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REMARKS ON THE UNIQUENESS OF RADIAL SOLUTIONS

by J. SMOLLER [1], A. WASSERMAN [1]

We are concerned with the uniqueness of radial solutions of the following boundary-value problem :

$$\begin{aligned} \Delta u(x) + f[u(x)] = 0\,, &\quad x \in D^n \\ \alpha u(x) - \beta\, du(x)/dn = 0\,, &\quad x \in \partial D^n\,. \end{aligned} \tag{1}$$

Here D^n is an n-ball, say of radius R, α and β are constants, $\alpha^2 + \beta^2 = 1$, and f is a C^1-function. Radial solutions of (1) are functions depending only on $r = |x|$, and thus satisfy

$$\begin{aligned} &u''(r) + \frac{n-1}{r} u'(r) + f[u(r)] = 0\,, \quad 0 < r < R \\ &u'(0) = 0 = \alpha u(R) - \beta u'(R)\,, \quad (r = d/dr)\,, \end{aligned} \tag{2}$$

where the condition $u'(0) = 0$ is needed in order that u be smooth. We can rewrite (2) as the first order system

$$u' = v\,, \quad v' = -\frac{n-1}{r} v - f(u)\,, \tag{3a}$$

$$v(0) = 0 = \alpha u(R) - \beta v(R)\,, \tag{3b}$$

The solution of the initial value poblem (3*a*) which satisfies $u(0) = p > 0$, $v(0) = 0$, will be denoted by $u(r, p)$; i.e., we can parametrize radial solutions by p. In order to be able to consider solutions having many zeros, we first define $\theta(r, p)$ by

$$\theta(r, p) = \tan^{-1}\,[v(r, p)/u(r, p)]\,,$$

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and set

$$\theta_0 = \tan^{-1}(\alpha/\beta), \quad -\pi \leq \theta_0 < 0$$

Then define, for any non-negative integer k, the function $T_k(p)$ by the condition

$$\theta[T_k(p), p] = \theta_0 - k\pi \tag{4}$$

A solution of (3a) which satisfies (4) will be said to belong to the *k-th-nodal-class* of f with respect to the given boundary conditions Thus in this framework, $T_k(p)$ plays the role of R, and R varies with p

Note that the function $T_k(p)$ plays a role in the uniqueness problem for radial solutions , indeed, if T_k is monotone in a neighbourhood of some $\bar{p}$, then the radial solution $u(\,.\,,\bar{p})$ is locally unique in the sense that for small enough $q > 0$, if $u(\,.\,,p)$ is a solution to (3) in the k-th nodal class and $|p - \bar{p}| < q$ then $p = \bar{p}$

The result that we discuss here states that near a hyperbolic zero, γ, of f (i e $f(\gamma) = 0$, $f'(\gamma) < 0$) and for each nodal class k, T_k is monotone near γ More precisely, there exists $q_k > 0$ such that $T_k' > 0$ on $(\gamma - q_k, \gamma)$

If $\theta_0 = -\pi/2$ and $k = 0$, i e , if we consider positive solutions to the Dirichet problem (1) then this result was obtained by Clement and Sweers [1] by entirely different methods

Uniqueness results for solutions of (2) are interesting in their own right but there are implications of this result concerning the existence of asymmetric solutions of (1) In particular, it was shown in [3] that near a hyperbolic zero of f and for any nodal class k and any $\delta > 0$ there exist bifurcation points $p \in (\gamma - \delta, \gamma)$ Taking δ to be q_k above, our result shows that there is no radial bifurcation, thus we must have symmetry breaking at p

We now list the hypotheses on f and F that we require (Here $F' = f$ and $F(0) = 0$)

$$\text{H}\begin{cases} \text{i)} & f \text{ is } C^1 \\ \text{ii)} & \text{there exists } 0 < \gamma \text{ with } f(\gamma) = 0 \text{ and } F(\gamma) > F(u) \\ & \quad \text{for } 0 \leq u \leq \gamma \\ \text{iii)} & f'(\gamma) < 0 \\ \text{iv)} & \text{there is a (greatest) } b < 0 \text{ with } F(b) = F(\gamma) \\ \text{v)} & \text{if } f(b) = 0 \text{ then } f'(b) < 0 \\ \text{vi)} & uf(u) + 2[F(\gamma) - F(u)] > 0 \text{ for } b < u < \gamma \end{cases}$$

Remark H i), ii) and iv) guarantee the existence of radial solutions of (3) in any nodal class Conditions iii), v) and vi) and are mild technical assumptions and could possibly be eliminated with further effort

The first result is proved in [3]

THEOREM 1 : *Suppose that f satisfies* H i), ii) *and* iv) *and let $k \in Z_+$. Then there exists $q_k > 0$ such that if $\gamma - q_k < p < \gamma$ then $u(\,.\,, p)$ is a solution to* (2) *in the k-th nodal class with $R = T_k(p)$. Furthermore, $T_k(p) \to \infty$ as $p \to \gamma$.*

It is easy to show that $T_k(p)$ is differentiable, see [2]. The main result is given by

THEOREM 2 : *If f satisfies hypotheses* (H) *and $k \in Z_+$ then there exists $\delta_k > 0$ $(\delta_k < q_k)$ such that $T_k'(p) > 0$ for $p - \delta_k < p < \gamma$.*

In the remainder of this note we sketch a proof of Theorem 2.

The equations (3*a*) define a flow, $\sigma_r(p)$ on (an open subset of) $\mathbb{R}^3$. If $q = (u, v, r) \in \mathbb{R}^3$ we define a vector field $X_q = \left[v, -\frac{n-1}{r} - f(u), 1\right]$; we then have $\frac{\partial}{\partial r}\sigma_r(q) = X_{\sigma_r(q)}$. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection $\pi(u, r\,;\gamma) = (u, v, 0)$.

LEMMA 1 : *If*

$$v^2 + \frac{n-1}{r} uv + uf(u) > 0 \tag{*}$$

along an orbit $\{\sigma_r(q) : r \geqslant 0\}$ of X then the vectors $\pi\sigma_r(q)$, $X_{\sigma_r(q)}$, and $(0, 0, 1)$ form a basis of $\mathbb{R}^3$ at each point of the orbit.

Let $\bar{p} = (p, 0, 0)$ and assume that (*) holds along the orbit $\{\sigma_r(\bar{p}) : r \geqslant 0\}$. Then we can write

$$\frac{\partial}{\partial p}\sigma_r(\bar{p}) = a\pi\sigma_r(\bar{p}) + bX_{\sigma_r(\bar{p})} + c(0, 1, 1) \tag{5}$$

where $a = a(r, p)$ etc.

LEMMA 2 : *If* H vi) *holds then (*) holds along orbits $\{\sigma_r(\bar{p}) : 0 \leqslant rT_k(p)\}$ for p sufficiently close to γ.*

The proof is a bit long and tedious.

Our reason for introducing the functions a, b, c is given by the following two propositions.

PROPOSITION 3 : *For p sufficiently close to γ we have $b[T_k(p), p] = -T_k'(p)$.*

PROPOSITION 4 : *For p near γ, the functions a and b satisfy the first order linear system of equations*

$$\left\{\begin{aligned} Ja' &= v\phi a - \frac{n-1}{r^2} v^2 b \\ Jb' &= -v\phi a + \frac{n-1}{r^2} uvb \end{aligned}\right. \tag{6}$$

with initial conditions $a(0) = \frac{1}{p}$, $b(0) = 0$. *Here* $a = a(r,p)$, $u = u(r,p)$, *etc.,* $J = uf(u) + (n-1)\frac{uv}{r} + v^2$, *and* $\phi = f(u) - uf'(u)$.

Both propositions are proved by differentiating (5) and equating components.

By Proposition 3 we must prove $b[p, T_k(p)] < 0$ and we use (6) to that end. We first note that for p near γ, we may assume $J > 0$ by Lemma 2. Furthermore, we have $\phi(\gamma) > 0$ by H iii) and thus $\phi(u) > 0$ in a neighbourhood U_1 of γ. Thus, if $p \in U$, the second equation of (6) yields $b'(p, 0) < 0$ and hence $b(p, r) < 0$ for small r. Furthermore, $a(p, r) > 0$ for small γ also by continuity. We set $z(p, r) = -b(p, r)/a(p, r)$ and (suppressing the dependence on p) note that $z(r) > 0$ for small r. We prove Theorem 2 by showing that $z(\gamma) > 0$ for $0 < \gamma \leq T_k(p)$. The theorem then follows since $[a(r), b(r)] \neq (0, 0)$ for any r ($(0, 0)$ is a rest point of (6)) and then noting that a cannot go to zero before b by the first of equations (6) (at $a = 0$ $\operatorname{sgn} a' = \operatorname{sgn} b$ if $v \neq 0$). Thus $z(p, r) > 0$ for $0 < r \leq T_k(p)$ implies $b[p, T_k(p)] < 0$.

We note that z satisfies the differential equation

$$z' = -\frac{(n-1)}{J}\frac{v^2}{r^2}z^2 + \left[\frac{n-1}{Jr^2}uv - \frac{v\phi}{J}\right]z + \frac{u\phi}{J}. \tag{7}$$

While (7) is obviously intractable we can compare (7) with an equation of the form $z' = -k_1 z^2 + k_2 z + k_3$ for various choices of k_1, k_2, k_3. Consider the projection of an orbit $\sigma_r(\bar{p}) = [u(p, r), r(p, r), r]$ to the (u, v) plane as depicted in figure 1. Here A is chosen sufficiently close to γ so that $\phi(u) < 0$ for $u > A$ and $T_s^A(p)$ is defined by $u[p, T_s^A(p)] = A$ (the s-th time orbit meets $u = A$) and T_s^N is defined by $v[p, T_s^N(p)] = 0$ (the s-th Neumann time). We assume $f(b) \neq 0$ (the case $f(b) = 0$, treated in [4], is more complicated notationally.

PROPOSITION 5 : *In region* I, $0 \leq r \leq T_1^A(p)$, *we have* $\lim_{p \to \gamma} z[T_1^A(p)] = \infty$. *Furthermore* $\lim_{p \to \gamma} T_1^A(p) = \infty$.

The proof follows by comparing (7) with an equation of the form $z' = -k_1 z^2 + k_3$.

The behavior of $z(r)$ in region II, $T_1^A \leq r \leq T_2^A$, can be controlled since $T_2^A(p) - T_1^A(p)$ is uniformly bounded (by M say) ; see [3].

PROPOSITION 6 : *There exists* $N > 0$ *such that* $z(T) \geq N$ *implies* $z(T + r) > 0$ *for* $0 \leq r \leq M$.

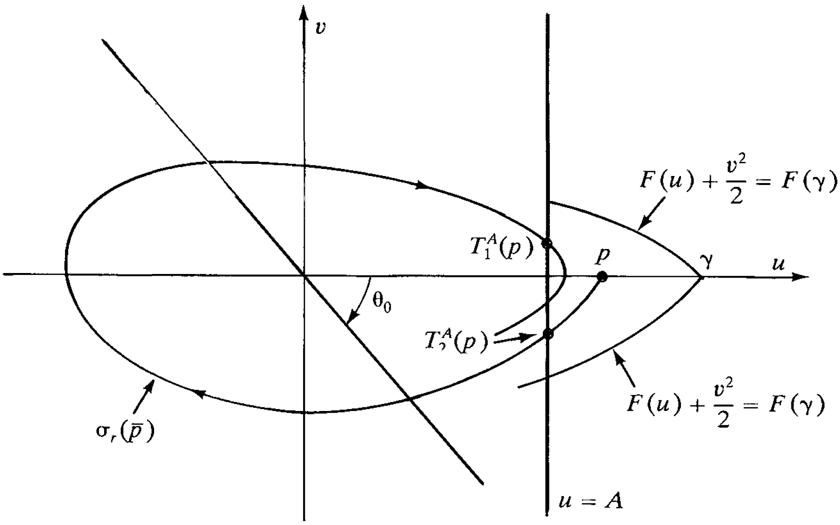

Figure 1. — The line $\theta = \theta_0$ represents the boundary conditions.

The proof compares (7) to $z' = -k_1 z^2 + k_2 z$ and several cases must be considered

Next we consider region III Here $z(r)$ decreases but we still have $z(r) > 0$ because $\phi > 0$ for $u \geqslant A$ hence $z' > 0$ In particular, $z(T_2^N) > 0$ The proof of Theorem 2 obviously must use induction on k and we have just shown that for p sufficiently close to γ, $z(p, r) > 0$ for $r \leqslant T_2^N(p)$ i e for one complete revolution of the orbit To continue the argument we need

PROPOSITION 7 *If* $z(r) > 0$ *for* $T_{2s+2}^A(p) \leqslant r \leqslant T_{2s+3}^A(p)$ *then* $z[T_{2s+3}^A(p)] \to \infty$ *as* $p \to \gamma$

The proof is again by comparison with an equation of the form $z' = -k_1 z^2 + k_3$

Proposition 7 allows us to repeat the argument again k times to conclude $z[T_k(p)] > 0$

Complete proofs will appear elsewhere, [4]

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