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# EXPLICIT UPPER AND LOWER BOUNDS ON THE NUMBER OF DEGREES OF FREEDOM FOR DAMPED AND DRIVEN CUBIC SCHRÖDINGER EQUATIONS

by J. M. GHIDAGLIA [1]

Abstract. — *We study space-periodic damped and driven cubic Schrodinger equations in one dimension. In a previous work, we have shown that the long time behavior of these equations was governed by a finite-dimensional attractor. Here our aim is to provide upper and lower bounds on this dimension which are explicit in terms of the data.*

## 1. INTRODUCTION

We consider the following cubic Schrödinger equation

$$i\,\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+|u|^2\,u+i\gamma u=f\,, \tag{1.1}$$

where $\gamma > 0$ is a damping factor and $f = f(x, t)$ is an external time-periodic driving force. We supplement (1.1) with space periodic boundary conditions i.e.

$$u(x+L,t)=u(x,t)\,,\quad \forall x\in\mathscr{R}\,,\quad \forall t\in\mathscr{R} \tag{1.2}$$

where $L$ is a *given* positive number. As it was shown in a previous work [4, 5], the long-time dynamics of the infinite dimensional dynamical system (1.1)-(1.2) is finite dimensional. More precisely we have shown that the trajectories of this evolution equation are captured by a finite dimensional attractor. This result contrasts with the case of the unperturbed equation (where $\gamma = 0$, $f = 0$) for which thanks to the Inverse Scattering Theory, one

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observes a totally different dynamics [11]. Moreover, our results on the perturbed equations agree with numerical and physical investigations [9, 2].

Our aim in this work is to derive explicit upper and lower bounds on the dimension of the global attractor associated to (1.1)-(1.2). The upper bounds are derived in Corollary 3.1 (third section) while the lower bounds are given in the fourth section. In the next section, we derive some estimates on the solutions to (1.1)-(1.4) and define the global attractor.

## 2. BOUNDED ABSORBING SETS AND ATTRACTORS

For the sake of simplicity in the exposition, we are going to assume in the remainder of this article that the driving term $f$ is time-independent (and depends periodically on the space variable $x$) : $f(x,t) = f(x)$. We denote by $\hat{v}(k)$ the $k$-th-Fourier-coefficient of a $L$-periodic function $v$ :

$$\hat{v}(k) = \frac{1}{2\pi}\int_0^L v(x)\exp\left(-i\,\frac{2\pi kx}{L}\right)dx\,, \quad k \in \mathscr{Z}\,;$$

and $H_L^m$, $m \in \mathscr{N}$, denotes the usual Sobolev space

$$H_L^m = \left(v \in L^2(0,L),\ \sum_{k\in\mathscr{Z}} (1+k^2)^m\,|\hat{v}(k)|^2 < \infty\right).$$

We assume that $f$ belongs to $L^2(0,L)$ and denote by $\{S(t), t \in \mathscr{R}\}$ the nonlinear group on $H_L^1$ that solves (1.1) with the boundary condition (1.2), i.e. $u(t) \equiv S(t)\,u_0$ solves (1.1)-(1.2) and $u(0) = u_0$, $u_0 \in H_L^1$. The mapping $(t, u_0) \to S(t)\,u_0$ is continuous on $\mathscr{R} \times H_L^1$ and $\mathscr{R} \times H_L^2$ ; and for fixed $t$, $S(t)$ is a homeomorphism of $H_L^1$ and $H_L^2$ ([5] and references therein).

An important feature of the damping in (1.1) is the existence of bounded absorbing sets for the group $S(t)$. We recall that given a norm $N$ on $\mathscr{H} = H_L^1$ or $H_L^2$, a set $\mathscr{B}_a \subset \mathscr{H}$ is a bounded absorbing set for $N$ if

(i) $\mathscr{B}_a$ is bounded with respect to $N$,

(ii) for every bounded (w.r. to $N$) set $\mathscr{B} \subset \mathscr{H}$, there exists a time $T(B) \in \mathscr{R}$ such that

$$S(t)\,\mathscr{B} \subset \mathscr{B}_a\,, \quad \forall t \geqslant T(B)\,.$$

Concerning (1.1)-(1.2) we have the following result.

PROPOSITION 2.1 : *The group $S(t)$ possess bounded absorbing sets in $H_L^2$ with respect to the norms of $L^2(0,L)$, $H_L^1$ and $H_L^2$.*

This result is proved in full details in [4]. We are going to sketch briefly some steps of the proof and derive a slightly stronger result. Our starting points are the two identities (mass evolution and energy evolution)

$$\frac{1}{2}\frac{d}{dt}\int |u|^2\,dx + \gamma\int |u|^2\,dx = \operatorname{Im}\int_0^L f\bar{u}\,dx\,, \tag{2.1}$$

$$\frac{1}{2}\frac{d}{dt}\varphi(u) + \gamma\psi(u) = 0\,, \tag{2.2}$$

where

$$\varphi(v) = \int_0^L \left\{|v_x|^2 + 2\operatorname{Re}(f\bar{v}) - |v|^4/2\right\}dx\,, \tag{2.3}$$

$$\psi(v) = \int_0^L \left\{|v_x|^2 + 2\operatorname{Re}(f\bar{v}) - |v|^4\right\}dx\,, \tag{2.4}$$

and Re $z$, Im $z$ denote the real and imaginary part of $z \in \mathscr{C}$. We denote by $|\cdot|_0$ the $L^2$-norm on $[0, L]$, and for a given $\varepsilon \geqslant 0$ we introduce the constant

$$\bar{\varphi}_\varepsilon \equiv 9|f|_0^6\gamma^{-6}(1+\varepsilon)^6/4 + 3|f|_0^4\gamma^{-4}L^{-1}(1+\varepsilon)^4/2\,. \tag{2.5}$$

With these notations, we can state the following result.

PROPOSITION 2.2 : *For every* $\varepsilon \geqslant 0$, *the sets*

$$\mathscr{B}_{0,\varepsilon} = \left\{v \in H_L^1,\ |v|_0 \leqslant |f|_0\gamma^{-1}(1+\varepsilon)\right\}\,, \tag{2.6}$$

$$\mathscr{B}_{1,\varepsilon} = \left\{v \in \mathscr{B}_{0,\varepsilon},\ \varphi(v) \leqslant \bar{\varphi}_\varepsilon\right\} \tag{2.7}$$

*are positively invariant by* $S(t)$ : $S(t)\,\mathscr{B}_{\iota,\varepsilon} \subset \mathscr{B}_{\iota,\varepsilon}$, $\forall t \in \mathscr{R}$. *Moreover, for* $\varepsilon > 0$, $\mathscr{B}_{0,\varepsilon}$ *(resp.* $\mathscr{B}_{1,\varepsilon}$*) is a bounded absorbing set for* $S(t)$ *in the* $L^2$ *(resp.* $H^1$*) norm.*

This last proposition is an easy consequence of (2.1), (2.2) and the following well-known estimate :

$$|v|_\infty^2 \equiv \sup_{0 \leqslant x \leqslant L} |v(x)|^2 \leqslant 2|v|_0\,|v_x|_0 + |v|_0^2 L^{-1}\,,\quad \forall v \in H_L^1\,. \tag{2.8}$$

Indeed, we deduce from (2.1) and the Cauchy-Schwarz inequality that

$$\frac{d}{dt}|u|_0 + \gamma|u|_0 \leqslant |f|_0\,,$$

and this implies that

$$|u(t)|_0 \leqslant |u_0|_0\,e^{-\gamma t} + |f|_0\gamma^{-1}(1 - e^{-\gamma t})\,,\quad \forall t \geqslant 0\,. \tag{2.9}$$

Now the properties of $\mathscr{B}_{0,\varepsilon}$, $\varepsilon \geqslant 0$ follow from this bound. Concerning $\mathscr{B}_{1,\varepsilon}$, we deduce from (2.2) that

$$\frac{d}{dt}\varphi(u) + \gamma\varphi(u) = -\gamma(|u_x|_0^2 - 3||u|^2|_0^2/2)\,.$$

By (2.8), the r.h.s. of this equation is less than $\gamma(3|u|_0^3\,|u_x|_0 - |u_x|_0^2 + 3\,L^{-1}|u|_0^4/2)$. Hence

$$\gamma^{-1}\frac{d}{dt}\varphi(u) + \varphi(u) \leqslant 9|u|_0^6/4 + 3\,L^{-1}|u|_0^4/2\,. \tag{2.10}$$

Taking $u_0 \in \mathscr{B}_{1,\varepsilon} \subset \mathscr{B}_{0,\varepsilon}$, we see that the r.h.s. of (2.10) is less than $\bar{\varphi}_\varepsilon$ since $u \in \mathscr{B}_{0,\varepsilon}$, $\forall t \geqslant 0$. This shows that the $\mathscr{B}_{1,\varepsilon}$ are invariant. Returning to (2.10), one sees easily that these sets are absorbing for the $H^1$-norm, provided $\varepsilon > 0$.

In order to achieve the proof of Proposition 2.1, it remains to show that $S(t)$ possesses an absorbing set in $H_L^2$ (w.r. to the $H_L^2$-norm). This result is slightly more technical than the previous ones. It relies on the study of the evolution of the quantity :

$$\int_0^L \left\{|u_{xx}|^2 - |u|^2\,|u_x|^2 - 2(\mathrm{Re}\,(u\bar{u}_x))^2 - 2\,\mathrm{Re}\,(f\bar{u}_{xx})\right\}\,dx\,,$$

and we refer to [4, 5] concerning the details.

We denote by $\mathscr{B}_a$ a bounded absorbing set in $H_L^2$, for the $H_L^2$-norm, and introduce its omega limit set

$$\mathscr{A} = \bigcap_{s>0} \mathrm{cl}\left(\bigcup_{t\geqslant s} S(t)\,\mathscr{B}_a\right)\,, \tag{2.11}$$

where cl denotes the closure with respect to the **weak** topology of $H_L^2$. This set is the global attractor for (1.1)-(1.2) in $H_L^2$ **(endowed with its weak topology) :**

THEOREM 2.1 : *The set $\mathscr{A}$ defined in* (2.11) *enjoys the following properties :*

$$\mathscr{A} \textit{ is not empty, compact and connected in } H_L^2\,, \tag{2.12}$$

$$\mathscr{A} \textit{ is invariant } : S(t)\,\mathscr{A} = \mathscr{A}\,,\quad \forall t \in \mathscr{R}\,, \tag{2.13}$$

$$\mathscr{A} \textit{ is attracting } : \textit{for every set } \mathscr{B} \textit{ in } H_L^2\,,$$

$$\textit{the sets } S(t)\,\mathscr{B} \textit{ converge in } H_L^2 \textit{ to } \mathscr{A} \textit{ as } t \to +\infty\,. \tag{2.14}$$

For a group (or semi-group) which admits an absorbing set, such a result is classical provided some compactness of the group (or semi-group) is obtained (Levinson [10], Billotti and La Salle [1]). Here, compactness is simply obtained from boundedness since we use the weak-topology. The counterpart is that one must show that the $S(t)$ are continuous w.r. to this topology, which is indeed the case [5].

By (2 14), the set $\mathscr{A}$ describes the long time dynamics of (1 1)-(1 2) In particular it contains the stationary solutions, the (time) periodic solutions or more generally every bounded invariant set in $H_L^2$, $X$, and its unstable set $\mathscr{M}^u(X)$

$$\mathscr{M}^u(X) = \{u_0 \in H_L^2, \text{ s t } S(t)\, u^0 \text{ weakly converges in } H_L^2 \text{ to } X \text{ as } t \to -\infty\} \quad (2\ 15)$$

As it is well-known, even for smooth finite dimensional dynamical systems, this set can be very complicated and therefore $\mathscr{A}$ can have a somewhat complex topological structure This could explain the chaotic behavior of solutions to (1 1)-(1 2) which has been observed [2], [9]

## 3. UPPER BOUNDS ON THE DIMENSION OF THE ATTRACTORS

In this section, we consider a bounded set $X$ in $H_L^1$, which is invariant under (1 1)-(1 2)

$$S(t)\,X = X\,, \quad \forall t \in \mathscr{R} \quad (3\ 1)$$

We introduce with Constantin, Foias and Temam [3] the global Lyapunov exponents on $X$ as follows Given $u_0 \in X$, the complete trajectory $u(t) = S(t)\,u_0$, $t \in \mathscr{R}$, lies in $X$ and we can solve the non autonomous linear equation

$$\iota \frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial x^2} + 2|u|^2 v + u^2 \bar{v} + \iota\gamma v = 0\,, \quad (3\ 2)$$

where $v$ is $L$-space periodic and

$$v(x, 0) = v_0(x)\,, \quad w \in\, ]0, L[ \quad (3\ 3)$$

is given in $H_2^1$ This linearized equation represents in a certain sense the differential of the mapping $S(t)$ at $u_0$ We denote by $L(t\ , u_0) \in \mathscr{L}(H_L^1)$ the linear operator $v_0 \to v(t)$, and set

$$\bar{\omega}_m(t) = \operatorname*{Sup}_{u_0 \in X} \omega_m(L(t\ , u_0))\,, \quad (3\ 4)$$

where $\omega_m(L) = \|\Lambda^M L\|_{\mathscr{L}(\Lambda^m H_L^1)}$ is the norm of the $m$-th exterior product of $L \in \mathscr{L}(H_L^1)$, $m \in \mathscr{N}^*$ (and $H_L^1$ is endowed with the norm $\|v\| \equiv (|v_x|_0^2 + L^{-2}|v|_0^2)^{1/2}$) Thanks to the differentiation chain rule and the fact that

$\omega_m(L_1 \circ L_2) \leqslant \omega_m(L_1)\,\omega_m(L_2)$, we have $\bar{\omega}_m(t_1+t_2) \leqslant \bar{\omega}_m(t_1)\,\bar{\omega}_m(t_2)$. Hence the following limit exists

$$\pi_m = \pi_m(X) = \lim_{t \to +\infty} \bar{\omega}_m(t)^{1/t} < \infty .$$

The global Lyapunov exponents on $X$ are then defined recursively from the $\pi_m$ by

$$\mu_1 = \operatorname{Log} \pi_1 \text{ and } \mu_j = \operatorname{Log} \pi_j - \log \pi_{j-1} \text{ for } j \geqslant 2 .$$

With these notations we can state :

THEOREM 3.1 : *We consider a subset $X$ which is bounded in $H_L^1$ and invariant by* (1.1)-(1.2). *The global Lyapunov exponents on $X$ satisfy*

$$\mu_1(X) + \cdots + \mu_m(X) \leqslant -\gamma m + C_0 m^{1/2}, \quad \forall m \geqslant 1 \tag{3.5}$$

*where $C_0$ is an explicit constant (see* (3.16)) *which depends only on $|f|_0$, $\gamma$ and $L$.*

This result provides explicit bounds on the Hausdorff and fractal dimension of the global attractor $\mathscr{A}$ :

COROLLARY 3.1 : *The Hausdorff dimension $d_{\mathscr{H}}(\mathscr{A})$ and the fractal dimension $d_{\mathscr{F}}(\mathscr{A})$ of the global attractor are finite and bounded as follows*

$$d_{\mathscr{H}}(\mathscr{A}) \leqslant 1 + C_0^2 \gamma^{-2}, \tag{3.6}$$

$$d_{\mathscr{F}}(\mathscr{A}) \leqslant 2(1 + 4 C_0^2 \gamma^{-2}), \tag{3.7}$$

*where $C_0$ is defined in* (3.16).

With regards to (3.5), we see that for $m_0 > C_0^2 \gamma^{-2}$, $\mu_1(X) + \cdots + \mu_{m_0}(X) < 0$. Hence the sum of the first $m$ global Lyapunov exponents on $X$ is negative so that the tangent flow (i.e. (3.2)) along a trajectory lying on $X$ shrinks the $m_0$-dimensional volumes. This implies according to [3, Theorem 3.3] that the Hausdorff dimension of $X = \mathscr{A}$ is less than $1 + C_0^2 \gamma^4$ (and an analogous bound on the fractal dimension). We note that since $\mathscr{A}$ is bounded and weakly closed in $H_L^2$, it is compact in $H_L^1$ and the previous result in [3] is applicable. More precisely we apply here an extension of this result [8], which is necessary here since the mappings $S(t)$ are not compact in $H_L^1$.

Let us now deal with the proof of (3.5). The key point here is a family of identities that satisfy the solutions to (3.2) :

$$\frac{d}{dt} q_\mu(t, v) + \gamma q_\mu(t, v) = r_\mu(t, v) \tag{3.8}$$

where $\mu \in \mathscr{R}$ is arbitrary and we have set for $w \in H_L^1$, $u_0 \in X$,

$$q_\mu(t, w) = \int_0^L \left\{ |w_x|^2 + \mu |w|^2 - |u|^2 |w|^2 - 2[\mathrm{Re}\,(u\bar{w})]^2 \right\} dx , \tag{3.9}$$

$$\begin{aligned} r_\mu(t, w) = & -4\mu \int_0^L \mathrm{Re}\,(u\bar{w})\,\mathrm{Im}\,(u\bar{w})\,dx - 2 \int_0^L \{ \mathrm{Re}\,(u\bar{u}_t)|w|^2 \\ & + 2\,\mathrm{Re}\,(u\bar{w})\,\mathrm{Re}\,(u_t\,\bar{w}) \}\,dx , \end{aligned} \tag{3.10}$$

where $u = u(t) = S(t)\,u_0$. The relations (3.8), $\mu \in \mathscr{R}$, are analogous to (2.1)-(2.2) and are obtained in a similar fashion [5]. For positive $\mu$, $q_\mu$ can be seen as a perturbation of the norm of $H_L^1$. More precisely we have

LEMMA 3.1 : *We take*

$$\mu = 18\,|f|_0^4\,\gamma^{-4} + 3\,|f|_0^2\,\gamma^{-2}\,L^{-1} + L^{-2}/2 , \tag{3.11}$$

*then for every* $u_0 \in X$, $t \in \mathscr{R}$ *and* $w \in H_L^1$,

$$\alpha \|w\|^2 \leqslant q_\mu(t, w) \leqslant |w_x|_0^2 + \mu |w|_0^2 , \quad \alpha = 1/2 . \tag{3.12}$$

Indeed, we have

$$\int_0^L (|u|^2\,|w|^2 + 2(\mathrm{Re}\,[u\bar{w}])^2)\,dx \leqslant 3\,|u|_0^2\,|w|_\infty^2 .$$

Since $X \subset B_{0,0}$, we deduce thanks to (2.8) that the r.h.s. of this inequality is less than $3\,|f|_0^2\,\gamma^{-2}(2\,|w|_0\,|w_x|_0 + |w|_0^2\,L^{-1})$. With the choice (3.11) for $\mu$ in (3.9), we find (3.12).

Concerning the quadratic form $r_\mu$ given in (3.10), we have the following estimate (whose proof is somewhat technical and postponed to the end of that of Theorem 3.1).

LEMMA 3.2 : *For every* $u_0 \in X$, *we have*

$$|r_\mu(t, w)| \leqslant C_1 \|w\|^{1/2}\,|w|_0^{1/2} \tag{3.14}$$

*where* $C_1$ *(given in* (3.21)) *depends only on* $|f|_0$, $\gamma$ *and* $L$.

Introducing the critical values of the quotient $|v|_0^2/\|v\|^2$ :

$$\kappa_p = \min_{F \subset V,\, \dim F = p} \; \max_{v \in F,\, v \neq 0} \frac{|v|_0^2}{\|v\|^2} ,$$

we deduce from (3.8), (3.12), (3.14) and an abstract result on Gram determinants ([6, Appendix]) that

$$\mu_1 + \cdots + \mu_m \leqslant -\gamma m + \frac{C_1}{2\alpha} \sum_{p=1}^m \kappa_p^{1/4} . \tag{3.15}$$

Here the $\kappa_p$ are explicitely known : they are the inverse of the eigenvalues of the operator $v \to \partial^2 v/\partial x^2 + L^{-2} v$ i.e. $\{(1 + 4\,\Pi^2\,k^2), k \in \mathscr{Z}\}$. Now since $\alpha = 1/2$ an easy computation shows (3.5) with

$$C_0 = 2\,C_1\,L^{1/2}/\Pi^{1/2}\,, \quad C_1 \text{ is given in (3.21)}\,. \tag{3.16}$$

It remains to show (3.14). Since $X$ is bounded in $H^1_L$ and is invariant under $S(t)$, we deduce from Proposition 2.2 that $X \subset B_{1,0}$ :

$$|u|_0 \leqslant |f|_0\,\gamma^{-1}\,, \quad \varphi(u) \leqslant \bar{\varphi}_0\,, \quad \forall u \in X\,. \tag{3.17}$$

On the other hand, using (2.8), we see that for every $v \in H^1_L$,

$$|v_x|_0^2 \leqslant 2\,\varphi(v) + |v|_0^6 + |v|_0^4\,L^{-1} + 4\,|f|_0\,|v|_0\,. \tag{3.18}$$

Hence by (3.17) we conclude using again (2.8) that

$$|u_x|_0^2 \leqslant \Omega_1 \equiv 11\,|f|_0^6\,\gamma^{-6} + 4\,|f|_0^4\,\gamma^{-4}\,L^{-1} + 4\,|f|_0^2\,\gamma^{-1}\,, \;\forall u \in X\,, \tag{3.19}$$

and

$$|u|_\infty^2 \leqslant \Omega_\infty \equiv 2\,\Omega_1^{1/2}\,|f|_0\,\gamma^{-1} + |f|_0^2\,\gamma^{-2}\,L^{-1}\,, \quad \forall u \in X\,, \tag{3.20}$$

Then, we estimate $r_\mu$ given in (3.10) by replacing $u_t$ by its value : $u_t = i u_{xx} + i\,|u|^2\,u - \gamma u - i f$. By using (3.17), (3.19) and (3.28), we finally deduce (3.14) with

$$C_1 = 12\left\{(\mu\,|f|_0^2\,\gamma^{-2} + 3\,|f|_0^2\,\gamma^{-1} + \Omega_1 + \Omega_\infty\,|f|_0^2\,\gamma^{-2})\,L^{1/2} + \right.$$
$$\left. + \sqrt{3}\,\Omega_\infty^{1/2}\,\Omega_1^{1/2}\right\}\,. \tag{3.21}$$

## 4. LOWER BOUNDS ON THE DIMENSION OF THE ATTRACTORS

In this section, we give a lower bound on the dimension of the global attractor by computing the dimension of the unstable set emerging from some particular stationary solutions. At the end of this section we compare these bounds to the previous upper bound in a case inspired by a situation arising from plasma physics.

We assume that for given $\rho > 0$ and $k_0 \in \mathscr{Z}$, the function

$$\tilde{u}(x) = \rho\,e^{2 i \Pi k_0 x/L} \tag{4.1}$$

is a stationary solution to (1.1), i.e. that

$$f(x) = \rho\,(\rho^2 - 4\,\Pi^2\,k_0^2/L^2 + i\gamma)\,e^{2 i \Pi k_0 x/L}\,. \tag{4.2}$$

In order to study the stability of this stationary solution, we write the solutions $u$ to (1.1) as

$$u(x,t)=\tilde{u}(x)(1+w(x,t)). \tag{4.3}$$

This leads to the following equation on $w$ :

$$iw_t+w_{xx}-2\,ikw_x-k_0^2 w+\rho^2\{|1+w|^2(1+w)-1\}+i\gamma w=0. \tag{4.4}$$

In order to compute the dimension of the local invariant manifolds in $w=0$, we linearize (4.4) :

$$iv_t+v_{xx}-2\,ikv_x-k_0^2 v+\rho^2\{v+2\operatorname{Re}v\}+i\gamma v=0. \tag{4.5}$$

The solutions to (4.5) can be expanded in terms of plane waves

$$v(x,t)=e^{\lambda t}(a\,e^{2i\Pi\ell x/L}+b\,e^{-2i\Pi\ell x/L}), \tag{4.6}$$

where $\lambda\in\mathscr{R}$, $a$, $b\in\mathscr{C}$ and $\ell\in\mathscr{Z}$. Inserting (4.6) in (4.5), we find the following dispersion relation

$$(\lambda+\gamma)^2=(3\,\rho^2-4\,\Pi^2(k_0+\ell)^2L^{-2})(4\,\Pi^2(k_0+\ell)^2L^{-2}-\rho^2). \tag{4.7}$$

Unstable modes correspond to $\lambda\in\mathscr{R}$, $\lambda>0$ and the dimension of the local unstable manifold is larger than the number of such $\lambda$'s : $N(\rho,\gamma)$. An easy computation shows that

$$N(\rho,\gamma)\geqslant(\rho^4-\gamma^2)_+^{1/4}L/(2^{1/2}\Pi), \tag{4.8}$$

where $x_+=(x+|x|)/2$. It follows that the Hausdorff dimension of the unstable set emerging from $\tilde{u}$, $\mathscr{M}^u(\tilde{u})$ (see (2.15)), is larger than $N(\rho,\gamma)$. Since this set is included in the global attractor $\mathscr{A}$ we deduce the following result (recall that the Hausdorff dimension of a set $\mathscr{E}$, $d_{\mathscr{H}}(\mathscr{E})$, is always smaller or equal to its fractal dimension $d_{\mathscr{F}}(\mathscr{E})$).

PROPOSITION 4.1 : *In the cases where the driving force is given by* (4.2), *the dimensions of the global attractor are bounded from below as follows :*

$$d_{\mathscr{F}}(\mathscr{A})\geqslant d_{\mathscr{H}}(\mathscr{A})\geqslant(\rho^4-\gamma^2)_+^{1/4}L/(2^{1/2}\Pi). \tag{4.9}$$

**An application to a situation arising from plasma physics.** Inspired by Nozaki and Bekki [9], we assume that we are given a small parameter $\varepsilon>0$ and that the damping term and the driving term (given in (4.2), with $k_0=0$) scale as follow

$$\gamma=\gamma_0\,\varepsilon^{-2},\quad \rho=\rho_0\,\varepsilon^{-1}. \tag{4.10}$$

In that case, provided $\rho_0^4 > \gamma_0^2$, we deduce from (4.9) that

$$d_{\mathscr{F}}(\mathscr{A}) \geq d_{\mathscr{H}}(\mathscr{A}) \geq \kappa_0 \, \varepsilon^{-1} \tag{4.11}$$

where $\kappa_0 = (\rho_0^4 - \gamma_0^2)_+^{1/4} \, L/(2^{1/2} \, \Pi)$.

On the other hand, since

$$|f|_0 = L^{1/2} \rho_0 (\rho_0^4 + \gamma_0^4)^{1/2} \, \varepsilon^{-3} , \tag{4.12}$$

we find thanks to (3.7) that

$$d_{\mathscr{F}}(\mathscr{A}) \leq \kappa_0' \, \varepsilon^{-8} , \quad 0 < \varepsilon \leq 1$$

where $\kappa_0'$ is independent of $\varepsilon$ and can be obtained explicitely in terms of $\rho_0$, $\gamma_0$ and $L$. In fact, in the case (4.10), the upper bound (3.14) can be improved (but we shall not give the details here) and this leads to a better estimate on the fractal dimension of $\mathscr{A}$, namely

$$d_{\mathscr{F}}(\mathscr{A}) \leq \kappa_1 \, \varepsilon^{-6} , \quad 0 < \varepsilon \leq 1 . \tag{4.13}$$

Summarizing (4.11) and (4.13) we see that

$$\kappa_0 \, \varepsilon^{-1} \leq d_{\mathscr{H}}(\mathscr{A}) \leq d_{\mathscr{F}}(\mathscr{A}) \leq \kappa_1 \, \varepsilon^{-6} , \quad 0 < \varepsilon \leq 1 . \tag{4.14}$$

We cannot conclude from these estimates that neither the lower bounds nor the upper bounds on the dimensions are asymptotically sharp, as it is the case for the Ginzburg-Landau equation [7]. However, as follows from (4.14), the long time dynamics of equations (1.1)-(1.2) is always finite dimensional but involves more and more degrees of freedom as $\varepsilon \to 0$.

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