (ii) of lemma 5 occurs then there follows

\[
w_h \equiv C_{10}(\mu_{\text{max}} + 1) \quad \text{on} \quad \Omega_{\mu_{\text{max}}}.
\]

Now we assume that we have case (i) of lemma 5 for all \( \delta \in [0, C_{10}(\mu_{\text{max}} + 1)] \).

For all \( \delta \) we denote by \( \mu_\delta \) one of the \( \mu \) which fulfil

\[
\text{meas } (E(\delta) \cap \Omega_{\mu}) \geq \frac{1}{2} \text{meas } (E(\delta) \cap \Omega_{\mu_{\text{max}}}).
\]

Then owing to (3.3) for all \( \beta > \delta \) there holds

\[
\left(\text{meas } (E(\beta) \cap \Omega_{\mu_\delta})\right)^{\frac{1}{2}} \leq \frac{C_9'}{\beta - \delta} \left(\text{meas } (E(\delta) \cap \Omega_{\mu_\delta})\right)^{\frac{1}{2} - \frac{1}{p}}.
\]

The values of \( \delta \) vary in an interval of length \( C_{10}(\mu_{\text{max}} + 1) \), and we have at most \( \mu_{\text{max}} \) different values of \( \mu_\delta \).

Hence there exists a \( \bar{\mu} \in \{1, \ldots, \mu_{\text{max}}\} \) s.t.

\[
\text{meas } M(\bar{\mu}) > C_{10}
\]

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where the notation $M(\mu) = \{ \delta \in [0, C_{10}(\mu_{max} + 1)] | \mu_\delta = \mu \}$ is used.

By virtue of (3.5) the function $\xi(\cdot, \cdot) = \text{meas}(E(\cdot, \cdot) \cap \Omega_\mu)$, restricted to $M(\bar{\mu})$, fulfills all assumptions of lemma 4 (with $\delta_0 = \inf \{ \delta | \delta \in M(\bar{\mu}) \}$, $\xi_0 = \text{meas} \Omega_0$, $\nu = q \left( \frac{1}{2} - \frac{1}{p} \right)$). Hence we have $\xi(\gamma) = 0$ for all $\gamma \in M(\bar{\mu})$ with

$$\text{meas} (M(\bar{\mu}) \cap (\delta_0, \gamma)) \geq C_4 C_5 \gamma \quad (= C_{10})$$

and owing to (3.6) the set of these $\gamma$ is not empty.

Thus there follows on $\Omega_\mu$

$$w_h \leq \gamma \leq C_{10}(\mu_{max} + 1) \quad (3.7)$$

(3.4) and (3.7) imply in both cases of lemma 5 the estimate

$$w_h \leq C_{10}(\mu_{max} + 1) \quad \text{on} \quad \Omega_{\mu_{max}}.$$ 

Because the assumption (H3) is also fulfilled for $-w_h$ instead of $w_h$ we get the same upper bound for $-w_h$. Hence we have proven

$$\|w_h\|_{0, \infty, \Omega_{\mu_{max}}} \leq C_{10}(\mu_{max} + 1).$$

Recalling the construction of $C_{10}$ and $\mu_{max}$ the assertion (3.1) follows. □

4. LOCAL $L^\infty$-ERROR ESTIMATES FOR THE HYBRID UPWIND FEM

Let the interpolation operator $I_h : C(\bar{\Omega}) \to V_h$ be defined by $(I_h z)(P_i) = z(P_i)$ for all nodes $P_i$.

$I_h$ is again the bilinearform constructed in Section 2.

Let $u, u_h$ be solutions of $(P), (P_h)$.

Applying theorem 1 we want to derive local estimates for $w_h = u_h - I_h u$ of the form $\|w_h\|_{0, \infty, \Omega'} \leq C(h^a + \varepsilon^B)$, $\Omega' \subset \Omega$. To this end we have to check the assumptions of theorem 1 with small $K_1, K_2, K_3, K_4$. Since $K_2, K_3$ and $K_4$ will prove themselves to be dependent on $W^{2,p}(\Omega_{\mu_{-1}})$-norms of $u$ we have to choose $\Omega_0$ such that boundary layers are excluded.

Of course $\Omega_{\mu_{max}}$ should be so large as possible. At least, to ensure that the numerical method « recognizes » boundary layers, we should fulfil

$$\sup \{ \text{dist} (x, \Omega_{\mu_{max}}) | x \in \Omega_0 \} \to 0 \quad \text{for} \quad h, \varepsilon \to 0.$$ 

For this property very special construction of the cut-off functions $\Psi_\mu$ is needed (the main problem is ensuring the $\Psi_\mu$-weighted pseudo-ellipticity (H4) with small $K_1$).
The $\Psi_\mu$, we use here are very similar to the cut-off function constructed by Nävert [4]. In [4] the cut-off technique is applied to the streamline diffusion method. There local estimates in energy norms are derived. The method we here propose for deriving $L^\infty$-estimates is, however, not applicable to the streamline diffusion method because the discrete bilinearform does not become of non-negative type, i.e. (H1) cannot be reached. For more special information about the streamline diffusion method see [4].

The hybrid upwind FEM gives the possibility to ensure (H1) and (H2). However, for (H2) special type of triangulation in the neighbourhood of the boundary layer is needed.

**Definition 1**: The triangulation is called of parallelogram type if it has arisen from the cut of three groups of parallels where the distance between two adjoining parallels is within each group constant.

![Figure 2. — Triangulation of parallelogram type.](image)

We use the notations

- $n(x)$ - outer normal vector to $\partial \Omega$ (there where it exists)
- $\Gamma^- = \{ x \in \partial \Omega \mid b(x) n(x) < 0 \}$.

With a fixed constant $d > 0$ ($d$ can be chosen small) we set

$$\Gamma^+ = \{ x \in \partial \Omega \mid b(x) n(x) \geq d \}$$
$$\Gamma^0 = \{ x \in \partial \Omega \mid 0 \leq b(x) n(x) < d \}.$$

Analogously we define e.g. $\Gamma^-\mu$, $\Gamma^+\mu$, $\Gamma^0\mu$.

Let $\sigma = \max (\varepsilon, h)$.

Now let $\Omega_\mu$ be a domain with boundary $\Gamma_\mu$ that fulfils the following assumption:

(H6)

(i) $\Omega_\mu \subset \Omega$, $\Gamma_\mu \subset \mathcal{C}^\infty$ piecewise, $\Gamma_\mu$ local Lipschitz-continuous.
(ii) $\Gamma^-\mu \subset \Gamma^-\mu$.
(iii) $|b| \geq C > 0$ in $\Omega_\mu$.
(iv) $|bn| \geq C > 0$ on $\Gamma_-\mu$.
(v) No characteristics of the reduced equation which start from \( \Gamma^0_\mu \) are contained in \( \Omega_\mu \).

(vi) The angles at edges of \( \Gamma^0_\mu \cup \Gamma^+_\mu \) are less than \( \pi \).

We construct our cut-off function in the following way. We begin with a one-dimensional cut-off function \( \Phi \) which is characterized by \( \Phi(t) = 0 \) for \( t \leq 0 \), \( \Phi(t) = 1 \) for \( t > C\sigma |\ln \sigma| \), and by an exponential behaviour for small positive \( t \).

![Figure 3.](image)

In figure 3 \( S, \gamma \) are certain positive constants; the value of \( t_0 \) is not essential (we can assume it as small in comparison with \( \sigma \)); essential is the behaviour of \( \Phi(t) \) in \( (t_0, t_0 + S\gamma\sigma|\ln \sigma|) \), there is

\[
\Phi(t) = \frac{1}{2} \exp \left( \frac{t - S\gamma\sigma|\ln \sigma| - t_0}{\gamma\sigma} \right).
\]

For \( t \geq t_0 + S\gamma\sigma|\ln \sigma| \) we can choose

\[
\Phi(t) = 1 - \Phi \left( 2t_0 + 2S\gamma\sigma|\ln \sigma| - t \right).
\]

Now we divide \( \Gamma^+_\mu \cup \Gamma^0_\mu \) into their smooth parts.

For such a smooth part \( \Gamma^k_\mu \) of \( \Gamma^0_\mu \) we define \( \Psi^k \) by \( \Psi^k(x) = \Phi(\text{dist} (x, \Gamma^k_\mu)) \) for \( x \in \Omega_\mu \) and \( \Psi^k(x) = 0 \) for \( x \in \Omega \setminus \Omega_\mu \).

For a smooth part \( \Gamma^k_\mu \) of \( \Gamma^0_\mu \) the construction of \( \Psi^k \) is similar; but we replace \( \sigma \) by \( \sqrt{\sigma} \) and \( \text{dist} (x, \Gamma^k_\mu) \) by \( \frac{1}{K} \hat{i}(x) \) where \( \hat{i}(x) = \text{dist} (\chi^- (x), \Gamma^k_\mu) \), \( \chi(x) \) is the characteristic through \( x \), \( \chi^- (x) \) is the part of \( \chi(x) \) from \( \Gamma^- \) to \( x \) and \( K \) is defined such that in a neighbourhood of \( \Gamma^k_\mu \) we have \( |\hat{i}(x) - \hat{i}(y)| \leq K|x - y| \), \( \hat{i}(x) \geq K \text{dist} (x, \Gamma^k_\mu) \). Finally we define our cut-
off function $\Psi_{\mu+1}$ by $\hat{\Psi}_{\mu+1} = \prod_k \Psi_k$ and $\Psi_{\mu+1} = I_h \hat{\Psi}_{\mu+1}$ (the use of $\Psi_{\mu+1}$ instead of $\hat{\Psi}_{\mu+1}$ is not essential and has only technical reasons).

For this $\Psi_{\mu+1}$ the following assumptions are valid:

**Lemma 6**: There is a constant $C < \infty$ s.t.:

- For all $\Omega_\mu'$ with
  - (i) $\Omega_\mu' \subset \Omega_\mu$,
  - (ii) $\text{dist} \ (\Omega_\mu', \Gamma_\mu^0) \geq C \sigma |\ln \sigma|$
  - (iii) $\text{dist} \ (\Omega_\mu', \Gamma_\mu^+ ) \geq C \sigma |\ln \sigma|

there holds $\Psi_{\mu+1} \equiv 1$ on $\Omega_\mu'$.

**Lemma 7** (Ensuring (H3)): Under (H1) there holds for all $z_h \in V_{0,h}$, $\delta \in R$

$$l_h(z_h, I_h(\Psi_\mu z_h, \delta)) = l_h(z_h, I_h(\Psi_\mu z_h, \delta)).$$

**Lemma 8** (Ensuring (H4)): Let $\Omega_{\mu-1}$ fulfil (H6).

Under (H2) with $\alpha_0 > 0$ and (2.3) there is a constant $C < \infty$ s.t. for sufficiently small $\sigma$, sufficiently large $\gamma, S$ and for a triangulation of $\Omega \cap \{ x | \Psi_\mu (x) < 1 \}$ which is of parallelogram type there holds for all $z_h \in V_{0,h}$, for all $\varepsilon, h$

$$\| z_h \|_V^2 \leq C \left( l_h(z_h, I_h(\Psi_\mu z_h)) + \sigma^{\frac{S+1}{2}} h^{-2} \| z_h \|_{0,2,\Omega_{\mu-1}}^2 \right).$$

**Lemma 9** (Ensuring (H5)): Let $1 < p \leq \infty$; let $u \in W^{2,p}(\Omega)$, $u_h \in V_{0,h}$ be solutions of $(P)$, $(P_h)$. We denote again $w_h = u_h - I_h u$. 

*Figure 4.*

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Then there is a constant $C < \infty$ s.t. for sufficiently small $\sigma$ and sufficiently large $\gamma$, $S$ there holds for all $z_h \in V_0$, for all $\varepsilon, h \approx \sigma$

\[ l_h(w_h, I_h(\Psi \mu z_h)) \leq C h \| u \|_{2, p, \Omega_{\mu-1}} \left( \| z_h \|_{1, p', \Psi_{\mu}} + \frac{\gamma}{2} h^{-1} \| z_h \|_{0, p', \Omega_{\mu-1}} \right). \]

If, beyond this, $\Omega_\mu$ is contained in a part of $\Omega$ the triangulation of which is of parallelogram type then we have

\[ l_h(w_h, I_h(\Psi \mu z_h)) \leq C h \| u \|_{2, p, \Omega_{\mu-1}} \times \]

\[ \times \left( \varepsilon \| z_h \|_{1, p', \Psi_{\mu}} + \| z_h \|_{0, p', \Psi_{\mu}} + \frac{\gamma}{2} h^{-1} \| z_h \|_{0, p', \Omega_{\mu-1}} \right). \]

Some remarks on the proofs of lemmata 6-9 (for detailed proofs see [6]) ; Lemma 6 follows from the construction of $\Psi_\mu + 1$. Lemma 7 even holds for all $\Psi_\mu \in C(\Omega)$, $\Psi_\mu = 0$. The straightforward proof is based on the fact that the discrete matrix $L_h$ is off-diagonal non-positive and that owing to $c \geq 0$ for all $i \sum_j (L_h)_{ij} \geq 0$. More extensive and rather technical are the proofs of lemma 9 and especially of lemma 8, the main steps of these proofs can be found in section 5. The constants in lemma 8, 9 depend on the $C^{0+1}(\Omega_{\mu-1})$-norms of $b$ and $c$.

Now we are able to formulate our local $L^\infty$ error estimate for the hybrid upwind FEM :

**THEOREM 2 :** If $2 < p \leq \infty$, $\Omega'' \subset \Omega$ fulfils (H6) (i)-(v), $u \in H^1_0(\Omega) \cap \mathcal{W}^{2,p}(\Omega)$, $u_h \in V_0$ solutions of $(P)$, $(P_h)$, $(H1)$, $(H2)$ with $\alpha_0 > 0$, $\sigma = \max (\varepsilon, h)$ sufficiently small, $q > \frac{2p}{p-2}$, then there are constants $C_{11}$, $C_{12} < \infty$ independent of $h$, $\varepsilon$ s.t. for all $\Omega' \subset \Omega''$ with

\[
\text{dist} (\Omega', (\Gamma'')^0) \geq C_{11} \sqrt{\sigma \ln \sigma \ln h}, \quad \text{dist} (\Omega', (\Gamma'')^+) \geq C_{11} \sigma \ln \sigma \ln h
\]

there holds :

(i) if $\Omega'' \setminus \Omega'$ is contained in a domain the triangulation of which is of parallelogram type then

\[
\| u_h - I_h u \|_{0, \infty, \Omega'} \leq C_{12} \ln h \left\{ \min \left( h^{\frac{q}{2}} \varepsilon^{-\frac{1}{2}}, h \varepsilon^{-1} \right) \| u \|_{2, p, \Omega''} + \right.
\]

\[ + \min \left( h^{\frac{q-1}{2}}, \varepsilon^{-\frac{1}{2}} \right) \sigma \frac{1}{2} h^{-1} \| u_h - I_h u \|_{0, \infty, \Omega''} \right\}.
\]
(ii) If $\Omega''$ is contained in a domain the triangulation of which is of parallelogram type then

$$\|u_h - I_h u\|_{0, \infty, \Omega'} \leq C_{12} |\ln h| \left( \min \left( h^\frac{2}{q} \epsilon^{-\frac{1}{2}}, h^{\frac{1}{p}} \right) \|u\|_{2, p, \Omega'} + \min \left( h^{\frac{q}{q-1}} \epsilon^{-\frac{1}{2}}, h^{\frac{1}{p}} \right) \sigma^{\frac{S}{2}} \|u_h - I_h u\|_{0, \infty, \Omega'} \right).$$

In both cases $C_{12}$ depends on the $C^{0,1}(\bar{\Omega}'')$-norms of $b$ and $c$. The positive constant $S$ can be made arbitrary large if we make $C_{11}$ large.

**Proof:** We can construct domains $\Omega_{\mu}$ with the following properties:

$$\Omega'' = \Omega_0 \supset \Omega_1 \supset \cdots \supset \Omega_{\mu_{\text{max}}} \supset \Omega'.$$

all $\Omega_{\mu}$, $\mu = 0, 1, \ldots, \mu_{\text{max}} - 1$ fulfil (H6) (i)-(v) ; for the above constructed $\Psi_{\mu + 1}$ it holds $\Psi_{\mu + 1} = 0$ on $\Omega_{\mu} \setminus \Omega_{\mu+1}$, $\Psi_{\mu + 1} = 1$ on $\Omega_{\mu + 1}$ ; the constants in lemma 6, 8, 9 are independent of $\mu$ ; $\mu_{\text{max}}$ is the same as in theorem 1.

Further, covering $\Omega'$ by a finite number of suitable domains which fulfil (H6) (i)-(vi) we may assume that all $\Omega_{\mu}$ fulfil (H6) (vi) too.

By virtue of the lemmata 6-9 then all assumptions of theorem 1 are fulfilled, where $C_{11}$ is the constant $C$ from lemma 8,

$$K_1 = O\left( \sigma^{\frac{S}{4}} h^{-1} \right), \quad K_2 = O(h \|u\|_{2, p, \Omega'}), \quad K_3 = O\left( \sigma^{\frac{S}{2}} \|u\|_{2, p, \Omega'} \right), \quad K_4 = O\left( \sigma^{\frac{S}{4}} h^{-1} \|u\|_{2, p, \Omega'} \right),$$

$$K_2 = O(h \|u\|_{2, p, \Omega'})$$

for case (i), $K_2 = O(h \epsilon \|u\|_{2, p, \Omega'})$ for case (ii), $C_4$ depends on $p$, $q$, meas $\Omega''$.

So we can apply theorem 1. Choosing in (3.2) $\kappa = \sigma^{\frac{S}{4}} \frac{1}{2} h$ we obtain the assertion.

Owing to the construction of the $\Psi_{\mu}$ there follows that $S$ is proportional to $C_{11}$ and so we can make $S$ large together with $C_{11}$. $\square$

**Remark 1:** What have we achieved by theorem 2 ?

To use the local estimates we first should ensure that dist $\Omega' \setminus (\Gamma'' \cup (\Gamma'')^*) \rightarrow 0$ for $\epsilon$, $h \rightarrow 0$. This can be guaranteed if we assume : $\epsilon^l \leq h$ with a certain positive constant $l$. The assumption $\epsilon^l \leq h$ also is sufficient to ensure that

$$g = \min \left( h^\frac{2}{q-1} \epsilon^{-\frac{1}{2}}, h^{\frac{1}{p}} \right) \sigma^{\frac{S}{4} + 1} h^{-1} \|u_h - I_h u\|_{0, \infty, \Omega'}$$

is small :

Indeed, ensuring that $h^\beta \|u_h - I_h u\|_{0, \infty, \Omega'}$ is bounded with a certain (e.g., positive !) constant $\beta$ and making $C_{11}$ (and with that also $S$) sufficiently large, for each desired positive $K$ we can obtain $g = O(\epsilon^K)$.
Finally, $\|u\|_{2,p,\Omega}$ has to be bounded. For this, we mention following known fact: under certain assumptions, where the most essential of which are $\Omega^\prime \subset \Omega^\prime \subset \Omega$, $\text{dist}(\Omega^\prime, (\Gamma^{\prime \prime})^0) \geq C \sqrt{\varepsilon} \ln \varepsilon$, $\text{dist}(\Omega^\prime, (\Gamma^{\prime \prime})^0) \geq C \varepsilon \ln \varepsilon$ and (H6) for $\Omega^\prime$, for sufficiently smooth data the norms $\|u\|_{k,2,\Omega}$ and with it (if $k > 3$) also $\|u\|_{2,\infty,\Omega}$ are bounded uniformly with respect to $\varepsilon$ (for exact assumptions, see e.g. [4].

Thus we can, somewhat simplified, say:

If $\varepsilon^l \leq h$, if $\Omega^\prime \subset \Omega$ is a domain where boundary layers and additional «numerical boundary layers» are excluded and if the triangulation is of parallelogram type then the local error $\|u_h - I_h u\|_{0,\infty,\Omega}$ is of order $h^\kappa$ with an arbitrary $\kappa < 1$.

If the triangulation is only in the neighbourhood of the boundary layer of parallelogram type we get at least the order $h^\kappa \varepsilon^{-\frac{1}{2}}$.

For a typical situation see following figure:

\[ \text{Figure 5.} \]

Remark 2: In [6] the results of theorem 2 are written for weakly coupled systems

\[- \varepsilon \Delta u^i + b^i(x) \nabla u^i + \sum_{j=1}^{m} c^{ij}(x) u^j = f^i(x), \quad i = 1(1) m.\]

For it, the most essential changes are:

Let $c$ be the $m \times m$-Matrix with the elements $c^{ij}$.

Assumptions (H1), (H2) are replaced by their (natural) generalizations:

(H1') $c$ is of non-negative type, that means

(i) $c^{ij} \leq 0$ for $i \neq j$
(ii) there is a vector \( s \in R^m, s \geq 0 \) s.t.
\[
\sum_i c^{ij}(x) s^j \geq 0 \quad \forall i = 1 \ldots m.
\]

(H2') The smallest eigenvalue of \( c + c^T \text{diag} \left( \text{div} \ b^i \right) \) is positive.

More restrictive is the following assumption:

(H7) The directions of the \( b^i \) do not differ « too strong » from each other: for all \( i = 1 \ldots m \) the \((\Gamma")^-\)-boundary corresponding to the \( b^i \) is the same (couplings between \((\Gamma")^{-}\) and \((\Gamma")^{+}\)-boundaries may appear).

5. PROOFS OF LEMMA 8 AND 9

First some notations and some properties needed below.

We denote for all \( i \)
\[
\Psi_i = \Psi_\mu(P_i), \\
z_i = z_h(P_i) \quad \text{for all} \quad z_h \in V_h.
\]

\( P_{iT}, i = 1, 2, 3 \) - edges of the triangle \( T \in \mathcal{T}_h \).
\( \lambda_i T, \) barycentric coordinates corresponding to the \( P_{iT}. \)
\( \varphi_i, \) basis functions of the \( V_h \) with \( \varphi_i(P_j) = \delta_{ij}. \)

\( E_i = \text{supp} \varphi_i \)
\( l_{ij} = \overline{P_i P_j} \)
\( \gamma_{ij} = \text{meas } \Gamma_{ij} \)

\( j^* \) (for triangulations of parallelogram type, for a given interior node \( P_i \) and \( j \in \Lambda_i \) is chosen such that \( j^* \in \Lambda_i \) and that \( P_{j^*}, P_i, P_j \) lie on the same straight line.

\( T_{ij}, \) one triangle \( \in \mathcal{T}_h \) containing \( P_i, P_j \) (for our purpose it is not essential which of these two triangles we choose).

\[
\Omega(> \sigma) = \Omega_{\mu-1} \cap \bigcup \left\{ T \in \mathcal{T}_h \left| \Psi_\mu(x) > \frac{\sigma}{2} \quad \forall x \in T \right. \right\}
\]
\[
\Omega(\leq \sigma) = \Omega_{\mu-1} \setminus \Omega(> \sigma).
\]

We use the following properties:

\[
\Psi_\mu(x) \leq C \sigma^2 \quad \text{for all} \quad x \in \Omega(\leq \sigma). \tag{5.1}
\]

For triangulations of parallelogram type for all vectors \( y \in R^2 \), all interior nodes \( P_i \) it holds
\[
\sum_{j \in \Lambda_i} n_{ij} \gamma_{ij}(l_{ij}, y) = 2 \text{meas } D_i y. \tag{5.2}
\]
For all $1 \leq q \leq \infty$, sufficiently large $\gamma$, $S$ there hold for all $z_h \in V_h$

\[ |I_h(\Psi \mu z_h)|_{0,q} \leq C \|z_h\|_{0,q,\Psi \mu} + C\sigma^{\frac{S}{2}} \|z_h\|_{0,q,\Omega_{\mu-1}} \]  

(5.3)

\[ |I_h(\Psi \mu z_h)|_{1,q} \leq C \left( \frac{1}{\gamma \sigma} \|z_h\|_{0,q,\Psi \mu} + |z_h|_{1,q,\Psi \mu} + \sigma^{\frac{S}{2}} h^{-2} \|z_h\|_{0,q,\Omega_{\mu-1}} \right) \]  

(5.4)

\[ |z_h|_{0,q,\Psi} \leq C |z_h|_{0,q,\Psi} + C\sigma^{\frac{S}{2}} |z_h|_{0,q,\Omega_{\mu-1}} \]  

(5.5)

\[ |z_h|_{k,q,\Psi} \leq C |z_h|_{k,q,\Psi} + C\sigma^{\frac{S}{2}} |z_h|_{0,q,\Omega_{\mu-1}} \]  

(5.6)

(5.1) follows from the construction of $\Psi \mu$ by use of Taylor’s formula, (5.2) is an elementary calculation and also (5.3)-(5.6) follow rather straightforward if the splitting $\Omega_{\mu-1} = \Omega(> \sigma) \cup \Omega(\leq \sigma)$ is used.

**Proof of lemma 8:**

A) By virtue of (5.1) it follows that both $|I_h(z_h, I_h(\Psi \mu z_h))|_{\Omega(\leq \sigma)}$ (i.e. the integral defining $I_h$ restricted to $\Omega(\leq \sigma)$) and $\|z_h\|_{\Psi \mu, \Omega(\leq \sigma)}$ can be estimated by $C\sigma^{\frac{S}{2} + 1} h^{-2} \|z_h\|_{0,2,\Omega_{\mu-1}}$.

Thus in the remaining investigations we may restrict us to $\Omega(> \sigma)$, i.e. in the following we will assume $\Psi \mu(x) > \sigma^{\frac{S}{2}}$.

B) Elementary transformations lead to

\[ b_h(z_h, I_h(\Psi \mu z_h)) = T_1 + T_2 + T_3 \]

where

\[ T_1 = -\frac{1}{2} \sum_i z_i^2 \Psi_i \sum_{j \in \Omega} \beta_{ij} \]

\[ T_2 = \frac{1}{4} \sum_{i,j} (\sqrt{\Psi_i} z_i - \sqrt{\Psi_j} z_j)^2 \times \]

\[ \times \left[ |\beta_{ij}| - \beta_{ij} \left( \frac{1}{\sqrt{\Psi_j}} - \frac{1}{\sqrt{\Psi_i}} \right) (1 - \lambda_{ij}) \sqrt{\Psi_i} + \lambda_{ij} \sqrt{\Psi_j} \right] \]

\[ T_3 = \frac{1}{4} \sum_{i,j} \beta_{ij} (z_i^2 \Psi_i + z_j^2 \Psi_j) \times \]

\[ \left( \frac{1}{\sqrt{\Psi_j}} - \frac{1}{\sqrt{\Psi_i}} \right) (1 - \lambda_{ij}) \sqrt{\Psi_i} + \lambda_{ij} \sqrt{\Psi_j} \right) . \]
C) We have for sufficiently small $h$

$$T_1 + (\bar{c} \bar{z}_h, \Psi_\mu \bar{z}_h) \geq \frac{\alpha_0}{2} \|z_h\|^2_{0,2, \Psi_\mu},$$

(5.7)

since (using (2.3))

$$T_1 + (\bar{c} \bar{z}_h, \Psi_\mu \bar{z}_h) = -\frac{1}{2} \sum_i \bar{z}_i^2 \Psi_i \left( \sum_{j \in \Lambda_t} \beta_{ij} - \int_{D_i} \text{div } b \, dx \right) +$$

$$+ \left( \left( c - \frac{1}{2} \text{div } b \right) \bar{z}_h, \Psi_\mu \bar{z}_h \right) + ((\bar{c} - c) \bar{z}_h, \Psi_\mu \bar{z}_h)$$

$$- Ch (\|b\|_{C^{1+1}} + \|c\|_{C^{0+1}})(\bar{z}_h, \Psi_\mu \bar{z}_h) + \left( \left( c - \frac{1}{2} \text{div } b \right) \bar{z}_h, \Psi_\mu \bar{z}_h \right)$$

(H1) with $\alpha_0 > 0$ now implies (5.7).

D) Estimation of $T_2$ and $T_3$. For the sake of simplicity we restrict us in the first instance to a subdomain $G \subset \Omega_{\mu-1}$ that is away from $\Gamma^0_{\mu-1} \cup \Gamma^+_{\mu-1}$ except of one smooth part $\Gamma$ of $\Gamma^+_{\mu-1}$.

So we have $\Psi_\mu = I_h(\Phi(t))^2$ with $t(x) = \text{dist}(x, \Gamma)$.

By Taylor's formula we get for sufficiently large $\gamma, S$, for all $t_1, t_2 > t_0$,

$$|t_1 - t_2| \leq h$$

$$\frac{\Phi(t_2)}{\Phi(t_1)} - 1 = \left( \frac{t_2 - t_1}{\gamma \sigma} + \delta \right) \Phi_{12}^\star + O(\sigma^5).$$

(5.8)

In formula (5.8) (and in the following) $\delta$ denotes an arbitrary function with $|\delta| \leq C \left| \frac{h}{\gamma \sigma} \right|^2$. For $t_2, t_1 \leq S \gamma \sigma |\ln \sigma| + t_0 + h$ is $\Phi_{12}^\star = 1$, otherwise $\Phi_{12}^\star$ may be an arbitrary value $\frac{1}{\Phi(t)} - 1$ with $|t - t_1| \leq h$ or $|t - t_2| \leq h$.

By use of (5.8) we can estimate

$$\left| \left( \frac{1}{\sqrt{\Psi_j}} - \frac{1}{\sqrt{\Psi_i}} \right) ((1 - \lambda_{ij}) \sqrt{\Psi_i} + \lambda_{ij} \sqrt{\Psi_j}) \right| \leq$$

$$\leq \max \left( \left| 1 - \frac{\sqrt{\Psi_i}}{\Psi_j} \right|, \left| 1 - \frac{\sqrt{\Psi_i}}{\Psi_i} \right| \right) \leq 1,$$

(5.9)

thus $T_2 \approx 0$.

For the investigation of $T_3$ we remark that, by help of (5.8) and with the notation $t_i = t(P_i)$,
By help of (5.10) and (2.3) we can estimate

\[ T_3 = \frac{1}{4} \sum_{i, \psi_i < 1} z_i^2 \psi_i \sum_{j \in \Lambda_i} (b(P_i) n_{ij} \gamma_{ij} + O(h^3)) \left\{ \left( \frac{t_i - t_j}{\gamma \sigma} + \delta \right) \Phi^*_{ij} + O(\sigma^5) \right\} \]

\[ = \frac{1}{4} \sum_{i, \psi_i < 1} z_i^2 \psi_i b(P_i) R_i - \frac{\alpha_0}{4} \| z_h \|_{0, 2, \psi}^2 \]  

(5.11)

where

\[ R_i = \Phi^*_{ij} \sum_{j \in \Lambda_i} n_{ij} \gamma_{ij} \left( \frac{t_i - t_j}{\gamma \sigma} + \delta \right) \]  

(5.12)

\[ \Phi^*_{ij} = 1 \quad \text{for} \quad \Phi(t_i) \leq \frac{1}{2} \quad \text{and} \quad \Phi^*_{ij} = \frac{1}{\Phi(t_i)} - 1 \quad \text{for} \quad \Phi(t_i) > \frac{1}{2}. \]  

(5.13)

Now, for the further treatment of \( R_i \), the parallelogram type of our triangulation is used. From (5.2) with \( y = \nabla t(P_i) \) we get

\[ b(P_i) R_i = b(P_i) \Phi^*_{ij} \left( - \frac{2}{\gamma \sigma} \text{meas } D_i \nabla t(P_i) + h\delta \right) \]  

(5.14)

and thus, by virtue of \( b(P_i) \nabla t(P_i) \leq -C < 0 \), for \( \gamma \) sufficiently large

\[ b(P_i) R_i \geq C \Phi^*_{ij} \frac{h^2}{\gamma \sigma} \quad \text{with} \quad C > 0. \]  

(5.15)

E) Estimation of \( \varepsilon(\nabla z_h, \nabla I_h(\Psi \mu z_h)) \). Elementary calculation give (again for the above introduced subdomain \( G \))

\[ \varepsilon(\nabla z_h, \nabla I_h(\Psi \mu z_h)) - \varepsilon |z_h|_{1, 2, \psi}^2 = T_4 \]

with (denoting \( \Psi_{k+3, T} = \Psi_{k, T} \) for \( k = 1, 2 \))

\[ T_4 = \frac{\varepsilon}{\Phi} \sum_{i} \sum_{T \in E_i} \left\{ \left( \text{meas } T \right) \Psi_i \nabla z_h \right\}_T \sum_{j=1}^{3} z_{jT} \nabla \lambda_{jT} \times \right\} \times \left( \frac{\Psi_j - \Psi_j + 1, T}{\Psi_i} + \frac{\Psi_j - \Psi_j + 2, T}{\Psi_i} \right) \]  

(5.16)
From (5.8) we get for all $P_i, P_k \in T \subset E_i$

$$\left| \frac{\Psi_i - \Psi_k}{\Psi_i} \right| \leq C \Phi_i^{\ast} \frac{h}{\gamma \sigma} + C \sigma^s,$$

where $\Phi_i^{\ast}$ is an arbitrary of the values $\Phi_i^{\ast}, \Phi_j^{\ast}, \Phi_k^{\ast}$. Thus it follows

$$| T_4 | \leq C T_5 + C \epsilon \sigma^s h^{-2} \| z_h \|_{0, 2, \Psi}^2,$$  \hspace{1cm} (5.17)

where

$$T_5 = \frac{\epsilon h^2}{\gamma \sigma} \sum_i \Psi_i \Phi_i^{\ast} \sum_{T \in E_i} | \nabla z_h | | z_h | | z_h | \infty, \cal T.$$

It holds for an arbitrary $\kappa > 0$

$$T_5 \leq C \left( \frac{\epsilon \kappa}{\gamma \sigma} \| \nabla z_h \|_{0, 2, \Psi}^2 + \frac{\epsilon h^2}{\gamma \sigma \kappa} \sum_i \Psi_i \Phi_i^{\ast} z_i^2 \right).$$  \hspace{1cm} (5.18)

Setting $\kappa = \sigma \sqrt{\gamma}$ we get

$$T_5 \leq \frac{C}{\sqrt{\gamma}} \| z_h \|_{\Psi^2}^2 + \frac{1}{\sqrt{\gamma}} \frac{h^2}{\gamma \sigma} \sum_i \Psi_i \Phi_i^{\ast} z_i^2.$$  \hspace{1cm} (5.19)

F) Lemma 8 now follows — on the subdomain $G$ — by combination of (5.7), (5.9), (5.11), (5.15), (5.19) for sufficiently large $\gamma$, (5.5), (5.6).

G) Now let $G$ be a subdomain of $\Omega_{\mu-1}$ that has a positive distance to $\Gamma_{\mu-1}^0 \cup \Gamma_{\mu-1}^+$ except of one smooth part $\tilde{\Gamma}$ of $\tilde{\Gamma}_{\mu-1}^0$. Then we have the following changes in comparison with D), E): $i(x)$ now denotes the following changes in comparison with D), E): $i(x)$ now denotes $\frac{1}{K} \text{dist} (\chi^- (x), \tilde{\Gamma})$ (see the construction of $\Psi^\mu$).

In the construction of $\Phi$, the definition of $\Phi$ and in the formulas (5.8)-(5.14), (5.17), (5.18) $\sigma$ is replaced by $\sqrt{\sigma}$.

Due to $b(P_i) \nabla i(P_i) \equiv 0$ we get now instead of (5.15) $b(P_i) R_i \equiv - C h | \Phi |$. Furthermore we set in (5.18) (with $\sigma$ replaced by $\sqrt{\sigma}$) $\kappa = \sqrt{\sigma}$ and get instead of (5.19)

$$T_5 \leq \frac{C}{\gamma} \| z_h \|_{\Psi^2}^2.$$  \hspace{1cm} (5.19)

Combination of (5.7), (5.9), (5.11), (5.5), (5.6) and the analoga to (5.15), (5.19) again leads to the assertion of lemma 8.

H) It remains to investigate subdomains of $\Omega_{\mu-1}$ which contain a neighborhood of an edge of $\Gamma_{\mu-1}^0 \cup \Gamma_{\mu-1}^+$. In this case the construction of $\Psi^\mu$ is based on a product of two functions $\Psi_k$.
Using the identity $ab - 1 = (a - 1) + (b - 1) + (a - 1)(b - 1)$ the term $\frac{\sqrt{\Psi_i}}{\sqrt{\Psi_j}} - 1$ becomes a sum of two terms of the form (5.8). Thus $T_3$ and $T_4$ become a sum of two terms of the already investigated forms of $T_3$ and $T_4$. On this way the assertion of lemma 8 follow in the whole domain $\Omega_{\mu - 1}$.

**Proof of lemma 9:**

A) We prove that for all $p > 1$ there is a constant $C = \infty$ s.t. for all $v \in W^{2,p}(\Omega)$, $v_h = I_h v$, $z_h \in V_{0h}$ the following approximation properties are valid:

\[ \| v - v_h \|_{j,p} \leq C h^{2-j} \| v \|_{2,p}, \quad j = 0, 1 \]  
(5.20)
\[ \| v_h - \bar{v}_h \|_{0,p} \leq C h \| v_h \|_{1,p} \]  
(5.21)
\[ \| (c v_h, z_h) - (\tilde{c} \bar{v}_h, \tilde{z}_h) \| \leq C h \| c \|_{C^{0,1}(\Omega)} \| v \|_{2,p} \| z_h \|_{1,p'} \]  
(5.22)
\[ \| (b \nabla v_h, z_h) - b_h (v_h, z_h) \| \leq C h \| b \|_{C^{0,1}(\Omega)} \| v \|_{2,p} \| z_h \|_{1,p'} \]  
(5.23)

Moreover, using special symmetry properties we get for triangulations of parallelogram type

\[ (v_h, \bar{z}_h) = (\bar{v}_h, z_h) \]  
(5.24)
\[ \| (c v_h, z_h) - (\tilde{c} \bar{v}_h, \tilde{z}_h) \| \leq C h \| c \|_{C^{0,1}(\Omega)} \| v \|_{2,p} \| z_h \|_{0,p'} \]  
(5.25)
\[ \| (b \nabla v_h, z_h) - b_h (v_h, z_h) \| \leq C h \| b \|_{C^{0,1}(\Omega)} \| v \|_{2,p} \| z_h \|_{0,p'} \]  
(5.26)

(5.20) is a known interpolation property.

For (5.21) and (with a slight modification) (5.23) see [2] (there lemma 2.1, and 4.6.1).

Using the splitting

\[ (c v_h, z_h) - (\tilde{c} \bar{v}_h, \tilde{z}_h) = (c v_h, z_h - \bar{z}_h) + (c (v_h - \bar{v}_h), \bar{z}_h) + ((c - \tilde{c}) \bar{v}_h, \bar{z}) \]  
(5.27)

(5.22) follows from (5.21) and (5.20).

For basis functions $\phi_i \in V_h$, $\phi_j \in V_{0h}$ and for triangulations of parallelogram type we can easily verify $(\phi_i, \tilde{\phi}_j) = (\tilde{\phi}_i, \phi_j)$. So (5.24) follows.

(5.25) results again from the splitting (5.27) since by virtue of (5.24)

\[ (c v_h, z_h - \bar{z}_h) = (c v_h - I_h (c v_h), z_h) - (c v_h - I_h (c v_h), \bar{z}_h) + (I_h (c v_h) - \bar{c} \bar{v}_h, z_h). \]  

It remains to prove (5.26). This estimate would be follow from Hölder’s inequality if we had shown that for all \( i = 1(1) N \)

\[
| (b \nabla v_h, \varphi_i) - b_h(v_h, \varphi_i) | \leq C h^{3 - \frac{2}{p}} \| v \|_{2, p, E_i} \cdot
\]

(5.28)

For the proof of (5.28) we may assume that \( b = b(P_i) = \text{const.} \) on \( E_i \). Indeed, let \( b \) be variable and \( \hat{b}(x) = b(P_i) \; \forall x \in E_i \) (and let \( \hat{\beta}_{ij}, \hat{\lambda}_{ij} \) and \( \hat{b}_h(\cdot, \cdot) \) be defined correspondingly to \( \hat{b} \)). Then we conclude from Hölder’s inequality

\[
\left| ((b - \hat{b}) \nabla v_h, \varphi_i) \right| \leq C h^{3 - \frac{2}{p}} \| b \|_{C^{0, 1}} \| v_h \|_{1, p, E_i}
\]

and also

\[
\left| (b_h - \hat{b}_h)(v_h, \varphi_i) \right| = \left| \sum_{j \in \Lambda_i} ((\beta_{ij} - \hat{\beta}_{ij})(\lambda_{ij} - 1) + \hat{\beta}_{ij}(\lambda_{ij} - \hat{\lambda}_{ij})) l_{ij} \nabla v_h |_{T_{ij}} \right|
\]

\[
\leq C h^{3 - \frac{2}{p}} \| b \|_{C^{0, 1}} \| v_h \|_{1, p, E_i}
\]

(for the second estimate it was used that for \( \lambda_{ij} \neq \hat{\lambda}_{ij} \) we have \( \text{sgn} \beta_{ij} \neq \text{sgn} \hat{\beta}_{ij} \) and so \( |\hat{\beta}_{ij}| \leq |\beta_{ij} - \hat{\beta}_{ij}| \).

By virtue of \( \| v_h \|_{1, p, E_i} \leq C \| v \|_{1, p, E_i} \) it is thus clear that we in the following may restrict us to the case \( b = b(P_i) = \text{const.} \) on \( E_i \).

First we transform \((b \nabla v_h, \varphi_i)\). There holds

\[
(b \nabla v_h, \varphi_i) = \frac{1}{3} \sum_{T \in E_i} \text{(meas } T \text{)} \cdot b \nabla v_h |_T .
\]

We now set \( y = \frac{1}{6} \sum_{T \in E_i} \nabla v_h |_T \) and get by use of (5.2)

\[
(b \nabla v_h, \varphi_i) = \frac{1}{2} \sum_{j \in \Lambda_i} b n_{ij} \gamma_{ij}(l_{ij} \cdot y)
\]

\[
= \frac{1}{2} \sum_{j \in \Lambda_i} \beta_{ij} l_{ij} (y - \nabla v_h |_{T_{ij}}) - \frac{1}{2} \sum_{j \in \Lambda_i} \beta_{ij}(v_i - v_j) .
\]

(5.29)

Next we transform \( b_h(v_h, \varphi_i) \). By virtue of \( \beta_{ij}^* = - \beta_{ij}, \lambda_{ij}^* = \lambda_{ij} - 1 \) we get

\[
b_h(v_h, \varphi_i) = \frac{1}{2} \sum_{j \in \Lambda_i} \{ \beta_{ij}(\lambda_{ij} - 1)(v_i - v_j) + \beta_{ij}^*(\lambda_{ij}^* - 1)(v_i - v_j^*) \}
\]

\[
= \frac{1}{2} \sum_{j \in \Lambda_i} \beta_{ij} \lambda_{ij}(2 v_i - v_j - v_j^*) - \frac{1}{2} \sum_{j \in \Lambda_i} \beta_{ij}(v_i - v_j) .
\]

(5.30)
For the treatment of the first summands of the right hand sides of (5.29) and (5.30) we will use an estimate of \( (\nabla v_h|_S - \nabla v_h|_T) \) where \( S \) and \( T \) are two triangles \( \subset E_i \) and \( S \) has arisen from \( T \) by a translation by a vector \( l \) (of course \( |l| \leq h \)). To this end we choose a function \( \hat{v} \in C^2(\overline{\Omega}) \) such that 
\[
\| \hat{v} - v \|_{2,p,E_i} \leq h \| v \|_{2,p,E_i}.
\]
Then we get
\[
\| \nabla v_h|_S - \nabla v_h|_T \|_{0,p,T} = \| \nabla v (x + l) - \nabla v (x) \|_{0,p,T} + hO (\| v \|_{2,p,E_i})
= \| \nabla \hat{v} (x + l) - \nabla \hat{v} (x) \|_{0,p,T} + hO (\| v \|_{2,p,E_i}).
\]

From \((5.33)\) and the definition of \( y \) it follows
\[
\| \nabla \hat{v} (x + l) - \nabla \hat{v} (x) \|_{0,p,T} \leq Ch \| \hat{v} \|_{2,p,E_i} \leq Ch \| v \|_{2,p,E_i}.
\]

(For \( p = \infty \) \((5.32)\) can be proven in a similar way.)

\((5.31)\) and \((5.32)\) yield
\[
\| \nabla v_h|_S - \nabla v_h|_T \| \leq Ch^{-\frac{2}{p}} \| \nabla v_h|_S - \nabla v_h|_T \|_{0,p,T}
\]
\[
\leq Ch^{-\frac{2}{p}} \| v \|_{2,p,E_i}.
\]

Now let's turn again to the investigation of (5.29) and (5.30). From (5.33) and the definition of \( y \) it follows
\[
\left| \sum_{j \in \Lambda_i} \beta_{ij} l_{ij} (y - \nabla v_h|_{T_{ij}}) \right| \leq C \| b \|_{\infty} h^{-\frac{2}{p}} \| v \|_{2,p,E_i}
\]
\[ \left| \sum_{i \in A} \beta_{ij} \lambda_{ij} (2 v_i - v_j - v_{j+}) \right| = \left| \sum_{i \in A} \beta_{ij} \lambda_{ij} l_{ij} (\nabla v_h|_{T_{ij}} - \nabla v_h|_{T_{ij}}) \right| \]
\[ \leq C \| b \|_{\infty} h^{-\frac{3}{2}} \| v \|_{2,p,E_i}. \]

Combination of (5.29), (5.30), (5.34), (5.35) give (5.26).

B) We have for \( w_h = u_h - I_h u, \) for all \( z_h \in V_{0h} \)

\[ l_h(w_h, z_h) = l(u - I_h u, z_h) + (l - I_h)(I_h u, z_h). \]

From that, using (5.20), (5.25), (5.26) for \( v_h = w_h \) and replacing \( z_h \) by \( I_h(\Psi_{\mu} z_h) \), we get for triangulations of parallelogram type

\[ l_h(w_h, I_h(\Psi_{\mu} z_h)) \leq \varepsilon \| u - I_h u \|_{1,p,\Omega_{\mu-1}} \| I_h(\Psi_{\mu} z_h) \|_{1,p'} + \left( \| b \|_{\infty} \| c \|_{\infty} \right) \| u - I_h u \|_{1,p,\Omega_{\mu-1}} \| I_h(\Psi_{\mu} z_h) \|_{0,p'} + Ch \left( \| b \|_{C^0} + \| c \|_{C^0} \right) \| u \|_{2,p,\Omega_{\mu-1}} \| I_h(\Psi_{\mu} z_h) \|_{0,p'} \leq Ch \| u \|_{2,p,\Omega_{\mu-1}} \| I_h(\Psi_{\mu} z_h) \|_{1,p'} + \| I_h(\Psi_{\mu} z_h) \|_{0,p'}. \]

Analogously, if we have a triangulation of parallelogram type only on \( \Omega_{\mu-1 \setminus \Omega_{\mu}} \), we get by (5.20), (5.22), (5.23) the weaker estimate

\[ l_h(w_h, I_h(\Psi_{\mu} z_h)) \leq Ch \| u \|_{2,p,\Omega_{\mu-1}} \left( \| z_h \|_{1,p',\Omega_{\mu}} + \varepsilon \| I_h(\Psi_{\mu} z_h) \|_{1,p',\Omega_{\mu-1 \setminus \Omega_{\mu}}} + \| I_h(\Psi_{\mu} z_h) \|_{0,p',\Omega_{\mu-1 \setminus \Omega_{\mu}}} \right) \]

Now the assertions of lemma 9 follow from (5.36), (5.37) by help of (5.3) and (5.4).
6. NUMERICAL EXAMPLE

From numerical examples we got the following statements.

Generally, with classical FEM and piecewise linear approximation we get useful numerical solutions for $\varepsilon \geq O(h)$ and senseless solutions for $\varepsilon < O(h^2)$. The hybrid upwind-FEM gives for all $\varepsilon$-$h$-relations useful numerical solutions, in all cases the qualitative behaviour of the exact solution is given correctly. Sharp contours may be smoothened somewhat but not too much (in the provided numerical experiments boundary layers were extended by not more than $2h$), and so the theoretical result from Figure 7.

Figure 7.
theorem 2 on the thickness $\sigma |\ln \sigma|^2$ of the numerical boundary layer generally seems to be too pessimistic.

In figure 7 we see the exact solution and the numerical solutions of the problem

$$-\varepsilon \Delta u + u_x + u_y + 2u = f \quad \text{in} \quad \Omega = (0, 1)^2$$

$$u = 0 \quad \text{on} \quad \partial \Omega$$

with

$$u_{\text{exact}} = xy \left( 1 - \exp \frac{1-x}{\varepsilon} \right) \left( 1 - \exp \frac{1-y}{\varepsilon} \right).$$

The solutions are drawn along the line $x = y$.

The full line stands for the exact solution, the clashed line for the solution obtained by classical FEM and the dotted line for the solution obtained by the hybrid upwind-FEM.

The triangulation was of the type as you can see in figure 6, we chose $36 \times 36$ nodal points and so the stepsize $h$ was $\sqrt{\frac{2}{35}}$.

REFERENCES


