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DISCRETE LJAPUNOV FUNCTIONALS AND ω-LIMIT SETS

by Bernold FIEDLER [1]

1. INTRODUCTION

This is a report on recent progress in our understanding of certain dynamical systems with discrete Ljapunov functionals. One-dimensional reaction diffusion equations are a main example. We emphasize our joint work with S. Angenent, P. Brunovský, and J. Mallet-Paret, giving a subjective point of view at the risk of some egocentric bias. As a principal topic, here, we will study infinite-dimensional systems with particularly simple dynamics. Focusing on one-dimensional reaction diffusion equations, we will have to determine just how « simple » the dynamics are.

Admittedly the dynamics of Lipschitz vector fields

$$\frac{d}{dt} u = f(u) , \quad u \in \mathbf{R}^n \tag{1.1}$$

is particularly « simple » in dimensions $n = 1, 2$. Let $\omega(u_0)$ denote the ω-limit set of the solution $u(t)$ through $u_0 = u(0)$, i.e. the set of accumulation points of $u(t)$ as $t \to +\infty$. The α-limit set is defined analogously with $t \to -\infty$. If $u(t)$ is uniformly bounded, then we know that $\alpha(u_0)$, $\omega(u_0)$ are nonempty, connected, compact, invariant (in both time directions) subsets of $\mathbf{R}^n$. For $n \leqslant 2$, of course, we know much more.

For $n = 1$, both $\alpha(u_0)$ and $\omega(u_0)$ consist entirely of equilibria. Even if $2 \leqslant n < \infty$, that statement still holds for gradient systems

$$f(u) = -\nabla_u F(u) ,$$

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because F serves as a Ljapunov functional

$$\frac{d}{dt} F(u(t)) = - |\nabla_u F(u)|^2$$

which decreases strictly along solutions, except at equilibria. In a way, gradient systems look somewhat like scalar equations in that respect. But there are also substantial differences, of course. For example, given an equilibrium v consider those equilibria w which v connects to. Here we say that v connects to w if there exists a u_0 such that $\alpha(u_0) = v$ and $\omega(u_0) = w$; the orbit $u(t)$ is called a heteroclinic connection. In dimension $n = 1$, v can connect to at most 2 distinct equilibria w, one above and one below v. In contrast, v can connect to arbitrarily many other equilibria in dimensions $n > 1$. Also $\omega(u_0)$ is just a single equilibrium in case $n = 1$. In certain degenerate situations, this is no longer true for gradient systems in dimension $n > 1$. Let us now return to the general, non-gradient case.

For $n = 2$ the Poincaré-Bendixson theorem holds.

THEOREM 1 : *Let $u(t)$ be uniformly bounded. Then $\omega(u_0)$ contains a periodic solution or an equilibrium. The same is true for $\alpha(u_0)$.*

For this version, as well as stronger ones, see [Poincaré], 1880-1886, [Bendixson], 1901, and e.g. the textbooks [Coddington & Levinson, Hale, Hartman, Lefschetz, Sansone & Conti]. The proof uses the Jordan curve theorem and is therefore strictly two-dimensional. Obviously, the theorem breaks down for ergodic flows on the 2-torus and, a forteriori, for flows in dimensions $n \geqslant 3$ which may contain strange attractors.

For $n = 1$, we have perceived the continuous Ljapunov functional F as a way of lifting an essential feature of ω-limit sets, namely to consist of equilibria, up to higher dimension n. In the following we attempt something vaguely analogous. We will use a discrete-valued Ljapunov functional, z, to lift the Poincaré-Bendixson theorem to higher, in fact infinite, dimension. Still, the equations remain genuinely high-dimensional as far as the global dynamics are concerned. The global attractor, for example, may have high dimension.

To be specific we fix the following setting for the rest of this paper. We consider the one-dimensional reaction diffusion equation

$$u_t = u_{xx} + f(x, u, u_x) , \tag{1.2}$$

with nonlinearity $f \in C^2$. For boundary conditions we admit the following alternatives. We consider separated boundary conditions $(1.3)^s$ first ; specifically of mixed type

$$(1.3)^s_m \quad \alpha_0 u(t, 0) + \beta_0 u_x(t, 0) = \alpha_1 u(t, 1) + \beta_1 u_x(t, 1) = 0, \ \alpha_\iota^2 + \beta_\iota^2 > 0,$$

including the special cases

$(1.3)_D^s$ $\qquad \beta_0 = \beta_1 = 0 \quad$ (Dirichlet) ,

$(1.3)_N^s$ $\qquad \alpha_0 = \alpha_1 = 0 \quad$ (Neumann) .

Later we select periodic boundary conditions

$$(1.3)^p \qquad \begin{aligned} u(t,0) &= u(t,1)\,, \\ u_x(t,0) &= u_x(t,1)\,, \end{aligned}$$

assuming f to be periodic in x. This amounts to considering equation (1.2) on the circle $x \in S^1 = \mathbf{R}/\mathbf{Z}$. Equation (1.2), together with boundary conditions $(1.3)^s$ or $(1.3)^p$, defines a strongly continuous local semiflow on a subspace X of the Sobolev space H^2 of functions $u_0 : [0, 1] \to \mathbf{R}$ with square integrable second derivative. The space X is given by those $u_0 \in H^2$ which satisfy the boundary conditions. The semiflow associates to any initial condition $u_0 \in X$ a maximal forward solution curve $u(t) \in X$, $t \in [0, \theta)$. For a reference see [Henry 1]. Let

$$(1.4) \qquad \gamma^+(u_0) := \{u(t) \mid t \in [0, \theta)\}$$

denote the (positive) trajectory through u_0.

If $\gamma^+(u_0)$ is a bounded subset of X then $\theta = \infty$ and the ω-limit set $\omega(u_0)$ in X can be defined as for (1.1) above, with the same general properties. In particular $\omega(u_0)$ is positively time invariant : for any $v_0 \in \omega(u_0)$ the associated trajectory $\gamma^+(v_0)$ remains in $\omega(u_0)$. Also $\omega(v_0) \subseteq \omega(u_0)$, since $\omega(u_0)$ is compact and $\omega(v_0) \subseteq \operatorname{clos} \gamma^+(v_0)$ by definition. For example, for a stationary or periodic solution $u(t)$ we trivially have

$$\omega(v_0) = \omega(u_0) = \gamma^+(u_0)\,,$$

for any $v_0 \in \omega(u_0)$. Below, we reserve the term « periodic » for nonstationary periodic solutions. By $E \subseteq X$ we denote the set of equilibria (stationary solutions) of (1.2). In other words $w \in E$, iff

$$(1.5) \qquad 0 = w_{xx} + f(x, w, w_x)$$

and $w \in X$ satisfies the underlying boundary conditions associated to X.

As early as 1968, [Zelenyak] proved the following result on ω-limit sets for separated boundary conditions $(1.3)_m^s$ and $f \in C^3$.

THEOREM 2 : *Assume that* $\gamma^+(u_0)$ *is a bounded trajectory of* (1.2) *with boundary conditions* $(1.3)^s$. *Then* $\omega(u_0) \subseteq E$ *consists of a single equilibrium.*

We mention that [Zelenyak] includes cases of separated nonlinear boundary conditions as well.

The proof of theorem 2 is based on a continuous Ljapunov functional F of the form

$$F(u(t,\,.\,)) := \int_0^1 \Phi(x, u(t,x), u_x(t,x))\,dx \tag{1.6}$$

with the property that

$$\frac{d}{dt} F(u(t,\,.\,)) < 0 \tag{1.7}$$

unless $u(t,\,.\,) \in E$ is already an equilibrium. Zelenyak constructs Φ as a solution of a certain hyperbolic equation such that (1.7) holds. The boundary conditions enter as boundary conditions for Φ. Note that (1.7) implies $\omega(u_0) \subseteq E$. Zelenyak completes the proof by direct estimates showing that $\omega(u_0)$ contains at most one equilibrium. For the case of Dirichlet or Neumann boundary conditions see also [Matano 1, 5].

Similarly to the gradient systems considered above, the semiflow defined by (1.2), $(1.3)^s$ is by no means genuinely one-dimensional. In fact, equilibria can have large (but finite) unstable dimension and can connect to many other equilibria. For more details we refer to our discussion in § 4, and in particular to theorem 6.

For periodic boundary conditions $(1.3)^p$, $x \in S^1$, the situation is quite different. For example consider

$$u_t = \varepsilon u_{xx} + u^2 u_x + u\,, \quad x \in S^1 = \mathbf{R}/\mathbf{Z}\,. \tag{1.8}$$

Looking for rotating waves, i.e. for solutions of the form $u = U(x - ct)$, one is lead to

$$\varepsilon U'' + (U^2 + c)\,U' + U = 0\,, \tag{1.9}$$

known as the van-der-Pol oscillator. In particular, periodic solutions U of (1.9) with (not necessarily minimal) period 1 yield periodic solutions $u(t,\,.\,)$ of (1.8). Therefore, a Ljapunov functional satisfying (1.7) cannot exist for (1.2) with periodic boundary conditions. However, an analogue of the Poincaré-Bendixson theorem 1 still holds.

THEOREM 3 : *Assume that* $\gamma^+(u_0)$ *is a bounded trajectory of* (1.2) *with periodic boundary conditions* $(1.3)^p$. *Then* $\omega(u_0)$ *contains a periodic solution or an equilibrium.*

This theorem, along with a stronger version mentioned as theorem 4 in § 4, is proved in [Fiedler & Mallet-Paret 2]. Our discussion in § 4 also covers related earlier results, other types of equations, and some results on connecting orbits.

As the principal tool in our proof of theorem 3, we monitor the number $z(\varphi)$ of sign changes of maps $x \mapsto \varphi(x)$, $\varphi \in H^2(S^1) \subseteq C^1(S^1)$. It is known that

$$t \mapsto z(u^1(t,\,.\,) - u^2(t,\,.\,)) \tag{1.10}$$

is nonincreasing along any difference of distinct trajectories $u^1(t,\,.\,)$, $u^2(t,\,.\,)$, see e.g. [Matano 2, Brunovský & Fiedler 1]. Due to this fact we call z a **discrete Ljapunov functional.** The proof of (1.10) uses the strong maximum principle and arguments given essentially in [Nickel], already. For $f \equiv 0$, i.e. for the standard heat equation, (1.10) was proved by Sturm [Sturm] in 1836, see also [Pólya], 1933. Beyond fact (1.10), our proof of theorem 3 depends on a much more subtle analysis due to [Angenent 2]. We use extensively that z drops whenever $u^1(t,\,.\,) - u^2(t,\,.\,)$ has a multiple zero. Sections 2 and 3 follow [Fiedler & Mallet-Paret 2]. In Section 2 we extract, from these crucial facts, some two-dimensionality which is then used in Section 3 to complete the proof of theorem 3.

2. SIGN CHANGES

In this section we study the dropping behavior of the zero number

$$z(u^1(t) - u^2(t))$$

along solutions $u^1(t)$, $u^2(t)$ of our reaction diffusion equation (1.2) with periodic boundary conditions $(1.3)^p$, see lemma 2.1 and corollary 2.2. In lemmata 2.3 and 2.4 we then draw the conclusion that single trajectories within ω-limit sets embed into the plane. This will be a crucial step towards proving the Poincaré-Bendixson type theorem 3 in § 3.

Following [Angenent 2] we study linear equations first, in a special setting which fits both our needs and the general framework of [Angenent 2]. Consider solutions $\varphi(t, x)$ of

$$\varphi_t = \varphi_{xx} + b\varphi_x + c\varphi\,, \quad x \in S^1 \tag{2.1}$$

with initial condition $\varphi_0 = \varphi(0,\,.\,) \in H^1(S^1)$. The coefficients b, c are allowed to depend on t and x such that

$$b,\, b_t,\, b_x,\, c \in L^\infty_{\mathrm{loc}}\,. \tag{2.2}$$

LEMMA 2.1 : *Under the above assumptions the following holds.*

(2.3*a*) $z(\varphi(t,\,.\,))$ *is finite for any* $t > 0$, *also when* $z(\varphi_0) = \infty$.

(2.3*b*) $z(\varphi(t,\,.\,))$ *drops strictly at* $t = t_0 > 0$ *if, and only if,* $\varphi(t_0,\,.\,) \neq 0$ *and* $x \to \varphi(t_0, x)$ *has a multiple zero at some* $x_0 \in S^1$ *(that is,* $\varphi(t_0, x_0) = \varphi_x(t_0, x_0) = 0$.)

(2.3*c*) *If* $\varphi(t_0,\,.\,) \equiv 0$ *then* $\varphi(t,\,.\,) \equiv 0$ *for all* t.

For a proof see [Angenent 2]. The special case of analytic coefficients was treated in [Angenent & Fiedler]. As a suggestive example consider the simplest case : $\varphi(t_0, .)$ has a precisely double zero at $x = x_0$, i.e. $\varphi(t_0, x_0) = \varphi_x(t_0, x_0) = 0$ and say $\varphi_{xx}(t_0, x_0) > 0$. Then $\varphi_t(t_0, x_0) > 0$ and hence two successive sign changes of $\varphi(t, .)$ cancel as t increases through t_0. Locally near x_0, the discrete Ljapunov functional z drops by 2. It is less obvious that z also drops at zeros of higher order. In the analytic case, this difficulty can be resolved via the Newton polygon associated to $(t, x) \to \varphi(t, x)$. The general case is played back to the analytic case by clever scaling and approximation arguments. Note that (2.3c) implies backward uniqueness for solutions φ : any $\varphi_0 \in H^1(S^1)$ can have at most one backward extension by a solution. We now return to our original nonlinear equation (1.2) with periodic boundary conditions $(1.3)^p$. Motivated by the peculiar role of multiple zeros, we define the following continuous linear projection π of our state space $X = H^2(S^1)$:

$$\begin{aligned} \pi : X &\to \mathbf{R}^2 \\ \varphi &\mapsto \pi\varphi = (\varphi(x_0), \varphi_x(x_0)) , \end{aligned} \tag{2.4}$$

where $x_0 \in S^1$ is arbitrary but, from now on, fixed. The planar projection π and the discrete Ljapunov functional z relate as follows.

COROLLARY 2.2 : *Let $u^1(t)$, $u^2(t)$ be solutions of* (1.2), $(1.3)^p$ *and let t_0 be positive. Then the following holds.*

(2.5a) *$z(u^1(t_0) - u^2(t_0))$ is finite.*

(2.5b) *If $\pi(u^1(t_0) - u^2(t_0)) = 0$ then either $t \mapsto z(u^1(t) - u^2(t))$ drops strictly at $t = t_0$, or else $u^1(t) - u^2(t) \equiv 0$, for all t.*

(2.5c) *If $t \mapsto z(u^1(t) - u^2(t))$ does not drop strictly at $t = t_0$, and if $u^1(t) - u^2(t) \not\equiv 0$, then $x \to u^1(t_0, x) - u^2(t_0, x)$ has only simple zeros.*

(2.5d) *Facts* (2.5a-c) *also hold for u_t replacing $u^1 - u^2$.*

Proof: Let $\varphi(t, x) := u^1(t, x) - u^2(t, x)$. Then φ solves a (formally) linear equation of the form (2.1). Indeed

$$\begin{aligned} &f(x, u^1(t, x), u_x^1(t, x)) - f(x, u^2(t, x), u_x^2(t, x)) = \\ &= \int_0^1 D_3 f(x, u^2(t, x) + \Theta\varphi(t, x), u_x^2(t, x) + \Theta\varphi_x(t, x))\, d\Theta \,.\, \varphi_x(t, x) + \\ &\quad + \int_0^1 D_2 f(x, u^2(t, x) + \Theta\varphi(t, x), u_x^2(t, x) + \Theta\varphi_x(t, x))\, d\Theta \,.\, \varphi(t, x) \\ &= b(t, x)\, \varphi_x(t, x) + c(t, x)\, \varphi(t, x) , \end{aligned}$$

with the obvious definitions of b, c. Subtracting equations (1.2) for $u = u^1$ resp. $u = u^2$ from each other therefore yields (2.1). Note that regularity assumptions (2.2) hold since u^1, u^2 are C^1 jointly in $t > 0$ and $x \in S^1$, and since $f \in C^2$.

We may therefore apply lemma 2.1. Claims (2.5*a-c*) are immediate from facts (2.3*a-c*) and the definition of π.

Differentiating (1.2) with respect to t, and putting $\varphi = u_t$, claim (2.5*d*) follows from lemma 2.1 in the same way. This completes the proof. ∞

For the rest of this section we consider an initial condition $u_0 \in X$ such that $\gamma^+(u_0)$ is bounded, as assumed in theorem 3. Let $v_0 \in \omega(u_0)$, i.e.

$$v_0 = \lim_{n \to \infty} u(t_n)$$

for some sequence $t_n \to \infty$. Then

$$v(t) := \lim_{n \to \infty} u(t_n + t), \quad t \in \mathbf{R}, \tag{2.6}$$

is the unique solution of (1.2), $(1.3)^p$ passing through v_0 at $t = 0$. Indeed, note that (2.5*b*) implies backward uniqueness of the solution through v_0. We denote the trajectory through v_0 by

$$\gamma(v_0) := \{v(t) \mid t \in \mathbf{R}\}.$$

LEMMA 2.3 : *The restriction*

$$\pi : \operatorname{clos} \gamma(v_0) \to \mathbf{R}^2 \tag{2.7}$$

is injective.

Proof: We give an indirect proof. Suppose there exist $v_0^1, v_0^2 \in \operatorname{clos} \gamma(v_0)$ such that $\pi v_0^1 = \pi v_0^2$ but $v_0^1 \neq v_0^2$. Denote the solution curves through v_0^1, v_0^2 by $v^1(\tau)$, $v^2(\tau)$. Then $z(v^1(\tau) - v^2(\tau))$ drops strictly at $\tau = 0$, by (2.5*b*), and hence

$$z(v^1(\varepsilon) - v^2(\varepsilon)) < z(v^1(-\varepsilon) - v^2(-\varepsilon)) \tag{2.8a}$$

for $\varepsilon > 0$. Because z is finite, it is also locally constant for negative resp. positive τ near 0. Hence we may assume $x \mapsto v^1(\pm\varepsilon, x) - v^2(\pm\varepsilon, x)$ to have only simple zeros, by (2.5*c*). Therefore there exist t^0, $\theta \in \mathbf{R}$ such that

$$z(v(t^0 + \theta \pm \varepsilon) - v(t^0 \pm \varepsilon)) = z(v^1(\pm\varepsilon) - v^2(\pm\varepsilon)). \tag{2.8b}$$

Note that we are using continuity of the semiflow in $X = H^2(S^1) \subseteq C^1(S^1)$

here Hence $x \mapsto v(t^0 + \theta \pm \varepsilon, x) - v(t^0 \pm \varepsilon, x)$ may still be assumed to have only simple zeros Since $v_0 = v(0) = \lim u(t_n)$, this implies that

$$(2\ 8c) \quad z(u(t_n + t^0 + \theta \pm \varepsilon) - u(t_n + t^0 \pm \varepsilon)) = \\ = z(v(t^0 + \theta \pm \varepsilon) - v(t^0 \pm \varepsilon)),$$

for all large enough t_n Together, (2 8*a*-*c*) imply that

$$t \mapsto z(u(t + t^0 + \theta) - u(t + t^0))$$

drops infinitely often, at $t_n \to \infty$ This contradicts finiteness of z, as stated in (2 5*a*), and thus completes the proof ∞

We caution the reader here that lemma 2 3 need not hold on clos $\{u(t) \mid t \geqslant t_0\}$, for large t_0 say, even if $u(t)$ solves a linear equation However, lemma 2 3 extends to the whole ω-limit set $\omega(u_0)$, see § 4, theorem 5

Omitting a proof, which parallels the previous one, we also state the following consequence of property (2 5*d*) of $z(u_t)$

LEMMA 2 4 *Let $v(t)$ be the solution curve through $v_0 \in \omega(u_0)$ and assume that*

$$\frac{d}{dt}\pi v(t) = 0 \quad at \quad t = 0$$

Then $v_0 \in E$ is an equilibrium

3. PROOF OF THEOREM 3

Let $v_0 \in \omega(u_0)$ We will show below that $\omega(v_0) \subseteq \omega(u_0)$ contains a periodic solution or an equilibrium In fact, we suppose that $\omega(v_0)$ does not contain any equilibrium We will then show that $\omega(v_0)$ contains a periodic solution

Let $w_0 \in \omega(v_0)$ and let $w_0^* \in \omega(w_0)$ We denote the solutions through v_0, w_0, w_0^*, defined as in (2 6), by $v(t)$, $w(t)$, $w^*(t)$, respectively

Lemma 2 4 implies that

$$\frac{d}{dt}\pi w^*(t) \neq 0 \quad \text{at} \quad t = 0,$$

because $w_0^* \in \omega(w_0) \subseteq \omega(v_0)$ is not an equilibrium Let S be a short straight section in $\mathbf{R}^2$ through πw_0^*, transverse to the segment $\pi w^*(t)$ for t near zero Choosing a small enough neighborhood U of πw_0^*, we claim that the following holds

(3 1) if $v(\,.\,)$ is a trajectory in clos $\gamma(v_0)$ such that $\pi v(0) \in U$, then $\pi v(t)$ crosses S in the same direction near $t = 0$ as $\pi w^*(t)$ does

Indeed, suppose this claim is not true. Then there exists a sequence $v^n(\,.\,)$ in clos $\gamma(v_0)$ such that $\pi v^n(0) \to \pi w_0^*$ but $\pi v_t^n(0) \not\to \pi w_t^*(0)$. By compactness, we may assume that the $v^n(0)$ converge to some $v_0^* \in \text{clos}\,\gamma(v_0)$. Obviously, $v_0^* \neq w_0^*$. Because $\pi(v_0^*) = \pi(w_0^*)$ and because v_0^*, $w_0^* \in \text{clos}\,\gamma(v_0)$, this contradicts lemma 2.3. This proves claim (3.1).

Let $t_n \to +\infty$ denote those positive times for which $\pi w(t_n) \in S$. If any two of the $\pi w(t_n)$ coincide, then $w(t)$ is a periodic solution by lemma 2.3 and we are done.

Suppose now that the $\pi w(t_n)$ are mutually distinct. Then we obtain a contradiction to the Jordan curve theorem. Indeed, consider the closed Jordan curve composed of $\pi w([t_n, t_{n+1}])$ and of the interval in S with endpoints $\pi w(t_n)$, $\pi w(t_{n+1})$. Watch $\pi v(t)$. As $t \to +\infty$, this curve has both these endpoints as accumulation points. By (3.1), $\pi v(t)$ has to cross S whenever it enters U. By lemma 2.3, $\pi v(t)$ cannot touch $\pi w([t_n, t_{n+1}])$ unless $\pi v(\,.\,)$, and hence $v(\,.\,)$, coincide with the trajectories $\pi w(\,.\,)$ resp. $w(\,.\,)$. Therefore $\pi v(t)$ cannot stay in the interior of the Jordan curve forever. Likewise, $\pi v(t)$ cannot stay in the exterior. But, by (3.1), $\pi v(t)$ can cross the Jordan curve at most once. This is a contradiction.

Therefore, $w(t)$ is a periodic solution in $\omega(v_0)$. This completes the proof.

4. DISCUSSION

We begin our discussion with a stronger version of a Poincaré-Bendixson type result, given as theorem 4 below. This result is put in perspective with work by Massatt and by Matano. We then widen our horizons a little to include other types of equations : monotone cyclic feedback systems and certain delay equations. Zooming in on reaction diffusion equations again we address dimension questions : the dimension of the global attractor may be large while, by theorem 5, each individual ω-limit set $\omega(u_0)$ is at most two-dimensional. We also survey some results on the global dynamics within the global attractor viz. on connecting orbits. We close with an open question concerning a viscosity limit.

To state a stronger version of theorem 3 we recall that (2.6) defines the solution curve $v(t)$ through any $v_0 \in \omega(u_0)$. Therefore we can associate to any $v_0 \in \omega(u_0)$ not only its ω-limit set $\omega(v_0)$ but also its α-limit set $\alpha(v_0)$, consisting of all accumulation points of $v(t)$ as $t \to -\infty$. Finally recall that E denotes the set of equilibria, *cf.* (1.5). The main result in [Fiedler & Mallet-Paret 2] is

THEOREM 4 : *Assume that* $\gamma^+(u_0)$ *is a bounded trajectory of* (1.2) *with periodic boundary conditions* $(1.3)^p$.

Then the ω-limit set $\omega(u_0)$ *satisfies exactly one of the following alternatives.*

(i) *Either* $\omega(u_0)$ *consists of precisely one periodic orbit, or*
(ii) $\alpha(v_0) \subseteq E$ *and* $\omega(v_0) \subseteq E$ *for any* $v_0 \in \omega(u_0)$.

As in the classical Poincaré-Bendixson theorem, alternative (ii) allows for $\omega(u_0)$ to consist of a chain of homoclinic or heteroclinic solutions joining equilibria.

Earlier results in this direction have dealt with nonlinearities $f = f(u, u_x)$ which are independent of x. This assumption makes (1.2), $(1.3)^p$ S^1-equivariant, i.e. the solution semiflow commutes with shifting x. Assuming also analyticity of f, the « soft » version of the Poincaré-Bendixson theorem, phrased in theorem 3 above, was proved in [Angenent & Fiedler, § 3] ; and stronger versions were suspected to hold. All periodic solutions turn out to be rotating waves, i.e. solutions of the form $u = U(x - ct)$. (Below we outline a reason for this.) Independently, [Massatt] has in fact proved that either $\omega(u_0)$ is a single rotating wave, or a set of equilibria which differ only by a shift in x. The same result has also been obtained by [Matano 4] who further shows that $\omega(u_0)$ is a single equilibrium if $f = f(u, u_x)$ is even in the second argument.

The result of [Massatt] fits into theorem 4 as follows. Due to S^1-equivariance, we may transform (1.2) into rotating coordinates $\hat{u}(t, x) = u(t, x + ct)$ and apply theorem 4 to the resulting equation

$$\hat{u}_t = \hat{u}_{xx} + f(\hat{u}, \hat{u}_x) + c\hat{u}_x\,, \quad x \in S^1\,. \tag{4.1}$$

Note that (4.1) turns out autonomous because f does not depend on x. Periodic solutions u are now seen to be rotating waves, for the following reason. Suppose $u(t, \,.\,)$ is periodic, but not a rotating wave. Then we can choose a small nonzero rotation speed c such that $\hat{u}(t, \,.\,)$ is a dense solution curve on the two-dimensional torus in X given by

$$T^2 = \{u(t, \xi + \,.\,) : t \in \mathbf{R}, \xi \in S^1\}\ .$$

Denoting by $\hat{\omega}(u_0)$ the ω-limit set with respect to (4.1), we obtain

$$\hat{\omega}(u_0) = T^2\,.$$

This clearly contradicts theorem 3, since the flow of (4.1) on T^2 is quasiperiodic. More directly, it contradicts theorem 4 since T^2 does not contain equilibria if $|c|$ is small enough. At any rate, periodic solutions are indeed rotating waves. Moreover, we can turn stationary solutions u into rotating waves $\hat{u}$ by picking $c \neq 0$. By theorem 4, $\hat{\omega}(u_0)$ can be at most a single periodic solution. Therefore, $\omega(u_0)$ is either itself a single rotating wave, or a set of equilibria differing only by phase shift in x. This proves the result of [Massatt].

Under the additional symmetry condition of [Matano 4], i.e. under $O(2)$-equivariance of (1.2), $(1.3)^p$, periodic solutions cannot occur. Therefore $\omega(u_0)$ is a single equilibrium, up to x-shift, by the above. In [Matano 4], additional symmetry arguments reveal that $\omega(u_0)$ is indeed a single equilibrium.

The reaction diffusion equation (1.2) is a special example of the much wider class of strongly monotone semiflows, investigated e.g. in [Hirsch 1-3, Matano 3, 6, H. Smith]. Monotone basically means that $u_0^1 > u_0^2$ in X implies $u^1(t) > u^2(t)$, for all $t \geq 0$, where $>$ is a suitable order relation on X. For such systems periodic solutions are always unstable. Even though $\omega(u_0)$ might be a periodic solution for some u_0, it will not be periodic for generic initial data u_0. Also, scalar reaction diffusion equations in higher space dimension of x define strongly monotone semiflows, while a discrete Ljapunov functional z does not seem to exist in general.

On the other hand, there are also examples which do not fit into the framework of strongly monotone semiflows, but which do admit discrete Ljapunov functionals z of the type studied above. Specifically, we mention monotone cyclic feedback systems

$$\frac{d}{dt} u_i = f_i(u_i, u_{i-1}) , \quad i \bmod n , \tag{4.2}$$

$u_i \in \mathbf{R}$, and the (related) monotone feedback delay equations

$$\frac{d}{dt} u(t) = f(u(t), u(t-1)) , \tag{4.3}$$

$u \in \mathbf{R}$. For a functional z to exist, it is required that

$$\delta_i \cdot D_2 f_i(\xi, \eta) > 0 , \quad \text{for all} \quad \xi, \eta, i ,$$

resp.

$$\delta \cdot D_2 f(\xi, \eta) > 0 , \quad \text{for all} \quad \xi, \eta ,$$

where $\delta_1, \ldots, \delta_n, \delta \in \{-1, +1\}$. These systems define strongly monotone semiflows only if $\delta_1 \cdot \ldots \cdot \delta_n = +1$ resp. if $\delta = +1$. In case these signs are reversed, stable periodic solutions can in fact occur.

For system (4.2) the functional z can be defined essentially as follows. For $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbf{R}^n$ with $\varphi_i \neq 0$ for all i let

$$z(\varphi) := \text{card } \{i \,(\bmod n) \mid \delta_i \, \varphi_{i-1} \, \varphi_i < 0\} .$$

For $u^1(t)$, $u^2(t)$ representing two different solutions of (4.2) it turns out that again

$$t \to z(u^1(t) - u^2(t))$$

is nonincreasing with t, and z drops precisely at those isolated times for which z is not defined. In [Mallet-Paret & H. Smith] this structure is used to prove that theorem 4 also holds for (4 2) The proof uses phase plane arguments which do not carry over to reaction diffusion equations In [Fiedler & Mallet-Paret 2] an axiomatic setting is pursued which allows a unified treatment of both results The analogue of the projection π for system (4 2) is given by

$$\begin{aligned} \pi . \mathbf{R}^n &\to \mathbf{R}^2 \\ \varphi &\to (\varphi_{\iota - 1}, \varphi_\iota) \end{aligned}$$

for some fixed ι

For monotone delay equations (4 3) a discrete Ljapunov functional z was found in [Mallet-Paret 2], notationally z was called V there Essentially $z(\varphi_0)$ counts the number of zeros of $\varphi(.)$ in $[\sigma - 1, \sigma]$, including multiplicity, in case φ is smooth, $\varphi(\sigma) = 0$ A candidate for the projection π is $\pi(\varphi_\sigma) = (\varphi(\sigma), \varphi(\sigma - 1))$ This projection was already used in 1975 by [Kaplan & Yorke] to find slowly oscillating periodic solutions ($z = 1$) Something like theorem 4 is expected to hold for (4 3), but most details are still in progress

Zooming in on our one-dimensional reaction diffusion equation again, we quote the following dimension result from [Fiedler & Mallet-Paret 2]

THEOREM 5 *Assume that* $\gamma^+(u_0)$ *is a bounded trajectory of* (1 2) *with periodic boundary conditions* $(1\ 3)^p$ *Then the projection*

$$\begin{aligned} \pi . \omega(u_0) &\to \pi\omega(u_0) \subseteq \mathbf{R}^2 \\ v &\mapsto (v(x_0), v_x(x_0)) \end{aligned}$$

is a homeomorphism onto its image In other words, $\omega(u_0)$ *embeds into the plane*

In light of theorem 5, the infinite-dimensional dynamical system (1 2) looks quite two-dimensional However, this perspective is somewhat deceptive For example, assume that all solutions of (1 2) are attracted eventually to some bounded region in X Define the global attractor A to consist of all limits of convergent sequences $u^n(t_n)$ where $t_n \to \infty$ and $u^n(0)$ is any bounded sequence in X, see [Billoti & LaSalle, Hale *et al*] In particular, A contains all equilibria and periodic solutions, and their unstable manifolds Therefore A can have arbitrarily high (but finite) dimension and it does, e g in the van-der-Pol type equation (1 8) In contrast, [R Smith 1-3] gives conditions on certain ODE-systems and delay equations which guarantee the global attractor A to be two-dimensional as a whole Clearly, this yields Poincare-Bendixson type theorems

A series of attempts have been made to understand the global dynamics of equation (1 2) For $f = f(x, u, u_x)$ in C^4 and mixed boundary conditions (excluding the Neumann case, but including certain nonlinear boundary conditions as well) [Henry 2] established that the semiflow is Morse-Smale as soon as all equilibria are hyperbolic (i.e. linearly nondegenerate). In other words, stable and unstable manifolds intersect **automatically** transversely, not just in « most » cases. Independently, this was also proved in [Angenent 1] for $f = f(x, u)$ of class C^2 and Dirichlet conditions $(1.3)_D^s$. [Henry 2] has used this transversality to establish all connecting orbits between equilibria for the so-called Chafee-Infante problem, i.e. for cubic-like $f = f(u)$, $f(0) = 0$, and Dirichlet boundary conditions. For an earlier approach to this problem using Conley's index see e.g. [Conley & Smoller]. By now, for general $f = f(u)$, $f \in C^2$ with suitable growth conditions and boundary conditions $(1.3)_D^s$ or $(1.3)_N^s$ all orbit connections between hyperbolic equilibria are also known, see [Brunovský & Fiedler 2,3]. A fundamental building block is the following result on heteroclinic orbits from [Brunovský & Fiedler 2], which we state here in a slightly generalized form.

THEOREM 6: *Let v be a hyperbolic equilibrium of* (1.2) *with mixed boundary conditions* $(1.3)_m^s$. *Let $i(v) > 0$ be the linear unstable dimension of v.*

Then v connects to at least $2\,i(v)$ other mutually distinct equilibria w. More precisely, for each $0 \leq k < i(v)$ there exists a pair of equilibria w, satisfying

$$z(v - w) = k\,,$$

which v connects to.

The proof monitors $z(u(t, .\,) - v)$ carefully along the unstable manifold W^u of v. A Borsuk-Ulam type argument yields existence of pairs of trajectories on W^u for which

$$z(u(t, .\,) - v) \equiv k\,, \quad \text{for all } t\,.$$

By theorem 2, [Zelenyak], these solutions converge to equilibria w in forward time, as desired. As we have indicated in Section 1, theorem 6 also implies that our semiflow is not quite one-dimensional.

Connecting orbits for the S^1-equivariant case (1.2), $(1.3)^p$, $f = f(u, u_x)$ were considered in [Angenent & Fiedler, § 4], using an S^1-equivariant version of the Borsuk-Ulam theorem. With theorem 3 replacing [Angenent & Fiedler, theorem 3.1] that result generalizes to

THEOREM 7: *Let $v(t)$ be a hyperbolic periodic solution of* (1.2), $(1.3)^p$ *with unstable dimension $i(v) > 0$. Then v connects to at least*

$$[i(v)/2] + 2$$

distinct periodic or stationary solutions Here $[\iota(v)/2]$ *denotes the maximal integer not exceeding* $\iota(v)/2$

In contrast to theorem 7 it is quite impossible in genuinely two-dimensional flows that a periodic solution connects to more than two distinct periodic solutions

For another example, we return to delay equation (4 3) We assume $f \in C^{\infty}$, negative feedback

$$\begin{aligned} &f(0,\eta) \,.\, \eta < 0 \qquad \text{for } \eta \neq 0\,,\\ &D_2 f(0,0) < 0\,,\\ &D_1 f(0,0) + D_2 f(0,0) < 0\,, \end{aligned}$$

linearized hyperbolicity of the trivial solution $u \equiv 0$ (with necessarily even unstable dimension ι), and dissipativeness Using the odd-valued functional z for (4 3) introduced above, the trivial solution connects to the maximal invariant subsets S_k of $\{z = 2k - 1\}$ as follows [Fiedler & Mallet-Paret 1]

THEOREM 8 *Under the above assumptions on delay equation* (4 3), *the trivial solution* $u \equiv 0$ *connects to each of the sets* S_k *with* $1 \leqslant k \leqslant \iota/2$

For related results on connecting orbits for this equation, using Conley's index, see [Mallet-Paret 1, Mischaikow]

Returning finally to reaction diffusion equations, we would like to close with an open question Consider

$$(1\ 2)_\varepsilon \qquad u_t = \varepsilon^2 u_{xx} + f(x, u, u_x)\,, \quad x \in S^1\,,$$

and let ε tend to zero Even for quite special f it seems to be unknown whether or not a Poincaré-Bendixson theorem holds for the limiting nonlinear hyperbolic equation Under suitable assumptions the case $f = g(u)_x$ has been studied, along with some obvious variants For this conservation law all solutions tend to equilibrium, see e g [Dafermos, Smoller] and the references there For an example which admits rotating shocks we happily refer the reader back to (1 8) in the introduction

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