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LOCAL LYAPUNOV EXPONENTS AND A LOCAL ESTIMATE OF HAUSDORFF DIMENSION

by Alp EDEN [1]

Abstract — *The Lyapunov dimension has already been used to give estimates of the Hausdorff dimension of an attractor associated with a dissipative ODE or PDE Here we give a slightly different version, utilizing local Lyapunov exponents, in particular we show the existence of a critical path along which the Hausdorff dimension is majorized by the associated Lyapunov dimension This result is then applied to Lorenz equations to deduce a better estimate of the dimension of the universal attractor We conclude with an example that shows some of the drawbacks of this estimate*

CONTENTS

1. INTRODUCTION

Various notions of dimension were introduced and studied in connection with dynamical systems : topological dimension, Hausdorff dimension, fractal dimension, information dimension just to name a few. It has been observed by R. Mañé [M] that the Hausdorff dimension gives an upper bound for the number of independent real variables that parametrizes the set. This nice result increases the significance of Hausdorff dimension. We will define and contrast two notions of dimension : fractal and Hausdorff,

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both notions were studied in the context of dissipative partial differential equations, since the upper bounds found for Hausdorff dimension are in general easier to assess we will modify one of these results. We turn our attention to the basic set-up now.

Let X be a compact subset of a metric space, define

$$\mu_{d,\varepsilon}(X) = \inf \left\{ \sum_{i=1}^{k} r_i^d : r_i \leqslant \varepsilon, X \subseteq \bigcup_{i=1}^{k} B_{r_i} \right\} \tag{1.1}$$

then

$$\mu_d(X) \doteq \lim_{\varepsilon \to 0} \mu_{d,\varepsilon}(X) = \sup_{\varepsilon > 0} \mu_{d,\varepsilon}(X) \tag{1.2}$$

and the Hausdorff dimension can uniquely be defined as

$$d_{\mathrm{H}}(\mathrm{X}) = \inf \{d > 0 : \mu_d(X) = 0\} . \tag{1.3}$$

We define the fractal dimension in a similar manner, let

$$n_\varepsilon(X) = \text{the minimum number of balls of radius less than } \varepsilon \text{ that can cover X} , \tag{1.4}$$

and

$$\mu_{d,F}(X) = \overline{\lim_{\varepsilon \to 0}}\ \varepsilon^d n_\varepsilon(X) \tag{1.5}$$

then the fractal dimension of X is defined by

$$d_F(X) = \inf \{d > 0 : \mu_{d,F}(X) = 0\} . \tag{1.6}$$

Although this definition is more involved, it is equivalent to the usual definition

$$d_F(X) = \overline{\lim_{\varepsilon \to 0}}\ \frac{\log n_\varepsilon(X)}{\log (1/\varepsilon)} \tag{1.7}$$

and makes the following inequality transparent,

$$d_H(X) \leqslant d_F(X) . \tag{1.8}$$

Considering $X = \left\{ \frac{1}{\log n} : n = 2, 3, \dots \right\} \cup \{0\}$ it is easy to see that $d_H(X) = 0$ whereas $d_F(X) = +\infty$. The uniform character of the fractal dimension makes it harder to obtain local estimates similar to ones that will be shown in the third section. Let us recall the CFT estimates [CFT].

2. CFT ESTIMATES

We will now assume that X is a compact subset of a separable Hilbert space H. Let $\{S_t\}$ be a continuous semigroup of nonlinear operators acting on X such that

$$S_t X = X \quad \text{for all} \quad t \geqslant 0 . \tag{2.1}$$

Moreover assume that for every $t > 0$ and u_0 in X, there exists, not necessarily unique, a compact linear operator $S'(t, u_0)$ satisfying

$$|S_t u - S_t u_0 - S'(t, u_0)(u - u_0)| \leqslant c(t)\, 0(|u - u_0|) . \tag{2.2}$$

For each t and u_0, we fix such a choice for $S'(t, u_0)$. Let

$$M(t, u_0) = (S'(t, u_0)^* S'(t, u_0))^{1/2} \tag{2.3}$$

then M is a positive compact operator on H, therefore it has a complete set of eigenvectors corresponding to the eigenvalues $\{m_j(t, u_0) : j = 1, 2, \dots\}$.
Set

$$P_N(t, u_0) = m_1(t, u_0) \,.\, m_2(t, u_0) \,.\, \dots \,.\, m_N(t, u_0) \tag{2.4}$$

and

$$P_N(t) = \sup_{u_0 \in X} P_N(t, u_0) \tag{2.5}$$

the (Global) Lyapunov Exponents are defined by

$$\mu_1 + \mu_2 + \cdots + \mu_N = \lim_{t \to \infty} t^{-1} \log P_N(t) \quad \text{for} \quad N = 1, 2, 3 \dots \tag{2.6}$$

we also define upper Lyapunov Exponents as

$$\bar{\mu}_N = \overline{\lim_{t \to \infty}}\; t^{-1} \log \left[\sup_{u_0 \in X} m_N(t, u_0) \right] . \tag{2.7}$$

THEOREM 1 [CFT] : *If N is the first integer such that*

$$\mu_1 + \mu_2 + \cdots + \mu_{N+1} < 0$$

then

$$d_H(X) \leqslant N + \frac{\mu_1 + \mu_2 + \cdots + \mu_N}{|\mu_{N+1}|} .$$

THEOREM 2 [CFT] : *If N is the first integer such that $\bar{\mu}_{N+1} < 0$ then*

$$d_F(X) \leqslant \max_{1 \leqslant \ell \leqslant N} \left\{ \ell + \frac{\mu_1 + \cdots + \mu_\ell}{|\bar{\mu}_{N+1}|} \right\} .$$

Remark : As we have already noticed before $d_H(X) \leqslant d_F(X)$, therefore the second theorem also gives an estimate on the Hausdorff dimension, in the case $\bar{\mu}_{N+1} < 0$ and $\mu_1 + \cdots + \mu_{N+1} \geqslant 0$ this estimate might give a better upper bound for the Hausdorff dimension. However, unless the eigenvalues $m_j(t, u_0)$ can be computed separately the only way to estimate $\bar{\mu}_{N+1}$ is to use the following well-known inequality [CFT]

$$(N+1)\, \bar{\mu}_{N+1} \leqslant \mu_1 + \cdots + \mu_{N+1} . \tag{2.8}$$

3. LOCAL LYAPUNOV EXPONENTS AND LOCAL LYAPUNOV DIMENSION

For each u_0 in X, the Local Lyapunov Exponents are defined by the relations

$$(\mu_1 + \mu_2 + \cdots + \mu_N)(u_0) = \overline{\lim_{t \to \infty}}\; t^{-1} \log P_N(t, u_0) . \tag{3.1}$$

There is an intimate relation between the Local and Global Lyapunov exponents, the following result is in that spirit.

THEOREM 3 [EFT] :

$$\mu_1 + \mu_2 + \cdots + \mu_N = \max_{u_0 \in X} \; (\mu_1 + \mu_2 + \cdots + \mu_N)(u_0) .$$

Remark : The above result is a corollary of the more refined result that is proved in [EFT].

There exists u_0 in X such that

$$t^{-1} \log P_N(t, u_0) \geqslant \mu_1 + \mu_2 + \cdots + \mu_N \quad \text{for every} \quad t \geqslant 1 . \tag{3.2}$$

The Lyapunov dimension d_L, defined by

$$d_L = N + \frac{\mu_1 + \mu_2 + \cdots + \mu_N}{|\mu_{N+1}|} \tag{3.3}$$

has a local analog,

$$d_L(u_0) = N + \frac{(\mu_1 + \cdots + \mu_N)(u_0)}{|\mu_{N+1}(u_0)|} \tag{3.4}$$

using the basic relations between local and global Lyapunov exponents it is easy to show that

$$d_L \geqslant \sup_{u_0 \in X} d_L(u_0) \tag{3.5}$$

then the following question is in the spirit of Theorem 3.

Question 1 : Does there exist a u_0 in X such that

$$d_L(u_0) = d_L .$$

THEOREM 4 [CFT] : *If N is the first integer such that $\mu_1 + \cdots + \mu_{N+1} < 0$ then*

$$d_H(X) \leqslant \sup_{u_0 \in X} \left\{ N + \frac{(\mu_1 + \cdots + \mu_N)(u_0)}{|\mu_{N+1}(u_0)|} \right\} .$$

THEOREM 5 [EFT] : *If N is as in Theorem 4, then there exists u_0 in X such that*

$$d_H(X) \leqslant N + \frac{(\mu_1 + \mu_2 + \cdots + \mu_N)(u_0)}{|\mu_{N+1}(u_0)|} .$$

Remark : If the answer to our first question turns out to be positive then the last theorem is a simple consequence of the previous theorem, yet in a more abstract setting it is possible to construct an example where $d_L(u_0) < d_L$ for every u_0 in X [E]. In either case, the following is an interesting question.

Question 2 : Is there another characterization for u_0's satisfying the conclusion of Theorem 5.

4. AN APPLICATION : LORENZ SYSTEM

We will consider the Lorenz system in the following form

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= -y - xz \\ \dot{z} &= -bz + xy - br \end{aligned} \tag{4.1}$$

where b, σ, r are real numbers satisfying $1 < b < \sigma$, $0 < r < +\infty$.

When $r > 1$, the system has three distinct stationary solutions, namely

$$u_{\pm} = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, -1) \quad \text{and} \quad u = (0, 0, -r) . \tag{4.2}$$

Since $S'(t, u_0)$ satisfies the ODE

$$v_t = N(u_0)\, v \tag{4.3}$$

where

$$N(u_0) = \begin{bmatrix} -\sigma & \sigma & 0 \\ -z & -1 & -x \\ y & x & -b \end{bmatrix} \quad \text{and} \quad u_0 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{4.4}$$

then using the Wronskian formula

$$(m_1 \,.\, m_2 \,.\, m_3)(t, u_0) = \det S'(t, u_0) = \exp\left(\int_0^t \operatorname{tr} N(u(z))\, dz \right) \tag{4.5}$$

therefore, using $\operatorname{tr} N(u_0) = -(\sigma + 1 + b)$ for every u_0 in X we deduce that

$$(\mu_1 + \mu_2 + \mu_3)(u_0) = -(\sigma + b + 1) \tag{4.6}$$

and by Theorem 3,

$$\mu_1 + \mu_2 + \mu_3 = -(\sigma + b + 1)\,. \tag{4.7}$$

Our goal is to estimate

$$d_L(u_0) = 2 + \frac{(\mu_1 + \mu_2)(u_0)}{|\mu_3(u_0)|}\,. \tag{4.8}$$

Since $\mu_1 + \mu_2 + \mu_3 = (\sigma + b + 1) < 0$, $d_L(u_0)$ will give an estimate of Hausdorff dimension. We only need to consider u_0's such that $(\mu_1 + \mu_2)(u_0) > 0$, for such u_0's the estimate

$$(\mu_1 + \mu_2)(u_0) \leq M \tag{4.9}$$

combined with (3.8) gives

$$d_L(u_0) \leq 2 + \frac{M}{1 + \sigma + b + M}\,. \tag{4.10}$$

In [EFT] the following estimate is found

$$M = \frac{1}{2}\left\{ -(\sigma + 2b + 1) + \sqrt{(\sigma - 1)^2 + \frac{2\, b\sigma r}{\sqrt{b - 1}}} \right\} \tag{4.11}$$

and in the case, where $u = (0, 0, -r)$

$$(\mu_1 + \mu_2)(u_0) = \frac{1}{2}\left\{ -(\sigma + 2b + 1) + \sqrt{(\sigma - 1)^2 + 4\,\sigma r} \right\} \tag{4.12}$$

using the values $b = 8/3$, $\sigma = 10$, $r = 28$ considered by Lorenz [L] we deduce that

$$M = 9.424 \tag{4.13}$$

$$(\mu_1 + \mu_2)(u) = 9.161 \tag{4.14}$$

which give,

$$d_L(u_0) \leqslant 2.4081 \qquad \text{for} \qquad u_0 \in X \tag{4.15}$$

$$d_L(u) = 2.40131 \qquad \text{when} \qquad u = (0, 0, -r)\,. \tag{4.16}$$

The first estimate gives a new upper bound for the Hausdorff dimension of the universal attractor and the second one evokes a new question.

Question 3 : Is $u = (0, 0, -r)$ the critical path along which the dimension is majorized ?

5. HOW GOOD IS THE ESTIMATE $d_H(X) \leqslant d_L(u_0)$

We consider a simple non-linear ODE, with a particularly simple attractor [ER]

$$\begin{aligned} \dot{x}_1 &= x_1 - x_1^3 \\ \dot{x}_2 &= -x_2 \end{aligned} \tag{5.1}$$

It is easy to see that $X = [-1, 1] \times \{0\}$ attracts all solutions and contains the three stationary points

$$u_{\pm} = (\pm 1{,}0) \quad u = (0, 0) \tag{5.2}$$

where $u_{\pm}$ are attracting fixed points and u is a repelling point. The linearized system is

$$\dot{v} = N(u)\, v \tag{5.3}$$

where $v = S'(t, u)$ and $N(u)$ is given by

$$N(u) = \begin{bmatrix} 1 - 3x_1^2 & 0 \\ 0 & -1 \end{bmatrix} \qquad \text{when} \qquad u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{5.4}$$

therefore

$$(m_1 m_2)(t, u_0) = \exp \int_0^t -3\, x_1^2(s)\, ds \tag{5.5}$$

and

$$(\mu_1 + \mu_2)(u) = \begin{cases} -3 & \text{if} \quad x_1 \neq 0 \\ 0 & \text{if} \quad x_1 = 0 \end{cases} \tag{5.6}$$

and writing

$$S'(t, u) = \begin{bmatrix} \exp \int_0^t (1 - 3\, x_1^2(s))\, ds & 0 \\ 0 & e^{-t} \end{bmatrix} \tag{5.7}$$

we can get

$$\mu_1(u) = \begin{cases} -1 & \text{if} \quad x_1 \neq 0 \\ 1 & \text{if} \quad x_1 = 0 \,. \end{cases} \tag{5.8}$$

Since $\mu_1 + \mu_2 = 0$, we can not apply the theory in this case, even though $d_H(X) = d_F(X) = 1$.

Concluding Remarks: This brief exposition might give the wrong impression that this theory is only applicable to ODE's, the scope of this theory is general enough to include the case where $S'(t, u_0)$ is the sum of a compact operator with a contraction [E] this case is treated with Global Lyapunov Exponents by Ghidaglia and Temam [GT and T]. On the other hand, by a simple argument it is possible to get an estimate of the topological entropy using a slightly modified Lyapunov exponent [EFT]

$$h(X) \leqslant \tilde{\mu}_1\, d_F(X) \,. \tag{5.9}$$

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