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ASYMPTOTICS OF THE SCATTERING FREQUENCIES FOR A THERMOELASTICITY PROBLEM WITH SMALL THERMAL CONDUCTIVITY (*)

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Abstract. — *We give a physical example of a system, depending on a parameter* ε, *such that, for* $\varepsilon = 0$, *it has an eigenvalue with infinite multiplicity, which, for* $\varepsilon > 0$, *splits into the set of an eigenvalue with infinite multiplicity and infinitely many scattering frequencies with finite multiplicity. The system is made of a thermoelastic heterogeneous medium and* ε *denotes the thermal conductivity of a bounded region of the medium.*

Résumé. — *Nous donnons un exemple physique d'un système dépendant d'un paramètre tel que, pour* $\varepsilon = 0$, *il a une valeur propre de multiplicité infinie, qui éclate, pour* $\varepsilon > 0$, *en l'ensemble d'une valeur propre de multiplicité infinie et d'une infinité de fréquences de scattering de multiplicité finie. Le système est formé par un milieu thermoélastique hétérogène et* ε *désigne la conductivité thermique d'une région bornée du milieu.*

1. GENERALITIES

It is known [6], [7] that the thermoelasticity system in a bounded domain has an eigenvalue at the origin when the thermal conductivity vanishes. This point of the spectrum splits into infinitely many eigenvalues for positive thermal conductivity. We consider here an analogous problem in an unbounded domain, which exhibits a more singular behaviour : for positive thermal conductivity there is an eigenvalue with infinite multiplicity plus infinitely many scattering frequencies, the corresponding scattering functions not belonging to the functional space where the problem is considered. We shall focus our study on the difference with respect to the case of bounded domain, and the reader is referred to [6], [7] for some questions. In

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this context, the density and specific heat will be taken equal to 1. Otherwise weighted spaces should be taken as in [6]. The thermal conductivity εk and the thermal-elastic coupling coefficient β will be considered isotropic (i.e. scalar).

Throughout this paper, the notations are standard. If u is a function, $u|_E$ denotes its restriction to the domain E. The same notation is used for traces. If H is a Hilbert space, H' denotes its dual ; $\mathscr{L}(H, V)$ denotes the space of the continuous operators from H into V, and $\mathscr{L}(H) = \mathscr{L}(H, H)$. Vectors in the physical space are noted with *boldface types :*

$$\mathbf{u} = (u_1, u_2, u_3) \tag{1.1}$$

and in the same way, $\mathbf{L}^2$ and $\mathbf{H}^1$ will denote $(L^2)^3$ and $(H^1)^3$, i.e. the space of « vectors » with components in the space of square integrable functions and in the space of functions with square integrable derivatives of order 0 and 1.

The convention of summation of repeated indices will be used, and δ_{ij} will denote the classical Kronecker tensor.

The linear thermoelasticity system (see [4] for instance) reads

$$\frac{\partial^2 u_i}{\partial t^2} - \frac{\partial \sigma_{ij}^T}{\partial x_j} = 0 \tag{1.2}$$

$$\frac{\partial}{\partial t} (\theta + \beta \operatorname{div} \mathbf{u}) - \varepsilon k \, \Delta\theta = 0 \tag{1.3}$$

where $\mathbf{u}$ and θ denote the displacement vector and the temperature, Δ is the Laplace operator, σ^T is the « total » stress tensor, which decomposes into two parts depending on $\mathbf{u}$ and θ according to

$$\sigma^T = \sigma(\mathbf{u}) + \tilde{\sigma}(\theta) \tag{1.4}$$

$$\sigma_{ij}(\mathbf{u}) = a_{ijlm} \, e_{lm}(\mathbf{u}) \, ; \quad e_{lm}(\mathbf{u}) = \frac{1}{2} \left(\frac{\partial u_l}{\partial x_m} + \frac{\partial u_m}{\partial x_l} \right) \tag{1.5}$$

$$\tilde{\sigma}_{ij}(\theta) = - \beta \, \delta_{ij} \, \theta \, . \tag{1.6}$$

The part σ is the classical elasticity tensor (or isothermal elasticity tensor) which depends on the strain tensor $e(\mathbf{u})$ according to (1.5), where a_{ijlm} are the elasticity coefficients, which satisfy the symmetry and positivity conditions :

$$a_{ijlm} = a_{ijml} = a_{mlij} \tag{1.7}$$

$$a_{ijlm} \, e_{lm} \, e_{ij} \geqslant C \, e_{ij} \, e_{ij} \quad \forall \, e_{ij} \text{ symmetric} \tag{1.8}$$

for some $C \geqslant 0$.

Of course, (1.2) should be considered in the distribution sense, and then it implies the transmission condition

$$[\sigma_{ij}^T n_j] = 0 \tag{1.9}$$

on the eventual discontinuities of the medium, where the bracket denotes the jump across a surface with normal $\mathbf{n}$.

Let us recall that the strain-stress relation (1.5) in the isotropic case becomes

$$\sigma_{ij} = \lambda\, e_{mm}\, \delta_{ij} + 2\,\mu\, e_{ij} \Leftrightarrow a_{ijlm} = \lambda\, \delta_{ij}\, \delta_{lm} + \mu\,(\delta_{il}\, \delta_{jm} + \delta_{jl}\, \delta_{im}) \tag{1.10}$$

where λ, μ are the Lamé constants of the material.

After these generalities on the thermoelasticity system, we consider the specific case where the space $\mathbf{R}^3$ is divided into two parts, a bounded one B, with boundary Γ, and the exterior region E. The medium is supposed to have constant properties on each of the regions B and E. Then, the coupling coefficient β, the conductivity εk and the elasticity coefficients a_{ijlm} are functions of x of the form :

$$\beta(x) = \begin{cases} \beta^B = \text{const.} > 0 & \text{if } x \in B \\ \beta^E = \text{const.} > 0 & \text{if } x \in E \end{cases} \tag{1.11}$$

$$k(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \in E \end{cases} \tag{1.12}$$

$$a_{ijlm} = \begin{cases} a_{ijlm}^B = \text{const.} & \text{if } x \in B \\ a_{ijlm}^E = \text{isotropic const.} & \text{if } x \in E \end{cases} \tag{1.13}$$

where it is understood that the a_{ijlm}^E are constants expressed in terms of Lamé constants λ^E, μ^E according to (1.10). The medium is not necessarily isotropic in B. On the other hand, we note that (1.12) expresses that the thermal conductivity vanishes in the exterior region E. More precisely the thermal conductivity will be $\varepsilon k(x)$, with $k(x)$ given by (1.12), where ε denotes a small parameter taking values ≥ 0. In fact, we shall also consider complex values of ε in a neighbourhood of the origin, in order to use techniques of holomorphic functions.

On the interface Γ we shall prescribe the boundary condition (1.9) and the continuity of the displacement vector ; moreover, for $\varepsilon \neq 0$, we shall prescribe a Neumann boundary condition for θ on the side B, expressing the fact that the heat cannot pass across Γ into the region E where the conductivity vanishes :

$$[\mathbf{u}] = 0\ ; \quad [\sigma_{ij}^T n_j] = 0 \quad \text{on } \Gamma \tag{1.14}$$

$$\frac{\partial \theta}{\partial n} = 0 \quad \text{on } \Gamma,\ \text{side } B,\ \text{for } \varepsilon \neq 0,\ \text{nothing for } \varepsilon = 0\,. \tag{1.15}$$

We note that the term $\varepsilon k(x)\,\Delta\theta$ in (1.3) may be written $\varepsilon\,\Delta_N^B\,\theta$, where Δ_N^B denotes the Laplace operator in B with Neumann boundary condition. This may be written in the distribution sense on $\mathbf{R}^3$ as :

$$\varepsilon \frac{\partial}{\partial x_i}\left(k(x)\frac{\partial\theta}{\partial x_i}\right). \tag{1.16}$$

In order to study the evolution system (1.2), (1.3) we shall reduce it to a first order system with respect to t by introducing the velocity field :

$$\mathbf{v}(x,t) = \frac{\partial \mathbf{u}(x,t)}{\partial t}. \tag{1.17}$$

We shall study the evolution of $\mathbf{U} = (\mathbf{u}, \mathbf{v}, \theta)$ in the space of configurations

$$\mathscr{H} = \mathbf{H}^1(R^3) \times \mathbf{L}^2(R^3) \times L^2(R^3). \tag{1.18}$$

We note that $\mathbf{u} \in \mathbf{H}^1$ shows that the first relation (1.14) is automatically satisfied in the trace sense. The system (1.2), (1.3) with (1.16) becomes :

$$\begin{cases} \dfrac{d\mathbf{U}}{dt} + \mathscr{A}_\varepsilon \mathbf{U} = 0 \\ \mathbf{U} = (\mathbf{u}, \mathbf{v}, \theta)\,; \end{cases} \quad \mathscr{A}_\varepsilon = \begin{bmatrix} 0 & -I & 0 \\ -\dfrac{\partial\sigma_{ij}}{\partial x_j} & 0 & \beta\dfrac{\partial}{\partial x_i} \\ 0 & \beta\,\mathrm{div} & -\varepsilon\dfrac{\partial}{\partial x_i}\left(k(x)\dfrac{\partial}{\partial x_i}\right) \end{bmatrix}. \tag{1.19}$$

We note that the derivatives are taken in the distribution sense on $\mathbf{R}^3$. Then, $\mathscr{A}_\varepsilon$ maps $\mathscr{H}$ into a larger space. In order to define $\mathscr{A}_\varepsilon$ as an unbounded operator of $\mathscr{H}$, we define its domain

$$D(\mathscr{A}_\varepsilon) = \left\{\mathbf{U} = (\mathbf{u}, \mathbf{v}, \theta) \in \mathscr{H}\,;\, -\frac{\partial\sigma_{ij}(\mathbf{u})}{\partial x_j} + \beta\frac{\partial \mathbf{u}}{\partial x_i} \in \mathbf{L}^2\,; \right. \tag{1.20}$$
$$\left. \beta\,\mathrm{div}\,\mathbf{v} - \varepsilon\frac{\partial}{\partial x_i}\left(k(x)\frac{\partial\theta}{\partial x_i}\right) \in L^2\right\}$$

and we emphasize that this domain depends on ε. We then have :

PROPOSITION 1.1 : *For real $\varepsilon \geqslant 0$, the operator $-\mathscr{A}_\varepsilon$ is generator of a strongly continuous semigroup of bounded operators in $\mathscr{H}$ which solve the system* (1.19).

Proof: For the sake of simplicity, we shall choose on $\mathbf{L}^2$ and L^2 the classical norm, and on $\mathbf{H}^1$:

$$\|\mathbf{u}\|_{H^1}^2 = a(\mathbf{u},\mathbf{u}) + \|\mathbf{u}\|_{L^2}^2, \text{ where} \tag{1.21}$$

$$a(\mathbf{u},\mathbf{v}) = \int_{\mathbf{R}^3} a_{ijlm}\, e_{lm}(\mathbf{u})\, e_{ij}(\bar{\mathbf{v}})\, dx \tag{1.22}$$

which is equivalent to the classical one by the Korn's inequality. Moreover, changing $\mathbf{U}(t) = \exp(\alpha t)\,\mathbf{V}(t)$, we have for $\mathbf{V}$ an analogous problem with $\mathscr{A}_\varepsilon + \alpha I$ instead of $\mathscr{A}_\varepsilon$. We shall prove that, for sufficiently large positive α, $-(\mathscr{A}_\varepsilon + \alpha I)$ is generator of a contraction semigroup (note that the corresponding semigroup generated by $-\mathscr{A}_\varepsilon$ is not necessarily of contraction). According to the Lumer-Phillips theorem ([2] or [6]), it sufficies to prove that $\mathscr{A}_\varepsilon + \alpha I$ is acretive and surjective on $\mathscr{H}$, i.e.

$$\operatorname{Re}((\mathscr{A}_\varepsilon + \alpha I)\,\mathbf{U},\mathbf{U})_{\mathscr{H}} \geqslant 0 \quad \forall \mathbf{U} \in D(\mathscr{A}_\varepsilon) \tag{1.23}$$

$$\left\{ \begin{array}{l} \text{there exists a solution } \mathbf{U} \in D(\mathscr{A}_\varepsilon) \text{ of} \\ (\mathscr{A}_\varepsilon + \alpha I)\,\mathbf{U} = \mathbf{U}^g \quad \text{for any given } \mathbf{U}^g \in \mathscr{H}\,. \end{array} \right. \tag{1.24}$$

In order to prove (1.23), using the definition (1.19), we obtain :

$$\begin{aligned} ((\mathscr{A}_\varepsilon + \alpha I)\,\mathbf{U},\mathbf{U}) = {} & (-\mathbf{v},\mathbf{u})_{H^1} + a(\mathbf{u},\mathbf{v}) - (\beta\theta, \operatorname{div}\mathbf{v})_{L^2} + \\ & + (\beta \operatorname{div}\mathbf{v}, \theta)_{L^2} + \varepsilon \int_B |\theta|^2\, dx + \alpha\|\mathbf{u}\|_{\mathbf{H}^1}^2 + \alpha\|\mathbf{v}\|_{\mathbf{L}^2}^2 + \alpha\|\theta\|_{L^2}^2 \end{aligned} \tag{1.25}$$

and taking the real part, we have (1.23) for $\alpha > 1$. We note that (1.25) amounts to formal integration by parts ; in fact it is rigorously obtained by taking the product in $\mathscr{H}$ of $\mathscr{A}_\varepsilon\,\mathbf{U}$ with $\mathbf{U}^n$:

$$\mathbf{U}^n = (\mathbf{u}^n, \mathbf{v}^n, \theta^n) \in \mathscr{D} \times \mathscr{D} \times \mathscr{D} \quad (= \text{space dense in } \mathscr{H})$$

and letting $\mathbf{U}^n \to \mathbf{U}$ in $\mathscr{H}$; as for $\mathbf{U}^n$, integration by parts is merely the interpretation of (1.19) in the distribution sense.

In order to prove the solvability of (1.24), we write it down, with $\alpha > 1$:

$$\begin{aligned} & -\mathbf{v} + \alpha\,\mathbf{u} = \mathbf{u}^g \\ & -\frac{\partial\sigma_{ij}(\mathbf{u})}{\partial x_j} + \beta\,\frac{\partial\theta}{\partial x_i} + \alpha\, v_i = v_i^g \\ & \beta \operatorname{div}\mathbf{v} - \varepsilon\,\frac{\partial}{\partial x_i}\left(k(x)\,\frac{\partial\theta}{\partial x_i}\right) + \alpha\theta = \theta^g\,. \end{aligned}$$

Eliminating $\mathbf{v}$ this is equivalent to

$$-\frac{\partial \sigma_{ij}(\mathbf{u})}{\partial x_j} + \beta \frac{\partial \theta}{\partial x_i} + \alpha^2 u_i = v_i^g + \alpha u_i^g \tag{1.26}$$

$$\alpha\theta - \varepsilon \frac{\partial}{\partial x_i}\left(k(x)\frac{\partial \theta}{\partial x_i}\right) + \alpha\beta \operatorname{div} \mathbf{u} = \theta^g - \beta \operatorname{div} \mathbf{u}^g . \tag{1.27}$$

In the case $\varepsilon = 0$, we solve (1.27) with respect to θ and we substitute into (1.26), which becomes a standard elasticity problem for the modified elasticity system :

$$-\frac{\partial}{\partial x_j}\left(\sigma_{ij}(\mathbf{u}) + \beta^2 \operatorname{div} \mathbf{u}\right) = \text{given terms} \tag{1.28}$$

which is solved by the standard Lax-Milgram method ; formal integration by parts is performed as above.

The case $\varepsilon > 0$ is in fact analogous ; but in order to solve (1.27) with respect to θ we must study separately the restrictions of θ to B and E. In B, (1.27) is solved with the Neumann boundary condition (1.15). In this case (1.24) is also solvable. ∎

Now, it is easily seen that zero is an eigenvalue of $\mathscr{A}_\varepsilon$. The eigenspace is the kernel of the operator, i.e. the (closed) subspace of the solutions of $\mathscr{A}_\varepsilon \mathbf{U} = 0$. We have immediatly :

PROPOSITION 1.2 : *The kernel of $\mathscr{A}_\varepsilon$ is*

$$\left\{(\mathbf{u}, \mathbf{v}, \theta) \in \mathscr{H} \,;\, \mathbf{v} = 0, -\frac{\partial \sigma_{ij}(\mathbf{u})}{\partial x_j} + \beta \frac{\partial \theta}{\partial x_i} = 0\right\} \quad \textit{for } \varepsilon = 0$$

$$\left\{(\mathbf{u}, \mathbf{v}, \theta) \in \mathscr{H} \,;\, \mathbf{v} = 0, \theta\,|_B = \text{const.}, -\frac{\partial \sigma_{ij}(\mathbf{u})}{\partial x_j} + \beta \frac{\partial \theta}{\partial x_i} = 0\right\} \quad \textit{for } \varepsilon \neq 0 .$$

∎

It is evident that this kernel for $\varepsilon \neq 0$ is a strict subspace of that for $\varepsilon = 0$. We shall see that as $\varepsilon \to 0$ there are « infinitely many » scattering frequencies converging to 0.

2. THE SCATTERING FREQUENCIES

Let us seek for solutions of (1.19) depending on t by the factor $\exp(-\zeta t)$, i.e. of the form

$$\mathbf{U}(x, t) = e^{-\zeta t}\, \mathbf{U}(x) . \tag{2.1}$$

We shall also denote $\zeta = i\omega$, and either ζ or ω will be called the corresponding frequency (in fact the genuine frequency is $\omega/(2\pi)$) : this leads to the system

$$(2.2) \quad \mathscr{A}_\varepsilon \mathbf{U} = \zeta \mathbf{U} \Leftrightarrow \begin{cases} -\mathbf{v} = \zeta \mathbf{u} \\ -\dfrac{\partial \sigma_{ij}(\mathbf{u})}{\partial x_j} + \beta \dfrac{\partial \theta}{\partial x_i} = \zeta v_i \\ \beta \operatorname{div} \mathbf{v} - \varepsilon \dfrac{\partial}{\partial x_i}\left(k(x)\dfrac{\partial \theta}{\partial x_i}\right) = \zeta \theta \end{cases}$$

and we note that, for $x \in E$, i.e. $k(x) = 0$, this is equivalent to

$$(2.3) \qquad -\frac{\partial}{\partial x_j}(\sigma_{ij}(\mathbf{u}) + (\beta^E)^2 \operatorname{div} \mathbf{u}\, \delta_{ij}) = -\zeta^2 u_i$$

which is a modified elasticity system with the Lamé coefficients

$$(2.4) \qquad \lambda + (\beta^E)^2, \mu \quad \text{instead of} \quad \lambda, \mu .$$

Then, the behaviour at infinity of the eventual eigenfunctions is the same as for the elasticity system. It is known that this system is much alike the Laplace equation [1] ; there are no eigenvectors (belonging to the space), and they must be replaced by scattering functions ; the corresponding frequencies are the scattering frequencies, which replace the classical eigenfrequencies. In order to define them, we consider the fundamental solution of the elasticity system (2.3), i.e. the solution $\mathbf{G}^l$ of

$$(2.5) \qquad -\frac{\partial}{\partial x_j}(\sigma_{ij}(\mathbf{G}^l) + (\beta^E)^2 \operatorname{div} \mathbf{G}^l\, \delta_{ij}) + \zeta^2 \mathbf{G}^l_i = \delta_{il}\, \delta$$

where δ denote the Dirac mass at the origin, and δ_{ij} is of course the Kronecker symbol. This function is given by ([1] or [3]) :

$$(2.6) \quad G^l_j(x, \zeta) = \frac{1}{4\pi}\left[\frac{\delta_{jl}}{b^2}\frac{e^{\pm \zeta r/b}}{r} + \frac{1}{\zeta^2}\frac{\partial^2}{\partial x_j\, \partial x_l}\left(\frac{e^{\pm \zeta r/a}}{r} - \frac{e^{\pm \zeta r/b}}{r}\right)\right]$$

for $\zeta \neq 0$, with

$$(2.7) \qquad a^2 = \lambda + (\beta^E)^2 + 2\mu \,; \quad b^2 = \mu \,; \quad r = |x|$$

where the sign + or − are used in the so-called outgoing or incoming fundamental solution, respectively. This denomination is obvious on account of the dependence $\exp(-\zeta t)$ on time. Each one of the solutions depend homomorphically on $\zeta \in \mathbf{C}$ and they both become for $\zeta = 0$ the fundamental solution of the static elasticity system :

$$G_j^l(x,0) = \frac{1}{4\pi}\left[\frac{\delta_{jl}}{b^2}\frac{1}{r} + \left(\frac{1}{a^2} - \frac{1}{b^2}\right)\frac{\partial^2 r}{\partial x_j\,\partial x_l}\right]. \tag{2.8}$$

Remark 2.1 : The fundamental solution for $\zeta = 0$ (2.8) is homogeneous of degree -1, and consequently it enjoys the behaviour at infinity

$$G(x,0) \approx r^{-1}\,; \quad \frac{\partial G(x,0)}{\partial x_i} \approx r^{-2}\,; \quad \frac{\partial^2 G(x,0)}{\partial x_i\,\partial x_j} \approx r^{-3}\,, \quad r \to \infty \tag{2.9}$$

but this is not true for $\zeta \neq 0$, (2.6).This property in the case $\zeta = i\omega$, ω real, is associated with the Sommerfeld radiation condition at infinity and energy flux at large distance, but the case $\zeta = 0$ is singular in this respect.

DEFINITION 2.2 : *The scattering functions and frequencies of the thermoelasticity system* (2.2) *are the solutions* $\mathbf{u} \neq 0$ *of the system of equations* (2.2) *which are outgoing, i.e. they are, for sufficiently large* $|x|$, *convolutions of the outgoing fundamental solution (i.e. (2.6) with sign +) with functions or distributions with compact support.* ■

In order to transform the problem of the scattering frequencies into a problem in the bounded domain B, we shall solve the Dirichlet problem in E :

$$\begin{cases} \text{Find } \mathbf{u} \text{ satisfying (2.3) in } E \text{ and} \\ \mathbf{u} = \varphi \text{ on } \Gamma \\ \mathbf{u} \text{ is outgoing} \end{cases} \tag{2.10}$$

where φ is a given element of $\mathbf{H}^{\frac{1}{2}}(\Gamma)$. Outgoing is understood as in Definition 2.2. This problem is well posed unless for some values ζ (the scattering frequencies of the Dirichlet problem in E) which form a discrete set contained in the halfplane Re $\{\zeta\} > 0$. For the values ζ for which (2.10) is solvable, we may compute

$$(\sigma_{ij}(\mathbf{u}) + (\beta^E)^2 \operatorname{div}\mathbf{u}\,\delta_{ij})\,n_j|_\Gamma = (\mathcal{T}(\zeta)\,\varphi)_i \tag{2.11}$$

where the right side constitutes a definition of the operator $\mathcal{T}$. In this connection we have the following proposition, which is the exact analogue of Proposition IX.3.8 of [6] :

PROPOSITION 2.3 : *The operator* $\mathcal{T}(\zeta)$ *defined in* (2.11) *is a meromorphic function of* ω *(with poles at the scattering frequencies of problem* (2.10), *with values in* $\mathcal{L}\left(\mathbf{H}^{\frac{1}{2}}(\Gamma),\ \mathbf{H}^{-\frac{1}{2}}(\Gamma)\right)$. *In particular, it is holomorphic in a neighbourhood of* $\zeta = 0$. ■

LEMMA 2.4 : *For* $\zeta = 0$ *the operator* $\mathcal{T}(0)$ *satisfies*

$$(2.12) \quad \langle \mathcal{T}(0)\, \varphi, \psi \rangle_{\mathbf{H}^{-\frac{1}{2}}, \mathbf{H}^{\frac{1}{2}}} = - \int_E [a^E_{ijlm}\, e_{lm}(\mathbf{u}^\varphi)\, e_{ij}(\mathbf{u}^\psi) + (\beta^E)^2 \operatorname{div} \mathbf{u}^\varphi \operatorname{div} \mathbf{u}^\psi]\, dx$$

for any $\varphi, \psi \in \mathbf{H}^{\frac{1}{2}}$, *where* $\mathbf{u}^\varphi$, $\mathbf{u}^\psi$ *denote the corresponding solutions of* (2.10).

Proof: As 0 is not a scattering frequency of (2.10), $\mathbf{u}^\varphi$ and $\mathbf{u}^\psi$ are well defined. Let us write (2.10) (and (2.3)) with φ, $\mathbf{u}^\varphi$. Taking the product with $\mathbf{u}^\psi$ and integrating by parts on $E_R = E \cap \{|x| < R\}$ for some large R, we obtain an expression analogous to (2.12) but with the right side integrated on E_R instead of E and the supplementary term

$$(2.13) \qquad \int_{|x| = R} [\sigma_{ij}(\mathbf{u}^\varphi) + (\beta^E)^2 \operatorname{div} \mathbf{u}^\varphi\, \delta_{ij}]\, n_j\, u_i^\psi\, dS$$

but $\mathbf{u}^\varphi$, $\mathbf{u}^\psi$, which are convolutions of $G(x, 0)$, behave at infinity as (2.9). Then, letting $R \to \infty$, we get (2.12). ■

In order to transform the scattering frequency problem into a problem on the bounded domain B, we shall define the sesquilinear form on $\mathbf{H}^1(B)$:

$$(2.14) \quad a(\zeta\, ; \mathbf{u}, \mathbf{w}) = \int_B a^B_{ijlm}\, e_{lm}(\mathbf{u})\, e_{ij}(\bar{\mathbf{w}})\, dx - \langle \mathcal{T}(\zeta)\, \mathbf{u}|_\Gamma, \mathbf{w}|_\Gamma \rangle_{\mathbf{H}^{-\frac{1}{2}}, \mathbf{H}^{\frac{1}{2}}}$$

and the corresponding operator of $\mathcal{L}(\mathbf{H}^1(B), \mathbf{H}^1(B)')$:

$$(2.15) \qquad \langle A(\zeta)\, \mathbf{u}, \mathbf{w} \rangle_{\mathbf{H}^1(B)', \mathbf{H}^1(B)} = a(\zeta, \mathbf{u}, \mathbf{w}) \quad \forall \mathbf{u}, \mathbf{w} \in \mathbf{H}^1(B)\,.$$

We then have :

LEMMA 2.5 : *For* $\zeta \in \mathbf{C}$ *(different from the scattering frequencies of the Dirichlet problem in E,* (2.10)), *the form* $a(\zeta)$ *defined in* (2.14) *is continuous on* $\mathbf{H}^1(B)$ *and depends holomorphically on* ζ *for fixed* $\mathbf{u}$, $\mathbf{w}$. *For* $\zeta = 0$ *the form is hermitian. For sufficiently small* $|\zeta|$, *zero belongs to the resolvent set of* $A(\zeta)$, *and* $A(\zeta)^{-1}$ *is holomorphic of* ζ *(with values either in* $\mathcal{L}(\mathbf{H}^1(B)', \mathbf{H}^1(B))$ *or in* $\mathcal{L}(\mathbf{L}^2(B))$. *The same result holds true for* $[A(\zeta) + \zeta^2]^{-1}$.

Proof: The first assertion follows from Proposition 2.3. For $\zeta = 0$, we have, from (2.14) and Lemma 2.2 :

$$(2.16) \quad a(0\, ; \mathbf{u}, \mathbf{w}) = \int_B a^B_{aijlm}\, e_{lm}(\mathbf{u})\, e_{ij}(\bar{\mathbf{w}})\, dx + \int_E [a^E_{ijlm}\, e_{lm}(\mathbf{u}^*)\, e_{ij}(\bar{\mathbf{w}}^*) + (\beta^E)^2 \operatorname{div} \mathbf{u}^* \operatorname{div} \bar{\mathbf{w}}^*]\, dx$$

where $\mathbf{u}^*$, $\mathbf{w}^*$ denote the extension of $\mathbf{u}$ and $\mathbf{w}$ to E defined by the solution of the outer Dirichlet problem (2.10) with $\zeta = 0$ and the data $\varphi = \mathbf{u}|_\Gamma$ or $\mathbf{w}|_\Gamma$. We note that the traces of $\mathbf{u}$ and $\mathbf{u}^*$ coincide on Γ (as well as those of $\mathbf{w}$ and $\mathbf{w}^*$). Thus, the form $a(0)$ is hermitian. Then, it is classical (see [2] or [6], chapter V, if necessary) that the corresponding operator $A(\zeta)$ is holomorphic with values in $\mathscr{L}(\mathbf{H}^1(B)', \mathbf{H}^1(B))$ or even in $\mathscr{L}(\mathbf{L}^2(B))$, when taking the restriction to $\mathbf{L}^2(B)$, i.e. considered as an unbounded operator in $\mathbf{L}^2(B)$. Moreover, because of the compact imbedding of $\mathbf{H}^1(B)$ into $\mathbf{L}^2(B)$, it is an operator with compact resolvent. As $a(0)$ is hermitian, $A(0)$ is selfadjoint. From (2.16) we see that $a(0, \mathbf{w}, \mathbf{w}) \geqq 0$ and consequently the eigenvalues of $A(0)$ are real and $\geqq 0$. In addition, 0 is not an eigenvalue. For, from (2.16),

$$a(0, \mathbf{u}, \mathbf{u}) = 0 \Rightarrow e_{ij}(\mathbf{u}) = 0\,, \quad e_{ij}(\mathbf{u}^*) = 0 \tag{2.17}$$

which shows that $\mathbf{u}^*$ and $\mathbf{u}$ are rigid motions in E and B. In fact they are the same rigid motion on $\mathbf{R}^3$ because the traces of $\mathbf{u}^*$ and $\mathbf{u}$ coincide on Γ. Moreover, as we saw at the end of the proof of Lemma 2.4, $\mathbf{u}^*$ tends to zero at infinity. Then, the rigid motion vanishes and (2.17) implies $\mathbf{u} = 0$. This shows that 0 is not an eigenvalue of $A(0)$ and then $A(0)^{-1}$ belongs to $\mathscr{L}(\mathbf{H}^{1'}, \mathbf{H}^1)$ or $\mathscr{L}(\mathbf{L}^2)$. According to classical holomorphic perturbation theory, $A(\zeta)^{-1}$ and $[A(\zeta) + \zeta^2]^{-1}$ are well defined and holomorphic in a neighbourhood of $\zeta = 0$. ■

Now we are able to write the scattering problem as a functional problem on B:

PROPOSITION 2.6 : *For $\zeta \in \mathbf{C}$ (different from the scattering frequencies of the Dirichlet problem in E* (2.10)), *the problem of finding the scattering frequencies and functions of the thermoelasticity problem* (*Definition* 2.2) *is equivalent to find $\zeta \in \mathbf{C}$, $\mathbf{u} \in \mathbf{H}^1(B)$, $\theta \in L^2(B)$ (not both vanishing) such that*

$$a(\zeta\,; \mathbf{u}, \mathbf{w}) - \int_B \beta^B\, \theta \operatorname{div} \bar{\mathbf{w}}\, dx = -\,\zeta^2 (\mathbf{u}, \mathbf{w})_{\mathbf{L}^2(B)} \quad \forall \mathbf{w} \in \mathbf{H}^1(B) \tag{2.18}$$

$$-\,\varepsilon\, \Delta\theta = \zeta\,(\theta + \beta^B \operatorname{div} \mathbf{u}) \quad in \quad B \tag{2.19}$$

$$\partial\theta/\partial n = 0 \quad on\, \Gamma\ for\ \varepsilon > 0 \tag{2.20}$$

where « equivalent » means that $\mathbf{u}$ *in* (2.18)-(2.20) *must be extended to E by the solution of* (2.10) *with the corresponding ζ and the datum $\varphi = \mathbf{u}|_\Gamma$.*

. *Proof:* Definition 2.2 is equivalent to

$$-\frac{\partial \sigma_{ij}(\mathbf{u})}{\partial x_j} + \beta\, \frac{\partial\theta}{\partial x_i} = -\,\zeta^2\, u_i \tag{2.21}$$

$$(2.22) \qquad - \varepsilon \frac{\partial}{\partial x_i} \left(k(x) \frac{\partial \theta}{\partial x_i} \right) = \zeta (\theta + \beta \operatorname{div} \mathbf{u})$$

in B and E, with the transmission and boundary conditions (1.14), (1.15) on Γ, and $\mathbf{u}$ outgoing. Then, solving in E, we see that this is equivalent to (2.21), (2.22) in B with the boundary conditions on Γ :

$$(2.23) \quad \sigma_{ij}^T n_j = (\mathcal{T}(\zeta)\, \mathbf{u}|_\Gamma)_i \quad \text{and} \quad \partial\theta/\partial n = 0 \quad \text{for} \quad \varepsilon > 0 ,$$

because $\sigma_{ij}^T n_j$ and $\mathbf{u}$ take the same value on both sides of Γ. Now, (2.18)-(2.20) is merely some kind of variational formulation of this last problem. ■

Let us think about (2.18), (2.19) with $\varepsilon = 0$, $\zeta \neq 0$. Solving (2.19) with respect to θ and inserting it into (2.18) we see that this problem is equivalent to the purely elastic problem with the coefficients

$$(2.24) \qquad a_{ijlm} + \delta_{ij}\, \delta_{lm}\, \beta^2 .$$

As these coefficients take in general different values in E and B, this is in fact a diffraction problem of elastic waves by the obstacle B. This problem has in general scattering frequencies ζ_j, which form a discrete set with $\operatorname{Re} \{\zeta_j\} > 0$.

The preceding Proposition furnishes in particular a description of the scattering frequencies in a neighbourhood of $\zeta = 0$ (as $\zeta = 0$ is not a scattering frequency of (2.10)). This will be used in next section to prove our main result. Nevertheless, we also have the following property of continuity of the scattering frequencies with respect to ε, which is proved exactly as Proposition VII, 9.6 of [6] :

PROPOSITION 2.7 : *Let Δ be an open domain of the complex plane the closure of which does not intersect the real axis and do not contain scattering frequencies of* (2.10). *Then, the scattering frequencies of the thermoelasticity problem $\zeta(\varepsilon)$ contained in Δ are continuous functions of ε which converge, as $\varepsilon \searrow 0$ to the scattering frequencies of the elasticity problem with the coefficients* (2.24). *Here continuous is taken in the classical sense of perturbation of eigenvalues : an eigenvalue may split into several ones.* ■

3. SCATTERING FREQUENCIES IN THE VICINITY OF THE ORIGIN

In order to study the scattering frequencies near $\zeta = 0$, we shall perform the dilatation

$$(3.1) \qquad \zeta = \varepsilon z$$

where z is the new spectral parameter, which we consider in any bounded region of $\mathbf{C}$ (and then ζ of order $O(\varepsilon)$). The equations (2.19), (2.20) become

$$-\Delta_N \theta = z(\theta + \beta^B \operatorname{div} \mathbf{u}) \quad \text{in } B \tag{3.2}$$

where Δ_N denotes the Laplacian with Neumann boundary condition. Now we shall « solve » (2.18) with respect to $\mathbf{u}$ and substitute into (3.2) to obtain a functional equation in θ. Let us define an operator $B \in \mathscr{L}(L^2(B), \mathbf{H}^1(B)')$ by :

$$\langle B\theta, \mathbf{w} \rangle_{\mathbf{H}^{1'}, \mathbf{H}^1} = \int_B \beta^B \theta \operatorname{div} \mathbf{w} \, dx \quad \forall \mathbf{w} \in \mathbf{H}^1(B) . \tag{3.3}$$

Then, (2.18) becomes :

$$[A(\zeta) + \zeta^2] \mathbf{u} = B\theta . \tag{3.4}$$

According to Lemma 2.5, this equation may be solved in a neighbourhood of $\zeta = 0$ by

$$\mathbf{u} = [A(\zeta) + \zeta^2]^{-1} B\theta . \tag{3.5}$$

In order to substitute this into (3.2), we define the operator $K(\zeta)$ by :

$$K(\zeta)\theta = \beta^B \operatorname{div} \{ [A(\zeta) + \zeta^2]^{-1} B\theta \} . \tag{3.6}$$

LEMMA 3.1 : *The operator $K(\zeta)$ is well defined for ζ in a neighbourhood of the origin. It is there a holomorphic function with values in $\mathscr{L}(L^2(B))$. The operator $I + K(\zeta)$ is an isomorphism of $L^2(B)$ for sufficiently small $|\zeta|$. Moreover, $K(0)$ is hermitian and*

$$(K(0)\theta, \theta)_{L^2(B)} \geqq 0 \quad \forall \theta \in L^2(B) . \tag{3.7}$$

Proof: The first part is obvious from Lemma 2.5. Let us study $K(0)$. Let θ and ϕ be arbitrary elements of $L^2(B)$. According to (3.5) with $\zeta = 0$, let $\mathbf{u}^\theta$, $\mathbf{u}^\varphi$ be the corresponding solutions of (3.4), i.e.

$$A(0)\mathbf{u}^\varphi = B\varphi ; \quad A(0)\mathbf{u}^\theta = B\theta \tag{3.8}$$

then, let us take the scalar product of the first one with $\mathbf{u}^\theta$ (in fact the duality product between $\mathbf{H}^{1'}$ and $\mathbf{H}^1$) : by virtute of (2.15) and (3.3), we have :

$$a(0 ; \mathbf{u}^\varphi, \mathbf{u}^\theta) = \int_B \beta^B \varphi \operatorname{div} \bar{\mathbf{u}}^\theta \, dx . \tag{3.9}$$

Moreover, from (3.6),

$$K(0)\theta = \beta^B \operatorname{div} \mathbf{u} \tag{3.10}$$

thus

$$(\varphi, K(0)\,\theta)_{L^2(B)} = \int_B \beta^B \varphi \operatorname{div} \bar{\mathbf{u}}^\theta \, dx \tag{3.11}$$

and by comparison with (3.9) and using Lemma 2.5, we see that this expression is hermitian. Moreover, taking in (3.9), (3.11) $\varphi = \theta$ we see that (3.7) follows from (2.16). It is then classical that $[I + K(0)]^{-1} \in \mathscr{L}(L^2(B))$ (see [6], Theorem III.6.5 if necessary). The same result remains valid for sufficiently small $|\zeta|$. Thus, $I + K(\zeta)$ is an isomorphism. ■

Now, (3.2) becomes

$$-\Delta_N \theta = z[I + K(\varepsilon z)]\,\theta \tag{3.2}$$

which is an implicit eigenvalue problem in $L^2(B)$. We shall write this under a more classical form by applying the isomorphism $[I + K(\zeta)]^{-1}$ (Lemma 3.1) to (3.12), which becomes :

$$\Lambda(\zeta)\,\theta = z\theta \; ; \quad \zeta = \varepsilon z \,, \tag{3.13}$$

where

$$\Lambda(\zeta) \equiv [I + K(\zeta)]^{-1}\,(-\Delta_N)\,. \tag{3.14}$$

We then have :

LEMMA 3.2 : *$\Lambda(\zeta)$ with sufficiently small $|\zeta|$ is a family of holomorphic unbounded operators of $L^2(B)$. For $\zeta = 0$, $\Lambda(0)$ has eigenvalues, noted $z_i(0)$ which are real and positive, tending to $+\infty$ as $i \to \infty$ and with finite multiplicity. According to classical holomorphic perturbation theory, $\Lambda(\zeta)$ has the eigenvalues $z_i(\zeta)$ which are algebroid functions of ζ, i.e. they are holomorphic functions of some fractional power $\zeta^{1/p}$ of ζ, p integer > 0.*

Proof: Let us consider the domain of $-\Delta_N$ as an unbounded operator of $L^2(B)$, i.e. :

$$D(-\Delta_N) = \{\theta \in H^2(B),\ \partial\theta/\partial n = 0 \quad \text{on } \Gamma\} \tag{3.15}$$

which is a Banach space either for the norm of H^2 or the graph norm of $-\Delta_N$.

Let us consider $\Lambda(\zeta)$ as an unbounded operator on $L^2(B)$ with domain $D(-\Delta_N)$. As $[I + K(\zeta)]^{-1}$ is an isomorphism, $\Lambda(\zeta)$ is closed, as it is easily seen.

Moreover, its domain is independent of ζ and $\Lambda(\zeta)\,\theta$ with a fixed $\theta \in D(-\Delta_N)$ is holomorphic. Thus, $\Lambda(\zeta)$ is a holomorphic family of unbounded operators [2] or [6], and as a consequence, the isolated eigenvalues of finite multiplicity are algebroid functions of the parameter ζ.

It only remains to prove the conclusions on the eigenvalues of $\Lambda(0)$. Using again the isomorphism, we write the eigenvalue problem for $\Lambda(0)$ under the form

$$(- \Delta_N)\,\theta = z[I + K(0)]\,\theta \quad \text{in } L^2(B) \tag{3.16}$$

but $-\Delta_N$ and $K(0)$ are selfadjoint and positive ; it then follows easily that the eigenvalues are real and positive (of course, $z = 0$ is an eigenvalue, with the eigenfunction $\theta = \text{const.}$). Now, in order to prove that the eigenvalues $z_i(0)$ actually exist and form an infinite sequence, it sufficies to prove that $\Lambda(0)$ is an operator with compact resolvent. Let us take $\mu > 0$ and consider

$$[\Lambda(0) + \mu I]\,\theta = f \quad \text{in } L^2(B)\,. \tag{3.17}$$

We shall see that $[\Lambda(0) + \mu I]^{-1}$ is well defined and compact in $L^2(B)$. We see that (3.17) is equivalent to

$$\{- \Delta_N + \mu[I + K(0)]\}\;\theta = [I + K(0)]\,f$$

and the left side is the operator associated by the classical Lax-Milgram theory to the form

$$(\text{grad}\,\theta, \text{grad}\,\tilde{\theta})_{L^2(B)} + \mu([I + K(0)]\,\theta, \tilde{\theta})_{L^2(B)}$$

on $H^1(B)$, which is coercive by (3.7). Then, the resolvent is well defined and continuous from $L^2(B)$ into $H^1(B)$ (and even into $D(-\Delta_N)$) and thus compact in $L^2(B)$, Q.E.D. ■

Coming back to (3.12), finding z as a function of ε is equivalent to « solve » the implicit equation

$$z = z_i(\varepsilon z) \tag{3.18}$$

for each i, where z_i are the functions quoted in Lemma 3.2. We shall disregard the first eigenvalue, $z_i(\zeta) \equiv 0$, corresponding to the eigenfunction $\theta = \text{const.}$ As we said in Lemma 3.2, $z_i(\zeta)$ has in general an algebraic singularity as $\zeta = 0$, i.e. it is a p-valued function which is expressed as a holomorphic function f of $\zeta^{1/p}$ with the p values of it. In order to use the implicit function theorem for holomorphic functions, we use the same device as in [5] : we write

$$\zeta^{1/p} = \varepsilon^{1/p}\, z^{1/p}\,, \quad z^{1/p} = \tilde{z}\,, \quad \varepsilon^{1/p} = \eta \tag{3.19}$$

and (3.18) becomes

$$F(\tilde{z}, \eta) = 0 \quad F(\tilde{z}, \eta) \equiv \tilde{z}^p - f_i(\eta\tilde{z})\,. \tag{3.20}$$

In order to solve in a neighbourhood of

$$\tilde{z} = z_i(0)^{1/p}\,, \quad \eta = 0\,,$$

we check that at this point, $\partial F/\partial \tilde{z} \neq 0$. Then we obtain the implicit function $\tilde{z}(\eta)$ and then

$$z = \tilde{z}^p(\varepsilon^{1/p})$$

which is a p-valued function of ε, which we shall denote $f_i(\varepsilon^{1/p})$. Finally, $\zeta = \varepsilon f(\varepsilon^{1/p})$, and we have proved the following :

THEOREM 3.3 : *The considered thermoelasticity problem, with small* ε, *has infinitely many scattering functions* ζ *near the origin, of the form*

$$\zeta = \varepsilon f_i(\varepsilon^{1/p}) \tag{3.20}$$

which have in general algebraic singularities (*i.e. each one is a holomorphic function of some root* $\varepsilon^{1/p}$ *of* ε). *The values* $f_i(0)$ *are real and positive, and form a sequence tending to* $+\infty$. *It should be noticed that all the functions* (3.20) *are not necessarily defined simultaneously for sufficiently small* ε ; *but, taking a finite number of them, they are well defined for sufficiently small* ε.

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