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SHARP MAXIMUM NORM ERROR ESTIMATES FOR GENERAL MIXED FINITE ELEMENT APPROXIMATIONS TO SECOND ORDER ELLIPTIC EQUATIONS (*)

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Abstract. — *In this paper we analyze the approximation properties in* L^∞ *of mixed finite elements for second order elliptic equations. The analysis relies on abstract assumptions on the finite element spaces involved and holds for the whole range of the index k of the discrete spaces. Sharp asymptotic* L^∞*-error estimates are derived for both the scalar and the vector fields. A superconvergence estimate is proved for the* L^2*-projection of the scalar unknown. As a result, the modified scalar field provided by a suitable element-by-element postprocessing is shown to be superconvergent in* L^∞*. A hybridization of the mixed method is introduced and the additional information provided by Lagrange multipliers is discussed in maximum norm. The required abstract properties are enjoyed by Raviart-Thomas-Nedelec mixed finite elements in 2-D and 3-D as well as by the new families of mixed finite elements recently introduced by Brezzi-Douglas-Marini in 2-D and by Brezzi-Douglas-Durán-Fortin in 3-D.*

Résumé. — *On analyse les propriétés dans* L^∞ *des approximations par éléments finis mixtes pour les équations elliptiques du deuxième ordre. L'analyse est basée sur des hypothèses abstraites pour les espaces d'éléments finis et s'applique dans des conditions très générales. On en déduit des estimations optimales en norme* L^∞ *pour les champs scalaire et vectoriel. On démontre une estimation de superconvergence pour la projection* L^2 *de l'inconnue scalaire. De plus le champ scalaire modifié, obtenu par un postprocessing effectué élément par élément, est superconvergent en* L^∞*. On introduit la formulation mixte-hybride de la méthode et on analyse dans* L^∞ *les informations supplémentaires fournies par les multiplicateurs de Lagrange. Les hypothèses abstraites que nous utilisons sont vérifiées par les éléments finis mixtes de Raviart-Thomas-Nedelec en 2-D et 3-D et aussi par les éléments récemment introduits par Brezzi-Douglas-Marini en 2-D et par Brezzi-Douglas-Durán-Fortin en 3-D.*

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1. INTRODUCTION

New families of mixed finite elements to approximate second order elliptic problems have been introduced recently as alternatives to the usual Raviart-Thomas-Nedelec spaces [15, 20]. Two families, one based on simplices and the other on rectangles, were proposed by Brezzi-Douglas-Marini in two dimensions [6] ; the natural analogues in three dimensions were presented by Brezzi-Douglas-Durán-Fortin [4]. The key idea in constructing these new discrete spaces was to modify the standard mixed finite elements of Raviart-Thomas-Nedelec but preserving their main property, which is expressed in the following commutative diagram :

$$\begin{array}{ccc} [H^1(\Omega)]^n & \xrightarrow{\text{div}} & L^2(\Omega) \\ \Pi_h^k \downarrow & & \downarrow P_h^k \\ \mathbf{X}_h^k & \xrightarrow{\text{div}} & \mathbf{M}_h^k \end{array}$$

Here $\mathbf{X}_h^k$ and $\mathbf{M}_h^k$ stand for the discrete spaces of order $k \geq 0$, $n = 2, 3$ denotes the dimension and Π_h^k, P_h^k are suitable interpolation operators [4, 5, 6, 7, 15, 20] ; a precise definition is given in section 5. This property implies the inf-sup condition of Brezzi [3] and, consequently, the stability of the discrete scheme. Moreover, it simplifies the L^2-error analysis giving also a superconvergence error estimate for the scalar field as shown by Douglas-Roberts [9] and Johnson-Thomee [13].

Within this framework the question whether the above diagram is the suitable abstract setting to analyze the L^∞-accuracy arises quite naturally. The primary aim of this paper is to derive sharp error bounds in L^∞ under this general setting and, next, to apply the abstract results already obtained to getting rates of convergence for the above families of mixed finite elements in 2 and 3-D. Moreover, we prove that the difference between the L^2-projection of the scalar field and the discrete solution superconverges in L^∞. This fact allows us to show that the modified scalar field produced element-by-element by a simple postprocess is asymptotically more accurate in L^∞. As a second by-product we can analyze the hybridization process and exploit the further information provided by Lagrange multipliers. Our basic tool in proving the abstract L^∞-results is, as usual, the Nitsche's method of weighted Sobolev norms [16, 17, 8, 14, 18].

Maximum norm error estimates for Raviart-Thomas-Nedelec spaces were obtained by Scholz [21, 22, 23], but excluding the lowest order approximation which is undoubtly the most important in practice. Moreover, Douglas-Roberts [9] and Johnson-Thomee [13] proved L^{∞}-error estimates for the scalar unknown in 2-D that hold for the entire range of the index k, but their techniques do not lead to an error bound for the vector unknown. A full error analysis in 2-D for both variables was recently done by Gastaldi-Nochetto [11, 12]. While this paper was being written, we learned that Durán [10] had derived sharp L^p-error estimates ($1 \leq p \leq \infty$) for the above families as well as for the Brezzi-Douglas-Marini spaces. However, his technique does not apply in more than two dimensions.

The new families of mixed finite elements introduced in [4, 5, 6,7] possess the same asymptotic accuracy for the vector unknown as the Raviart-Thomas-Nedelec ones but at lower computational cost. So they are designed to be competitive in approximating the variable for which mixed methods are known to work better. We shall be mainly concerned with these new families because the standard ones in 2-D were recently treated in [11, 12]. Moreover the present ideas are a natural extension of those in [12] where a rather general second order elliptic operator was considered ; hence we restrict ourselves to the Laplacian. For u and p being the scalar and vector fields associated with $-\Delta u = f$, our main results are summarized in table 1.1.

TABLE 1.1
Asymptotic L^{∞}-Error Estimates.

	R–T–N	B–D–M & B–D–D–F
$\|p - p_h\|_{L^\infty}$	$h^{k+1}\lvert\log h\rvert \, \|f\|_{W^{k,\infty}}$	$h^{k+1}\lvert\log h\rvert \, \|f\|_{W^{k,\infty}}$
$\|u - u_h\|_{L^\infty}$	$h^{k+1}\lvert\log h\rvert \, \|f\|_{W^{k,\infty}}$	$h^{k} \, \|f\|_{W^{k,\infty}}$
$\|u - u_h^*\|_{L^\infty}$	$h^{k+2}\lvert\log h\rvert^2 \, \|f\|_{W^{k,\infty} \cap W^{1,n}}$	$h^{k+2-\delta_{k1}}\lvert\log h\rvert^{2-\delta_{k1}} \, \|f\|_{W^{k,\infty}}$

There u_h and p_h stand for the discrete scalar and vector fields and u_h^* indicates the modified scalar field. Some logarithmic factors can be removed under slightly stronger assumptions on u and f (see section 5).

The paper is organized as follows. In section 2 we state the notation and the abstract assumptions under which the error analysis holds. In Sections 3 recall some technical results for weighted Sobolev norms and we prove other new ones in n dimensions. The proof of the L^{∞}-error estimates is carried out in section 4 ; we demonstrate that the abstract framework provides optimal error bounds according to the approximation theory. This is so for the whole range of the index k with exception of the lowest order method for which a logarithmic factor occurs. Moreover we construct a modifed scalar field which is asymptotically more accurate in L^{∞}. These abstract results are applied in section 5 to the new families and the standard ones in 2-D and 3-D. Finally in section 6 we analyze in L^∞ the hybrid formulation.

2. NOTATION AND ABSTRACT ASSUMPTIONS

Let Ω be a regular bounded domain in $\mathbf{R}^n (n \geq 2)$ and let u be the unique solution of the following *model problem*

$$u \in H_0^1(\Omega) : -\Delta u = f \text{ in } \Omega . \tag{2.1}$$

The associated vector unknown p is defined by

$$p = -\operatorname{grad} u . \tag{2.2}$$

Let $\{\tau_h\}_h$ be a family of *regular* and *quasi-uniform* decompositions of Ω into triangles or rectangles in 2-D, and the corresponding generalizations in higher dimensions ; here $h > 0$ denotes the mesh-size. Boundary finite elements are allowed to have one curvilinear edge [4, 6, 7] ; so we are implicitly assuming that $\Omega = U\{T : T \in \tau_h\}$. This simplification does not yield a loss of generality [13].

Let us now introduce the functional spaces we shall work with, namely

$$\begin{aligned} &\mathbf{X} := H(\operatorname{div} ; \Omega) = \{q \in [L^2(\Omega)]^n : \operatorname{div} q \in L^2(\Omega)\} , \\ &\tilde{\mathbf{X}} := \{q \in X : q|_T \in [H^1(T)]^n \quad \forall T \in \tau_h\} , \\ &\mathbf{M} := L^2(\Omega) . \end{aligned} \tag{2.3}$$

The mixed formulation of problem (2.1) is the following first order system : seek a pair $\{u, p\} \in \mathbf{M} \times \mathbf{X}$, such that

$$\begin{aligned} \langle p, q \rangle - \langle \operatorname{div} q, u \rangle &= 0 , \quad \forall q \in \mathbf{X} , \\ \langle \operatorname{div} p, v \rangle &= \langle f, v \rangle , \quad \forall v \in \mathbf{M} , \end{aligned} \tag{2.4}$$

where $\langle . , . \rangle$ stands for the inner product in $\mathbf{M}$. Let $\mathbf{X}_h^k$ and $\mathbf{M}_h^k$ be the finite dimensional approximating spaces of order $k \geq 0$ [4, 6, 7, 15, 20] ; namely they satisfy globally $\mathbf{X}_h^k \subset \mathbf{X}$, $\mathbf{M}_h^k \subset \mathbf{M}$ and locally

$$[\mathbf{P}^k(T)]^n \subset \mathbf{X}_h^k|_T , \quad \mathbf{P}^j(T) \subset \mathbf{M}_h^k|_T \tag{A1}$$

for all $T \in \tau_h$ where either $j = k$ or $j = k - 1$ provided $k \geq 1$. Here $\mathbf{P}^l(T)$ denotes the set of polynomials of total degree l restricted to $T \in \tau_h$. The mixed finite element approximation of (2.1) reads as follows : find a pair $\{u_h, p_h\} \in \mathbf{M}_h^k \times \mathbf{X}_h^k$ such that

$$\begin{aligned} \langle p_h, q_h \rangle - \langle \operatorname{div} q_h, u_h \rangle &= 0, \quad \forall q_h \in \mathbf{X}_h^k , \\ \langle \operatorname{div} p_h, v_h \rangle &= \langle f, v_h \rangle , \quad \forall v_h \in \mathbf{M}_h^k . \end{aligned} \tag{2.5}$$

Let P_h^k denote the L^2-projection operator onto $\mathbf{M}_h^k$; since $\mathbf{M}_h^k$ is defined without continuity constraints, P_h^k is local. Let Π_h^k be a local interpolation operator which satisfies the following commutative diagram [4, 6, 7, 15, 20] :

$$\begin{array}{ccccc} \tilde{\mathbf{X}} & \xrightarrow{\text{div}} & \mathbf{M} & & \\ \Pi_h^k \downarrow & & \downarrow P_h^k & & \\ \mathbf{X}_h^k & \xrightarrow{\text{div}} & \mathbf{M}_n^k & \longrightarrow & 0 . \end{array}$$

In other words, div $(\mathbf{X}_h^k) = \mathbf{M}_h^k$ and

$$\text{(2.6)} \qquad \operatorname{div} \Pi_h^k q = P_h^k \operatorname{div} q , \quad \forall q \in \tilde{\mathbf{X}} .$$

In particular, (2.6) holds for all $q \in [H^1(\Omega)]^n$. These operators are assumed to satisfy *L^p-approximation properties* [4, 6, 7, 15, 20], namely for all $v \in W^{j+1,p}(\Omega)$ and $q \in [W^{k+1,p}(\Omega)]^n$ with $2 \leqslant p \leqslant \infty$

$$\text{(A3)} \qquad \begin{aligned} &\| v - P_h^k v \|_{L^p(\Omega)} \leqslant C h^{j+1} \| v \|_{W^{j+1,p}(\Omega)} , \\ &\| q - \Pi_h^k q \|_{L^p(\Omega)} + h \| \operatorname{div} (q - \Pi_h^k q) \|_{L^p(\Omega)} \leqslant C h^{k+1} \| q \|_{W^{k+1,p}(\Omega)} . \end{aligned}$$

The primary tool in our error analysis will be the use of weighted Sobolev norms. This technique was introduced by Nitsche [16, 17] and Natterer [14] for conforming finite element methods, and was first applied by Scholz [21, 22, 23] and more recently by Gastaldi-Nochetto [11, 12] for Raviart-Thomas-Nedelec mixed methods in 2-D. Let us now introduce the corresponding notation and recall some elementary properties. The weight function σ is defined by

$$\text{(2.7)} \qquad \sigma(x) := (|x - x_0|^2 + \theta^2)^{1/2}, \quad x, x_0 \in \Omega ,$$

where $|.|$ denotes the Euclidean distance in $\mathbf{R}^n$ and $\theta = C^* h$ with $C^* \geqslant 1$ being a constant to be specified later on. Then σ satisfies the following non-oscillation property [17, p. 295] :

$$\text{(2.8)} \qquad \max_{x \in T} \sigma(x) \leqslant C \min_{x \in T} \sigma(x), \quad \forall T \in \tau_h .$$

For $\alpha \in \mathbf{R}$ and $i \in \mathbf{N}$, the weighted Sobolev norms are defined by

$$\text{(2.9)} \qquad \| D^i v \|_{\sigma^\alpha}^2 := \sum_{|\lambda| = i} \langle \sigma^\alpha \partial^\lambda v, \partial^\lambda v \rangle , \quad \forall v \in H^i(\Omega) .$$

A trivial consequence of (2.7), (A3) and the local character of P_h^k and Π_h^k is that they approximate well also in weighted norms ; more precisely

$$\|v - P_h^k v\|_{\sigma^\alpha} \leq C h^{j+1} \|D^{j+1} v\|_{\sigma^\alpha}, \tag{2.10}$$

$$\|q - \Pi_h^k q\|_{\sigma^\alpha} + h \|\operatorname{div}(q - \Pi_h^k q)\|_{\sigma^\alpha} \leq C h^{k+1} \|D^{k+1} q\|_{\sigma^\alpha}, \tag{2.11}$$

for all $\alpha \in \mathbf{R}$, $v \in H^{j+1}(\Omega)$ and $q \in [H^{k+1}(\Omega)]^n$. A further *superapproximation* property of Π_h^k is required namely,

$$\|\sigma^\beta q_h - \Pi_h^k(\sigma^\beta q_h)\|_{\sigma^\alpha} \leq C(h/\theta) \|q_h\|_{\sigma^{\alpha+2\beta}}, \tag{A4}$$

for all $q_h \in \mathbf{X}_h^k$, and $\alpha, \beta \in \mathbf{R}$ where $C > 0$ does not depend on q_h, h, θ, α and β. This property was proved by Scholz [22] for Raviart-Thomas-Nedelec mixed methods using the Bramble-Hilbert lemma. The same ideas apply in this context.

3. ON WEIGHTED SOBOLEV NORMS

Let us start by recalling two important properties of the weight function σ. The first one relates derivatives of σ^α ($\alpha \in \mathbf{R}$) with powers of σ [17, p. 298] namely,

$$|\partial^i \sigma^\alpha(x)| \leq C(i, \alpha)\, \sigma^{\alpha - i}(x), \quad \forall x \in \Omega. \tag{3.1}$$

The second result is the elementary estimate [17 ; 8, p. 149] :

$$\int_\Omega \sigma^{-\alpha} \leq C \begin{cases} \theta^{n-\alpha}, & \text{if } \alpha > n \\ |\log \theta|, & \text{if } \alpha = n. \end{cases} \tag{3.2}$$

The relations between weighted and L^∞-norms follow then from the previous ones and inverse inequalities. In fact, we have

$$\|v\|_{\sigma^{-\alpha}} \leq C \|v\|_{L^\infty} \begin{cases} \theta^{(n-\alpha)/2}, & \text{for } \alpha > n \\ |\log \theta|^{1/2}, & \text{for } \alpha = n \end{cases}, \quad v \in L^\infty(\Omega), \tag{3.3}$$

$$\|\chi\|_{L^\infty} \leq C(\theta^\alpha / h^n)^{1/2} \|\chi\|_{\sigma^{-\alpha}}, \quad \text{for } \alpha \in \mathbf{R}, \quad \chi \in \mathbf{M}_h^k \text{ (or } \mathbf{X}_h^k), \tag{3.4}$$

where x_0 in (2.7) is chosen in such a way that $|\chi(x_0)| = \|\chi\|_{L^\infty}$. The key idea of Nitsche's method of weighted norms is that through the previous relations it allows one to work in $L^2(\Omega)$ instead of $L^\infty(\Omega)$ and to use duality arguments. Let us conclude by establishing some a priori estimates in weighted norms. The first result is due to Nitsche ; see [16, p. 266 ; 8, p. 160] for $n = 2$ and [18, p. 74] for $n > 2$.

LEMMA 3.1 : *For every* $v \in H_0^1(\Omega) \cap H^2(\Omega)$ *we have*

$$\|D^2 v\|_{\sigma^n} + \|Dv\|_{\sigma^{n-2}} \leqslant C(|\log\theta|^{1/2}/\theta)\|\Delta v\|_{\sigma^{n+2}}. \tag{3.5}$$

The second estimate was proved by Rannacher-Scott in two dimensions [19, p. 442]. We are going to demonstrate it now for any dimension and to extend it also to a critical value of the exponent α, say $\alpha = 2$.

LEMMA 3.2 : *Let* $b \in \mathbf{X}$ *be given and let* $v \in H_0^1(\Omega) \cap H^2(\Omega)$ *be the solution of* $-\Delta v = \operatorname{div} b$. *Then for any* $0 < \alpha < 2$ *we have*

$$\|D^2 v\|_{\sigma^{n+\alpha}} \leqslant C(\alpha)(\|\operatorname{div} b\|_{\sigma^{n+\alpha}} + (1/\theta)\|b\|_{\sigma^{n+\alpha}}). \tag{3.6}$$

Moreover, for $\alpha = 2$ *this estimate degenerates as*

$$\|D^2 v\|_{\sigma^{n+2}} + \|Dv\|_{\sigma^n} \leqslant C(\|\operatorname{div} b\|_{\sigma^{n+2}} + (|\log\theta|^{1/2}/\theta)\|b\|_{\sigma^{n+2}}). \tag{3.7}$$

Proof: By a simple calculation as in [19, p. 443] using the fact that $\partial\Omega$ is regular, and thus $\|w\|_{H^2(\Omega)} \leqslant C\|\Delta w\|_{L^2(\Omega)}$ for any $w \in H_0^1(\Omega) \cap H^2(\Omega)$, we easily arrive at

$$\|D^2 v\|_{\sigma^{n+\alpha}} + \|Dv\|_{\sigma^{n+\alpha-2}} \leqslant C(\|\operatorname{div} b\|_{\sigma^{n+\alpha}} + \|v\|_{\sigma^{n+\alpha-4}}), \quad 0 < \alpha \leqslant 2.$$

Therefore it only remains to estimate $\|v\|_{\sigma^{n+\alpha-4}}$. To this aim let us use the integral representation of v in terms of the Green function $G(.,.)$, namely

$$v(x) = \int_\Omega G(x,y)\operatorname{div} b(y)\,dy = -\int_\Omega DG(x,y)\,.\,b(y)\,dy.$$

Moreover, it is well known that

$$|DG(x,y)| \leqslant C/|x-y|^{n-1}, \quad \text{for all } x, y \in \Omega.$$

Then using Cauchy-Schwarz inequality and Fubini's theorem, in this order, we can write

$$\begin{aligned}\|v\|^2_{\sigma^{n+\alpha-4}} &\leqslant \int_\Omega \sigma(x)^{n+\alpha-4}\left(\int_\Omega \frac{|b(y)|}{|x-y|^{n-1}}\,dy\right)^2 dx \\ &\leqslant \int_\Omega \sigma(y)^{n+\alpha}|b(y)|^2 \times \\ &\quad\times\left\{\int_\Omega \frac{\sigma(x)^{n+\alpha-4}}{|y-x|^{n-1}}\left(\int_\Omega \frac{\sigma(z)^{-(n+\alpha)}}{|z-x|^{n-1}}\,dz\right)dx\right\}dy.\end{aligned}$$

So the assertions follow from the estimate

$$\int_\Omega \frac{\sigma(x)^{n+\alpha-4}}{|y-x|^{n-1}} \left(\int_\Omega \frac{\sigma(z)^{-(n+\alpha)}}{|z-x|^{n-1}} dz \right) dx \leqslant C \begin{cases} 1/\theta^2, & \text{for } 0 < \alpha < 2 \\ |\log \theta|/\theta^2, & \text{for } \alpha = 2 \end{cases}$$

which holds uniformly in $y \in \Omega$. The rest of the proof consists of showing this bound. We proceed in two steps.

Step 1 :

$$\int_\Omega \frac{\sigma(z)^{-(n+\alpha)}}{|z-x|^{n-1}} dz \leqslant C \frac{\sigma(x)^{1-n}}{\theta^\alpha}, \quad \text{for } 0 < \alpha \leqslant 2 .$$

Let us first assume that $|x - x_0| \leqslant \theta$ where $x_0 \in \Omega$ was introduced in (2.7). Then $|x - z|^2 \leqslant 2(\theta^2 + |x_0 - z|^2)$ and using polar coordinates we get

$$\begin{aligned} \int_\Omega \frac{\sigma(z)^{-(n+\alpha)}}{|z-x|^{n-1}} dz &\leqslant \frac{C}{\theta^{n+\alpha}} \int_0^\theta d\rho + C \int_\theta^d \frac{d\rho}{\rho^{n+\alpha}} \\ &\leqslant C/\theta^{\alpha+n-1} \leqslant C\theta^{-\alpha} \sigma(x)^{1-n} \end{aligned}$$

where $d > 0$ stands for the diameter of Ω. Suppose now that $|x - x_0| > \theta$ and, in addition, let Ω be decomposed into the sets

$$\Omega_1 := \{ z \in \Omega : |z - x| \leqslant |x - x_0|/2 \}, \quad \Omega_2 := \Omega \backslash \Omega_1 .$$

For $z \in \Omega_1$ we have $|x_0 - z| \geqslant |x - x_0|/2$; hence

$$\begin{aligned} \int_{\Omega_1} \frac{\sigma(z)^{-(n+\alpha)}}{|z-x|^{n-1}} dz &\leqslant \frac{C}{|x - x_0|^{n+\alpha}} \int_0^{|x-x_0|/2} d\rho \\ &= \frac{C}{|x - x_0|^{\alpha+n-1}} \leqslant C\theta^{-\alpha} \sigma(x)^{1-n} . \end{aligned}$$

For Ω_2 instead we can write

$$\begin{aligned} \int_{\Omega_2} \frac{\sigma(z)^{-(n+\alpha)}}{|z-x|^{n-1}} dz &\leqslant \frac{C}{|x - x_0|^{n-1}} \int_0^d \frac{\rho^{n-1}}{(\theta^2 + \rho^2)^{(n+\alpha)/2}} d\rho \\ &\leqslant \frac{C}{\theta^\alpha |x - x_0|^{n-1}} \leqslant C\theta^{-\alpha} \sigma(x)^{1-n} . \end{aligned}$$

This completes the first step.

Step 2 :

$$\int_\Omega \frac{\sigma(x)^{\alpha-3}}{|y-x|^{n-1}} dx \leqslant \begin{cases} \theta^{\alpha-2}, & \text{for } 0 < \alpha < 2 , \\ |\log \theta|, & \text{for } \alpha = 2 . \end{cases}$$

Let us now decompose Ω into the sets

$$\Omega_3 := \{x \in \Omega : |x - y| \leq |x - x_0|\}, \quad \Omega_4 := \Omega \backslash \Omega_3 .$$

Taking polar coordinates centered at y yields

$$\int_{\Omega_3} \frac{\sigma(x)^{\alpha - 3}}{|x - y|^{n-1}} \, dx \leq C \int_0^d \frac{d\rho}{(\theta^2 + \rho^2)^{(3-\alpha)/2}} \leq C \begin{cases} \theta^{\alpha - 2}, & \text{for } 0 < \alpha < 2 \\ |\log \theta|, & \text{for } \alpha = 2 . \end{cases}$$

The same holds for Ω_4 because the argument is symmetric (take now x_0 as origin). This ends the lemma. □

4. THE ABSTRACT ERROR ANALYSIS

This section is devoted to the error analysis in $L^\infty(\Omega)$ for both the scalar and the vector unknown. The following *error equations* are easily obtained from (2.4) and (2.5) :

$$\langle p - p_h, q_h \rangle - \langle \operatorname{div} q_h, u - u_h \rangle = 0 , \quad \forall q_h \in \mathbf{X}_h^k , \tag{4.1}$$

$$\langle \operatorname{div} (p - p_h), v_h \rangle = 0 , \quad \forall v_h \in \mathbf{M}_h^k . \tag{4.2}$$

Our analysis relies solely on the abstract assumptions (A1) to (A4). The key argument is the combined use of the commutative diagram (A2) and the lemmas 3.1 and 3.2. In particular lemma 3.2 is the suitable tool to avoid logarithmic factors for the vector field. Indeed, this is so for all $j \geq 1$ but for $j = 0$ a logarithmic term still occurs. We also obtain a superconvergence estimate for $P_h^k u - u_h$. This allows us to prove that the modified scalar field provided by a local postprocess superconverges with an optimal rate up to a logarithmic factor.

Let us now state our main result.

THEOREM 4.1 : *Let* $\{u, p\}$ *and* $\{u_h, p_h\}$ *be the solutions of* (1.4) *and* (1.5) *respectively. Then for every* $k \geq 0$ *there exists a positive constant* $C > 0$ *independent of h, such that the following estimates hold*

$$\|p - p_h\|_{L^\infty(\Omega)} \leq C \left\{ \|p - \Pi_h^k p\|_{L^\infty(\Omega)} + h |\log h|^{\delta_{j0}} \|f - P_h^k f\|_{L^\infty(\Omega)} \right\} , \tag{4.3}$$

$$\|P_h^k u - u_h\|_{L^\infty(\Omega)} \leq C \left\{ h |\log h| \, \|p - \Pi_h^k p\|_{L^\infty(\Omega)} + h^2 |\log h|^{1 + \delta_{j0}} \|f - P_h^k f\|_{L^\infty(\Omega)} + \delta_{j0} \, h |\log h| \, \|f - P_h^k f\|_{L^n(\Omega)} \right\} , \tag{4.4}$$

where δ_{j0} *is the Kroenecker symbol,* P_h^k, Π_h^k *are the local projection operators introduced in section* 1 *and* j *is defined in* (A1).

The proof will be carried out in a series of lemmas. As in the L^2-error analysis, the use of assumption (A2) leads to a separation of the error estimates for each unknown. So we first analyze the vector variable and next the scalar one. Since the index k will be kept fixed along this section, it will be omitted in the notation.

LEMMA 4.1 : *Let $j \geq 1$ and let the assumptions* (A1) *to* (A4) *hold. Then there exists a constant $C > 0$ independent of h such that*

$$\|p - p_h\|_{L^\infty(\Omega)} \leq C\left(\|p - \Pi_h p\|_{L^\infty(\Omega)} + h\|f - P_h f\|_{L^\infty(\Omega)}\right). \tag{4.5}$$

Proof: Obviously it is enough to bound the error $\|\Pi_h p - p_h\|_{L^\infty}$. Let $x_0 \in T_0 \in \tau_h$ be chosen so that $\|\Pi_h p - p_h\|_{L^\infty} = \max_{i=1,2} |(\Pi_h p - p_h)_i(x_0)|$.

Since $\Pi_h p - p_h \in \mathbf{X}_h$ we use (3.4) to obtain

$$\|\Pi_h p - p_h\|_{L^\infty} \leq C(\theta^{n+\alpha}/h^n)^{1/2}\|\Pi_h p - p_h\|_{\sigma^{-(\alpha+n)}}, \quad \text{for } 0 < \alpha < 2. \tag{4.6}$$

Let us set $\psi := \sigma^{-(\alpha+n)}(\Pi_h p - p_h)$ and then write

$$\begin{aligned} \|\Pi_h p - p_h\|^2_{\sigma^{-(\alpha+n)}} &= \langle \Pi_h p - p, \psi\rangle \\ &\quad + \langle p - p_h, \psi - \Pi_h \psi\rangle + \langle p - p_h, \Pi_h \psi\rangle \\ &= I + II + III. \end{aligned} \tag{4.7}$$

Notice that $\Pi_h \psi$ is well defined. Applying now Cauchy-Schwarz inequality yields for I

$$\begin{aligned} I &\leq \|\Pi_h p - p\|_{\sigma^{-(\alpha+n)}}\|\psi\|_{\sigma^{\alpha+n}} \\ &\leq \frac{1}{2}\|\Pi_h p - p_h\|^2_{\sigma^{-(\alpha+n)}} + C\|\Pi_h p - p\|^2_{\sigma^{-(\alpha+n)}}. \end{aligned}$$

For the second term we use again Cauchy-Schwarz inequality combined now with the superapproximation property (A4) as follows (recall that $\theta \geq h$):

$$\begin{aligned} II &\leq C(h/\theta)\|p - p_h\|_{\sigma^{-(\alpha+n)}}\|\Pi_h p - p_h\|_{\sigma^{-(\alpha+n)}} \\ &\leq C(h/\theta)\|\Pi_h p - p_h\|^2_{\sigma^{-(\alpha+n)}} + C\|\Pi_h p - p\|^2_{\sigma^{-(\alpha+n)}}. \end{aligned}$$

For the remaining term III, the error equation (4.1) implies

$$III = \langle \operatorname{div} \Pi_h \psi, u - u_h\rangle = \langle \operatorname{div} \Pi_h \psi, P_h u - u_h\rangle.$$

Here we have employed that $\operatorname{div} \mathbf{X}_h = \mathbf{M}_h$, which is implicit in (A2), and that P_h is the L^2-projection onto $\mathbf{M}_h$. To proceed further, we use a duality argument. Let φ be the solution of the auxiliary problem

$$\varphi \in H_0^1(\Omega) : -\Delta\varphi = \operatorname{div} \Pi_h \psi .$$

Since $\operatorname{div} \Pi_h \psi \in \mathbf{M}_h \subset \mathbf{M} = L^2(\Omega)$ by (A1) and (A2), we get $\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$. Then, using again (A2) and (4.1), we easily arrive at

$$\begin{aligned} III &= -\langle \operatorname{div}(\Pi_h \operatorname{grad} \varphi), P_h u - u_h \rangle = -\langle p - p_h, \Pi_h \operatorname{grad} \varphi \rangle \\ &= \langle p - p_h, \operatorname{grad} \varphi - \Pi_h(\operatorname{grad} \varphi) \rangle - \langle p - p_h, \operatorname{grad} \varphi \rangle = IV + V . \end{aligned}$$

Applying Cauchy-Schwarz inequality, the interpolation error estimate (2.11) and the a priori estimate (3.6) yields

$$\begin{aligned} IV &\leq \|p - p_h\|_{\sigma^{-(\alpha+n)}} \|\operatorname{grad} \varphi - \Pi_h(\operatorname{grad} \varphi)\|_{\sigma^{\alpha+n}} \\ &\leq Ch \|p - p_h\|_{\sigma^{-(\alpha+n)}} \|D^2\varphi\|_{\sigma^{\alpha+n}} \\ &\leq Ch \|p - p_h\|_{\sigma^{-(\alpha+n)}} \left(\|\operatorname{div} \Pi_h \psi\|_{\sigma^{\alpha+n}} + (1/\theta) \|\Pi_h \psi\|_{\sigma^{\alpha+n}} \right) . \end{aligned}$$

Let us now analyze the last two terms in IV. First notice that (4.2) and (2.6) imply $\operatorname{div} \Pi_h p = \operatorname{div} p_h$. Then using again (2.6) coupled now with the definition of ψ gives after straightforward calculations

$$\|\operatorname{div} \Pi_h \psi\|_{\sigma^{\alpha+n}} \leq (C/\theta) \|\Pi_h p - p_h\|_{\sigma^{-(\alpha+n)}} .$$

By the superapproximation property (A4) and the fact that $\theta \geq h$, we get

$$\|\Pi_h \psi\|_{\sigma^{\alpha+n}} \leq \|\Pi_h \psi - \psi\|_{\sigma^{\alpha+n}} + \|\psi\|_{\sigma^{\alpha+n}} \leq C \|\Pi_h p - p_h\|_{\sigma^{-(\alpha+n)}} .$$

So we conclude that

$$\begin{aligned} IV &\leq C(h/\theta) \|p - p_h\|_{\sigma^{-(\alpha+n)}} \|\Pi_h p - p_h\|_{\sigma^{-(\alpha+n)}} \\ &\leq C(h/\theta) \|\Pi_h p - p_h\|^2_{\sigma^{-(\alpha+n)}} + C \|\Pi_h p - p\|^2_{\sigma^{-(\alpha+n)}} . \end{aligned}$$

Since $\operatorname{div} p_h \in \mathbf{M}_h$ and $\operatorname{div}(p - p_h) = f - P_h f$ is orthogonal to $\mathbf{M}_h$, we can rewrite V as follows

$$V = \langle \operatorname{div}(p - p_h), \varphi - P_h \varphi \rangle = \langle f - P_h f, \varphi - P_h \varphi \rangle .$$

We then combine the fact that $j \geq 1$, and so $\mathbf{P}^1 \subset \mathbf{M}_h$ locally (see (A1)), with the error estimate (2.10) and the a priori estimate (3.6) to arrive at

$$\begin{aligned} V &\leq \|f - P_h f\|_{\sigma^{-(\alpha+n)}} \|\varphi - P_h \varphi\|_{\sigma^{\alpha+n}} \leq Ch^2 \|f - P_h f\|_{\sigma^{-(\alpha+n)}} \|D^2\varphi\|_{\sigma^{\alpha+n}} \\ &\leq C(h^2/\theta) \|f - P_h f\|_{\sigma^{-(\alpha+n)}} \|\Pi_h p - p_h\|_{\sigma^{-(\alpha+n)}} \\ &\leq C(h/\theta) \|\Pi_h p - p_h\|^2_{\sigma^{-(\alpha+n)}} + Ch^2 \|f - P_h f\|^2_{\sigma^{-(\alpha+n)}} . \end{aligned}$$

Next inserting all the estimates already obtained in (4.7) and choosing $C^* = \theta/h$ big enough (but independent of h), the term $\|\Pi_h p - p_h\|_{\sigma^{-(\alpha+n)}}$ which appears on the right hand side can be hidden in the left. So the final estimate reads as follows.

$$\|\Pi_h p - p_h\|^2_{\sigma^{-(\alpha+n)}} \leqslant C\,\|\Pi_h p - p\|^2_{\sigma^{-(\alpha+n)}} + Ch^2\|f - P_h f\|^2_{\sigma^{-(\alpha+n)}}\,.$$

Replacing this into (4.6) and going back to L^∞-norms by means of (3.3), we easily get the desired estimate (4.5) because $\alpha > 0$. This completes the lemma.□

The most interesting case $j = 0$ can also be handled as before with minor changes in the proof, but at the expense of a logarithmic factor.

LEMMA 4.2 : *Let $j = 0$ and let the assumptions* (A1) *to* (A4) *hold. Then there exists a constant $C > 0$ independent of h such that*

$$(4.8) \quad \|p - p_h\|_{L^\infty(\Omega)} \leqslant C\left(\|p - \Pi_h p\|_{L^\infty(\Omega)} + h\,|\log h|\,\|f - P_h f\|_{L^\infty(\Omega)}\right).$$

Proof: The proof proceeds along the same lines as the previous one except for treating term V. So let us now explain the necessary changes. Using the error estimate (2.10) with $j = 0$ and the a priori estimate (3.7) yields

$$\begin{aligned} V &\leqslant \|f - P_h f\|_{\sigma^{-n}}\|\varphi - P_h \varphi\|_{\sigma^n} \leqslant Ch\,\|f - P_h f\|_{\sigma^{-n}}\|D\varphi\|_{\sigma^n} \\ &\leqslant Ch\,\|f - P_h f\|_{\sigma^{-n}}\left(\|\operatorname{div}\Pi_h \psi\|_{\sigma^{n+2}} + (|\log\theta|^{1/2}/\theta)\,\|\Pi_h \psi\|_{\sigma^{n+2}}\right). \end{aligned}$$

The same analysis as before for the last two terms leads to

$$\begin{aligned} V &\leqslant C\,(h/\theta)\,|\log\theta|^{1/2}\|f - P_h f\|_{\sigma^n}\|\Pi_h p - p_h\|_{\sigma^{2-n-2\alpha}} \\ &\leqslant C\,(h/\theta^\alpha)\,|\log\theta|\,\|f - P_h f\|_{L^\infty}\|\Pi_h p - p_h\|_{L^\infty}\,. \end{aligned}$$

Here we have chosen $1 < \alpha < 2$ and used (3.3) twice. Once we have inserted all the estimates in (4.7) and chosen $C^* = \theta/h$ so big as to absorb into the left hand side the remaining terms $\|\Pi_h p - p_h\|^2_{\sigma^{-(\alpha+n)}}$, we get

$$\begin{aligned} &\|\Pi_h p - p_h\|^2_{\sigma^{-(\alpha+n)}} \\ &\qquad \leqslant C\,\|\Pi_h p - p\|^2_{\sigma^{-(\alpha+n)}} + Ch^{1-\alpha}|\log h|\,\|f - P_h f\|_{L^\infty}\|\Pi_h p - p_h\|_{L^\infty}\,. \end{aligned}$$

By (4.6) and (3.3) we transform this bound in an L^∞-one as follows

$$\begin{aligned} &\|\Pi_h p - p_h\|^2_{L^\infty} \\ &\qquad \leqslant C\,\|\Pi_h p - p\|^2_{L^\infty} + Ch\,|\log h|\,\|f - P_h f\|_{L^\infty}\|\Pi_h p - p_h\|_{L^\infty}\,. \end{aligned}$$

This obviously implies the assertion of the lemma. □

The next step in our analysis is to prove an L^∞-superconvergence error estimate for $P_h u - u_h$ in terms of $\|p - p_h\|_{L^\infty}$. This will be done in the following two lemmas.

LEMMA 4.3 : *Let $j \geqslant 1$ and let the assumptions* (A1) *to* (A4) *hold. Then there exists a constant $C > 0$ independent of h such that*

$$(4.9) \quad \|P_h u - u_h\|_{L^\infty(\Omega)} \leqslant Ch|\log h|\left(\|p - p_h\|_{L^\infty(\Omega)} + h\|f - P_h f\|_{L^\infty(\Omega)}\right).$$

Proof: We first reduce the estimate of $\|P_h u - u_h\|_{L^\infty}$ to an estimate in weighted Sobolev norms. However the current weight function σ is different from that one used before due to a different choice of x_0 and θ occurring in (2.7). Indeed, let $x_0 \in T_0 \in \tau_h$ be chosen now in such a way that $|(P_h u - u_h)(x_0)| = \|P_h u - u_h\|_{L^\infty}$; the parameter $\theta = C^* h$ will be specified later on. From (3.4) we then have

$$(4.10) \qquad \|P_h u - u_h\|^2_{L^\infty} \leqslant C(\theta^{2+n}/h^n)\|P_h u - u_h\|^2_{\sigma^{-(2+n)}}.$$

As usual, to estimate the right hand side in (4.10) we apply a duality argument. Let $z \in H^2(\Omega)$ be the unique solution of the following auxiliary problem

$$z \in H_0^1(\Omega): -\Delta z = \sigma^{-(2+n)}(P_h u - u_h).$$

Lemma 3.1 gives the following bound for second derivatives of z,

$$(4.11) \qquad \|D^2 z\|_{\sigma^n} \leqslant C(|\log \theta|^{1/2}/\theta)\|P_h u - u_h\|_{\sigma^{-(2+n)}}.$$

Since $P_h u - u_h \in \mathbf{M}_h$, using the relation (2.6) for $\operatorname{grad} z$ yields

$$\begin{aligned}(4.12) \quad \|P_h u - u_h\|^2_{\sigma^{-(2+n)}} &= -\langle \operatorname{div} \operatorname{grad} z, P_h u - u_h\rangle \\ &= -\langle \operatorname{div} \Pi_h \operatorname{grad} z, u - u_h\rangle \\ &= \langle p - p_h, \operatorname{grad} z - \Pi_h \operatorname{grad} z\rangle - \langle p - p_h, \operatorname{grad} z\rangle \\ &= I + II.\end{aligned}$$

We now proceed to estimate separately each term following the same strategy as for term III in Lemma 4.1. So the error estimate (2.11), together with (4.11), implies

$$\begin{aligned} I &\leqslant \|p - p_h\|_{\sigma^{-n}}\|\operatorname{grad} z - \Pi_h \operatorname{grad} z\|_{\sigma^n} \leqslant Ch\|p - p_h\|_{\sigma^{-n}}\|D^2 z\|_{\sigma^n} \\ &\leqslant C(h/\theta)|\log \theta|^{1/2}\|P_h u - u_h\|_{\sigma^{-(2+n)}}\|p - p_h\|_{\sigma^{-n}} \\ &\leqslant (h/\theta)\|P_h u - u_h\|^2_{\sigma^{-(2+n)}} + C|\log \theta|\,\|p - p_h\|^2_{\sigma^{-n}}.\end{aligned}$$

Integrating by parts in II and using the fact that div $(p - p_h) = f - P_h f$ is orthogonal to $\mathbf{M}_h$ lead to

$$II = \langle \operatorname{div}(p - p_h), z \rangle = \langle f - P_h f, z - P_h z \rangle$$
$$\leq \|f - P_h f\|_{\sigma^{-n}} \|z - P_h z\|_{\sigma^n}$$
$$\leq C(h^2/\theta) |\log \theta|^{\frac{1}{2}} \|f - P_h f\|_{\sigma^{-n}} \|P_h u - u_h\|_{\sigma^{-(2+n)}}$$
$$\leq (h/\theta) \|P_h u - u_h\|^2_{\sigma^{-(2+n)}} + Ch^2 |\log \theta| \|f - P_h f\|^2_{\sigma^{-n}} .$$

Here we have used that $j \geq 1$, which means that the local interpolant polynomials contain $\mathbf{P}^1$ (see (A1)), and the error estimate (2.10). Inserting the bounds obtained for I and II in (4.12) we find that a suitable choice of $C^* = \theta/h$ allows the term $\|P_h u - u_h\|^2_{\sigma^{-(2+n)}}$ to be absorbed into the left hand side. The resulting expression reads as follows

$$\|P_h u - u_h\|_{\sigma^{-(2+n)}} \leq C |\log h|^{\frac{1}{2}} (\|p - p_h\|_{\sigma^{-n}} + h \|f - P_h f\|_{\sigma^{-n}}) .$$

This estimate can be easily rewritten in terms of L^∞-norms by means of (4.10) and (3.3) for $\alpha = n$. This implies the assertion (4.9) and finishes the lemma. □

For the remaining case $j = 0$ we may have a loss of superconvergence order as happens also in the L^2-analysis.

LEMMA 4.4 : *Let $j = 0$ and let the assumptions* (A1) *to* (A4) *hold. Then there exists a constant $C > 0$ independent of h such that*

$$\|P_h u - u_h\|_{L^\infty(\Omega)} \leq Ch |\log h| (\|p - p_h\|_{L^\infty(\Omega)} + \|f - P_h f\|_{L^\infty(\Omega)}) . \tag{4.13}$$

Proof: This proof differs from the previous one only in treating term II. So we focus our attention on this term. Since now M_h is made basically of piecewise constants (see (A1)), from (2.10) and (3.5) we get

$$II \leq \|f - P_h f\|_{\sigma^{2-n}} \|z - P_h z\|_{\sigma^{n-2}}$$
$$\leq C(h/\theta) |\log \theta|^{\frac{1}{2}} \|f - P_h f\|_{\sigma^{2-n}} \|P_h u - u_h\|_{\sigma^{2+n}}$$
$$\leq (h/\theta) \|P_h u - u_h\|^2_{\sigma^{2+n}} + C |\log \theta| \|f - P_h f\|^2_{\sigma^{2-n}} .$$

We can further bound the last term by using Cauchy-Schwarz inequality and

(3.2) as follows

$$\|f - P^h f\|_{\sigma^{2-n}} \leq \|\sigma^{-1}\|_{L^n}^{(n-2)/2} \|f - P_h f\|_{L^n}$$
$$\leq C\,|\log \theta|^{(n-2)/2n} \|f - P_h f\|_{L^n}.$$

A suitable choice of $C^* = \theta/h$ allows the term $\|P_h u - u_h\|^2_{\sigma^{-(2+n)}}$ to be hidden in the left hand side of (4.12) and the argument concludes as before. □

Although the difference between the L^2-projection of the scalar field and the discrete solution superconverges in L^∞, the estimate for the scalar field cannot be better than $\|u - P_h u\|_{L^\infty(\Omega)}$. Therefore, starting from the evaluated discrete solution one can modify u_h through an *element-by-element postprocess* to produce a new approximation u_h^* to u, which is asymptotically more accurate. Let us now describe one of these procedures. The approximation u_h^* to u is such that $u_h^*|_T \in P^{k+1}(T)$ and on each triangle $T \in \tau_h$ is the solution of the following Neumann problem

$$\langle \text{grad}\, u_h^*, \text{grad}\, v\rangle_T = \langle f, v\rangle_T - \langle\!\langle p_h \cdot n_T, v\rangle\!\rangle_{\partial T}, \quad v \in P^{k+1}(T) \tag{4.14}$$

$$\langle u_h^* - u_h, 1\rangle_T = 0 \tag{4.15}$$

where $\langle f, g\rangle_T := \int_T fg$, $\langle\!\langle f, g\rangle\!\rangle_{\partial T} := \int_{\partial T} fg$ and n_T is the outward normal to ∂T. Since the compatibility condition $\langle f, 1\rangle_T - \langle\!\langle p_h \cdot n_T, 1\rangle\!\rangle_{\partial T} = 0$ holds (use (2.5) with $v_h =$ characteristic function of T) such a function u_h^* actually exists.

LEMMA 4.5 : *There exists a positive constant $C > 0$ such that for every $T \in \tau_h$*

$$\|u - u_h^*\|_{L^\infty(T)} \leq C\,(h\,\|D(u - \hat{u})\|_{L^\infty(T)} + h\,\|p - p_h\|_{L^\infty(T)} + h^2\|f - P_h f\|_{L^\infty(T)} + \|P_h u - u_h\|_{L^\infty(T)}), \tag{4.16}$$

where $\hat{u}$ is a suitable approximation to u in $P^{k+1}(T)$.

Proof: Since the argument is local, it is enough to prove the analogous estimate of (4.16) with L^2 instead of L^∞ and then use inverse inequalities. So let us write the error equation

$$\langle \text{grad}\,(u - u_h^*), \text{grad}\, v\rangle_T = - \langle\!\langle (p - p_h) \cdot n_T, v\rangle\!\rangle_{\partial T}, \quad \forall v \in P^{k+1}(T). \tag{4.17}$$

Straightforward calculations lead to

$$\|D(u - u_h^*)\|^2_{L^2(T)} \leq \|D(u - \hat{u})\|^2_{L^2(T)} + 2\,\langle\!\langle (p - p_h) \cdot n_T, \hat{u} - u_h^*\rangle\!\rangle_{\partial T},$$

where $\hat{u} \in P^{k+1}(T)$ approximates u. Notice that from (4.2) the normal component of $p - p_h$ has mean value zero. Then a trace inequality together with a scaling argument yield

$$\begin{aligned} &\left| \langle\!\langle (p - p_h) \cdot n_T, \hat{u} - u_h^* \rangle\!\rangle_{\partial T} \right| \\ &\quad = \left| \left\langle\!\!\left\langle (p - p_h) \cdot n_T, (\hat{u} - u_h^*) - |\partial T|^{-1} \int_{\partial T} (\hat{u} - u_h^*) \right\rangle\!\!\right\rangle_{\partial T} \right| \\ &\quad \leq C \left(\|p - p_h\|_{L^2(T)} + h \|\operatorname{div}(p - p_h)\|_{L^2(T)} \right) \|D(\hat{u} - u_h^*)\|_{L^2(T)} . \end{aligned}$$

Consequently

$$\text{(4.18)} \quad \begin{aligned} \|D(u - u_h^*)\|^2_{L^2(T)} \leq C (&\|D(u - \hat{u})\|^2_{L^2(T)} + \|p - p_h\|^2_{L^2(T)} \\ &+ h^2 \|f - P_h f\|^2_{L^2(T)}) . \end{aligned}$$

Now, using a local version of the standard duality argument, we get the L^2-estimate for $u - u_h^*$. Let φ be any solution of the auxiliary Neumann problem

$$\text{(4.19)} \quad \begin{aligned} - \Delta \varphi &= u - u_h^* - |T|^{-1} \int_T (u - u_h^*) , \quad \text{in } T , \\ \frac{\partial \varphi}{\partial n} &= 0 , \quad \text{on } \partial T , \end{aligned}$$

where $|T|$ stands for the measure of T. Solutions of (4.19) clearly exist because the proper compatibility condition holds. Since the right hand side of (4.19) has mean value zero we can write

$$\begin{aligned} \|D\varphi\|_{L^2(T)} &\leq Ch \left\| u - u_h^* - |T|^{-1} \int_T (u - u_h^*) \right\|_{L^2(T)} \\ &\leq Ch^2 \|D(u - u_h^*)\|_{L^2(T)} . \end{aligned}$$

Hence, the definition of u_h^* and P_h yields

$$\begin{aligned} \|u - u_h^*\|^2_{L^2(T)} &= \langle \operatorname{grad}(u - u_h^*), \operatorname{grad} \varphi \rangle_T + |T|^{-1} \left(\int_T (u - u_h^*) \right)^2 \\ &\leq C \left(h^2 \|D(u - u_h^*)\|^2_{L^2(T)} + \|P_h u - u_h^*\|^2_{L^2(T)} \right) \end{aligned}$$

which together with (4.18) gives

$$\begin{aligned} \|u - u_h^*\|_{L^2(T)} \leq C (&h \|D(u - \hat{u})\|_{L^2(T)} \\ &+ h \|p - p_h\|_{L^2(T)} + h^2 \|f - P_h f\|_{L^2(T)} + \|P_h u - u_h\|_{L^2(T)}) . \end{aligned}$$

Finally, application of inverse inequalities element-by-element concludes the proof. □

5. APPLICATIONS

In this section we shall apply the previous abstract results to some mixed finite element methods that fulfill the abstract assumptions (A1) to (A4). Let us first describe briefly how the new families recently introduced in [4, 6, 7] look like.

5.1. 2-D Mixed Finite Elements of Brezzi-Douglas-Marini

Let τ_h be a decomposition of Ω into triangles T. The discrete functional spaces are defined locally by

$$(5.1)\quad \mathbf{M}_h^k|_T := \mathbf{P}^{k-1}(T)\,,\quad \mathbf{X}_h^k|_T := [\mathbf{P}^k(T)]^2 \quad \text{for all } T \in \tau_h\,,\quad k \geqslant 1\,;$$

so (A1) holds. An easy calculation shows that the local degrees of freedom of these elements are $2k+2$ less than that corresponding to Raviart-Thomas-Nedelec spaces with same accuracy. The operator Π_h^k is defined locally by the following degrees of freedom when $T \in \tau_h$ has three straight edges e_i, $i = 1, 2, 3$

$$(5.2)\quad \begin{aligned} &\langle\!\langle (q - \Pi_h^k q) \cdot n_{e_i}, w \rangle\!\rangle_{e_i} = 0\,, && w \in \mathbf{P}^k(e_i)\,,\quad i = 1, 2, 3 \\ &\langle q - \Pi_h^k q, \operatorname{grad} v \rangle_T = 0\,, && v \in \mathbf{P}^{k-1}(T)\,; \\ &\langle q - \Pi_h^k q, \operatorname{curl} b \rangle_T = 0\,, && b \in \mathbf{B}^{k+1}(T) := \lambda_1 \lambda_2 \lambda_3 \mathbf{P}^{k-2}(T)\,, \\ & && k \geqslant 2\,. \end{aligned}$$

Here λ_i stands for the barycentric coordinates of T. A suitable change in (5.2) is needed when T has a curve side [6, 7]. Then, the construction of Π_h^k guarantees that the assumptions (A2) and (A3) (with $j = k-1$) hold. The remaining hypothesis (A4) is straightforward from (A3).

Now, let τ_h be a decomposition of Ω into rectangles R. The discrete functional spaces are defined locally by

$$(5.3)\quad \begin{aligned} &\mathbf{M}_h^k|_R := \mathbf{P}^{k-1}(R)\,, \\ &\mathbf{X}_h^k|_R := [\mathbf{P}^k(R)]^2 \oplus \operatorname{span}\,\{\operatorname{curl} x^{k+1} y, \operatorname{curl} x y^{k+1}\} \end{aligned}$$

for all $R \in \tau_h$ and $k \geqslant 1$; so again (A1) is satisfied. The dimension of $\mathbf{X}_h^k|_R$ is essentially twice less than that of similar Raviart-Thomas-Nedelec spaces; the dimension of the scalar space is less, as well [6]. For R having no curved edges, Π_h^k is defined through the following degrees of freedom [6]:

$$(5.4)\quad \begin{aligned} &\langle\!\langle (q - \Pi_h^k q) \cdot n_{e_i}, w \rangle\!\rangle_{e_i} = 0\,, && w \in \mathbf{P}^k(e_i)\,,\quad i = 1, 2, 3, 4\,; \\ &\langle q - \Pi_h^k q, v \rangle_R = 0\,, && v \in [\mathbf{P}^{k-2}(R)]^2\,. \end{aligned}$$

The definition varies to consider one curved side [6]. As before, the assumptions (A2), (A3) obviously hold while (A4) can be proved by means of the Bramble-Hilbert lemma arguing as in Scholz [22]. Note finally that the discrete spaces $\mathbf{M}_h^k$ and $\mathbf{X}_h^k$ may also be based on mixing rectangular and triangular elements, because they are designed to fit across straight-edges.

5.2. 3-D Mixed Finite Elements of Brezzi-Douglas-Durán-Fortin

These 3-D families were introduced in [4]. For Ω being decomposed into simplices T, the discrete spaces are

$$\mathbf{M}_h^k|_T := \mathbf{P}^{k-1}(T), \quad \mathbf{X}_h^k|_T := [\mathbf{P}^k(T)]^3 \quad \text{for all } T \in \tau_h, \quad k \geq 1. \tag{5.5}$$

The local dimension of Nedelec spaces [15] exceeds that of the present spaces by $(k+2)\ (k+1)$. The operator Π_h^k is defined locally by the following relations :

$$\begin{aligned} &\langle\!\langle (q - \Pi_h^k q) \cdot n_e, w \rangle\!\rangle_e = 0, \quad w \in \mathbf{P}^k(e), \quad \text{for each face } e \text{ of } \partial T, \\ &\langle (q - \Pi_h^k q), \operatorname{grad} v \rangle_T = 0, \quad v \in \mathbf{P}^{k-1}(T), \\ &\langle (q - \Pi_h^k q), r \rangle_T = 0, \\ &\qquad r \in \{ t \in [\mathbf{P}^k(T)]^3 : t \cdot n = 0 \quad \text{on } \partial T \\ &\qquad\quad \text{and} \quad \langle t, \operatorname{grad} v \rangle_T = 0, v \in \mathbf{P}^{k-1}(T) \}. \end{aligned} \tag{5.6}$$

If, instead, Ω is decomposed into cubic elements T we then have for all $T \in \tau_h$:

$$\begin{aligned} &\mathbf{M}_h^k|_T := \mathbf{P}^{k-1}(T), \\ &\mathbf{X}_h^k|_T := [\mathbf{P}^k(T)]^3 \oplus \text{span } \{\text{curl } (0, 0, x^{k+1} y), \text{curl } (0, xz^{k+1}, 0), \\ &\qquad \text{curl } (y^{k+1} z, 0, 0), \text{curl } (0, 0, xy^{i+1} z^{k-i}), \\ &\qquad \text{curl } (0, x^{i+1} y^{k-i} z, 0), \text{curl } (x^{k-i} yz^{i+1}, 0, 0), \\ &\qquad i = 1, \ldots, k\}. \end{aligned} \tag{5.7}$$

The operator Π_h^k is defined through the following degrees of freedom

$$\begin{aligned} &\langle\!\langle (q - \Pi_h^k q) \cdot n_e, w \rangle\!\rangle_e = 0, w \in \mathbf{P}^k(e), \quad \text{for each face } e \text{ of } T, \\ &\langle q - \Pi_h^k q, r \rangle_T = 0, \ r \in [\mathbf{P}^{k-2}(T)]^3. \end{aligned} \tag{5.8}$$

The definitions (5.6) and (5.8) correspond to straight-sided elements since, in fact, for curved boundary elements they are somewhat different [4]. The local dimension of $\mathbf{X}_h^k$ is about one half that of Raviart-Thomas-Nedelec spaces of equivalent accuracy [4]. Moreover it is easily seen that assumptions (A1) to (A4) hold again.

5.3. Raviart-Thomas-Nedelec Mixed Finite Elements

Let $n = 2$ or 3. Let τ_h be a decomposition of Ω into n-simplices T. The discrete functional spaces are defined locally by

$$\mathbf{M}_h^k|_T := \mathbf{P}^k(T)\,, \quad \mathbf{X}_h^k|_T := [\mathbf{P}^k(T)]^n \oplus \underset{\sim}{x}\mathbf{P}^k(T)\,, \quad k \geqslant 0\,, \tag{5.9}$$

where $\underset{\sim}{x} := (x_1, \ldots, x_n)$. Then (A1) is fulfilled. The existence of an operator Π_h^k satisfying assumptions (A2) and (A3) with $j = k$ is well known [15, 20] ; (A4) also holds (see Scholz [22]).

Let now τ_h be made up of n-rectangles T and let $\mathbf{Q}^k(T)$ be the set of polynomials of degree $\leqslant k$ in each variable restricted to T. The discrete spaces are defined by

$$\mathbf{M}_h^k|_T := \mathbf{Q}^k(T)\,, \quad \mathbf{X}_h^k|_T := \prod_{i=1}^{n} [\mathbf{Q}^k(T) \oplus x_i\, \mathbf{Q}^k(T)]\,, \quad k \geqslant 0\,. \tag{5.10}$$

Then the assumptions (A1) to (A4) hold [15, 20, 22].

5.4. Asymptotic L^∞-Error Estimates

Let us now state the L^∞-rates of convergence for the families of mixed finite elements just recalled. So, assume that $n = 2, 3$.

COROLLARY 5.1 : *Let $k \geqslant 1$ and $u \in W^{k+2,\infty}(\Omega)$. Let $\mathbf{M}_h^k$ and $\mathbf{X}_h^k$ be the BDM or BDDF spaces defined in either* (5.1), (5.3) *or* (5.5), (5.7). *Then there exists a constant $C > 0$ independent of h such that*

$$h\,\|u - u_h\|_{L^\infty(\Omega)} + |\log h|^{-\delta_{k1}}\|p - p_h\|_{L^\infty(\Omega)} \leqslant C h^{k+1} \|u\|_{W^{k+2,\infty}(\Omega)}\,. \tag{5.11}$$

Furthermore, the following superconvergence error estimates hold

$$\|P_h^k u - u_h\|_{L^\infty(\Omega)} \leqslant C h^{k+2-\delta_{k1}} |\log h|\, \|u\|_{W^{k+2,\infty}(\Omega)}\,, \tag{5.12}$$

$$\|u - u_h^*\|_{L^\infty(\Omega)} \leqslant C h^{k+2-\delta_{k1}} |\log h|\, \|u\|_{W^{k+2,\infty}(\Omega)}\,, \tag{5.13}$$

where $u_h^* \in \mathbf{P}^{k+1}(T)$ for each $T \in \tau_h$ is the modified scalar field defined in (4.14), (4.15). □

The proof is an immediate consequence of Theorem 4.1 and Lemma 4.5.

Note that, according to the degree of the interpolant polynomials, the orders of convergence in (5.11) are *optimal for* $k \geq 2$ and *quasioptimal for* $k = 1$ due to the presence of logarithmic factors. Since the regularity property $u \in W^{k+2,\infty}(\Omega)$ is not easy to be verified in practical cases, let us consider the weaker assumption $f \in W^{k,\infty}(\Omega)$. We can still derive sharp error estimates arguing as in Johnson-Thomee [13] and Gastaldi-Nochetto [12] ; so we only sketch the proof.

COROLLARY 5.2 : *Let* $k \geq 1$ *be given and let* $\mathbf{M}_h^k$ *and* $\mathbf{X}_h^k$ *be defined in either* (5.1), (5.3), (5.5) *or* (5.7). *Assume that* $f \in W^{k,\infty}(\Omega)$. *Then*

$$(5.14) \quad h \|u - u_h\|_{L^\infty(\Omega)} + |\log h|^{-1} \|p - p_h\|_{L^\infty(\Omega)} \leq C h^{k+1} \|f\|_{W^{k,\infty}(\Omega)}$$

$$(5.15) \qquad \|P_h^k u - u_h\|_{L^\infty(\Omega)} \leq C h^{k+2-\delta_{k1}} |\log h|^{2-\delta_{k1}} \|f\|_{W^{k,\infty}(\Omega)}$$

$$(5.16) \qquad \|u - u_h^*\|_{L^\infty(\Omega)} \leq C h^{k+2-\delta_{k1}} |\log h|^{2-\delta_{k1}} \|f\|_{W^{k,\infty}(\Omega)}$$

where $C > 0$ *does not depend on* h.

Proof: The argument is based on tracing constants in [1] and using a well-known interpolation error estimate [8, p. 124]. So, for any $r < \infty$ we can write

$$\|p - \Pi_h^k p\|_{L^\infty} \leq C h^{k+1-n/r} \|u\|_{W^{k+2,r}(\Omega)} \leq C h^{k+1} r h^{-n/r} \|f\|_{W^{k,\infty}(\Omega)} .$$

The desired result follows from taking $r = |\log h|$. □

Another consequence of Theorem 4.1 is the following l^∞-superconvergence result.

COROLLARY 5.3 : *Let* $k = 1$ *and* $f \in W^{1,s}(\Omega)$, $s > n$. *For any* $T \in \tau_h$ *being either a simplex or an n-rectangle, let* x_T *denote its barycenter. Then the following superconvergence estimate holds*

$$(5.17) \quad |(u - u_h)(x_T)| \leq C h^2 |\log h| \, \|f\|_{W^{1,s}(\Omega)}, \quad \text{for all } T \in \tau_h .$$

Proof: Since $f \in W^{1,s}(\Omega)$ and $\partial\Omega$ is regular, Sobolev inequalities combined with regularity elliptic theory imply, [1] :

$$\|u\|_{W^{2,\infty}(\Omega)} \leq C \|u\|_{W^{3,s}(\Omega)} \leq C \|f\|_{W^{1,s}(\Omega)} .$$

Thus $\|p - \Pi_h^1 p\|_{L^\infty} = 0(h)$. In addition, the following interpolation error estimate is well known [8, p. 124]

$$\|f - P_h^1 f\|_{L^\infty} \leq C h^{1-n/s} \|f\|_{W^{1,s}(\Omega)} .$$

Inserting these estimates in (4.4) we get

$$\|P_h^1 u - u_h\|_{L^\infty} \leqslant Ch^2 |\log h| \, \|f\|_{W^{1,s}(\Omega)} .$$

To conclude the argument we split the error $(u - u_h)(x_T)$ for each $T \in \tau_h$ as follows

$$(u - u_h)(x_T) = (u - P_h^2 u)(x_T) + (P_h^2 u - P_h^1 u)(x_T) + (P_h^1 u - u_h)(x_T) .$$

Then, since $\|u - P_h^2 u\|_{L^\infty} = 0(h^2)$, we only have to estimate the middle term. However, this is a trivial task because $(P_h^2 u - P_h^1 u)(x_T) = 0$, as can be easily checked. The proof is thus complete. □

Remark 5.1 : For $f \in W^{1,n}(\Omega) \cap L^\infty(\Omega)$ the error bound (5.17) becomes $0(h^2 |\log h|^2)$. To see this one has to proceed as before using now a similar argument to that in Corollary 5.2.

Let us now state the analogues of corollaries 5.1, 5.2 and 5.3 for Raviart-Thomas-Nedelec mixed elements. As before assume $n = 2, 3$.

COROLLARY 5.4 : *Let $k \geqslant 0$ and $u \in W^{k+2,\infty}(\Omega)$. Let $\mathbf{M}_h^k$ and $\mathbf{X}_h^k$ be the RTN spaces defined in either* (5.9) *or* (5.10). *Then there exists a constant $C > 0$ independent of h such that*

$$(5.18) \quad \|u - u_h\|_{L^\infty(\Omega)} + \|p - p_h\|_{L^\infty(\Omega)} \leqslant Ch^{k+1} |\log h|^{\delta_{k0}} \|u\|_{W^{k+2,\infty}(\Omega)} ,$$

$$(5.19) \quad \|P_h^k u - u_h\|_{L^\infty(\Omega)} \leqslant Ch^{k+2-\delta_{k0}} |\log h|^{1+\delta_{k0}} \|u\|_{W^{k+2,\infty}(\Omega)} ,$$

$$(5.20) \quad \|u - u_h^*\|_{L^\infty(\Omega)} \leqslant Ch^{k+2-\delta_{k0}} |\log h|^{1+\delta_{k0}} \|u\|_{W^{k+2,\infty}(\Omega)} .$$

where $u_h^* \in \mathbf{P}^{k+1}(T)$ for each $T \in \tau_h$ is the modified scalar field defined in (4.14)-(4.15). □

Next the weaker assumption $f \in W^{k,\infty}(\Omega)$ still yields the following sharp error estimates.

COROLLARY 5.5 : *Let $k \geqslant 0$ be given and let $\mathbf{M}_h^k$ and $\mathbf{X}_k^h$ be defined in* (5.9) *or* (5.10). *Assume that $f \in W^{k,\infty}(\Omega)$. Then*

$$(5.21) \quad \|u - u_h\|_{L^\infty(\Omega)} + \|p - p_h\|_{L^\infty(\Omega)} \leqslant Ch^{k+1} |\log h| \, \|f\|_{W^{k,\infty}(\Omega)},$$

$$(5.22) \quad \|P_h^k u - u_h\|_{L^\infty(\Omega)} \leqslant Ch^{k+2-\delta_{k0}} |\log h|^2 \|f\|_{W^{k,\infty}(\Omega)},$$

$$(5.23) \quad \|u - u_h^*\|_{L^\infty(\Omega)} \leqslant Ch^{k+2-\delta_{k0}} |\log h|^2 \|f\|_{W^{k,\infty}(\Omega)} .$$

If, in addition for $k = 0$, we assume that $f \in W^{1,n}(\Omega)$, then

$$(5.24) \quad \|P_h^0 u - u_h\|_{L^\infty(\Omega)} \leqslant Ch^2 |\log h|^2 (\|f\|_{L^\infty(\Omega)} + \|f\|_{W^{1,n}(\Omega)}),$$

$$(5.25) \quad \|u - u_h^*\|_{L^\infty(\Omega)} \leqslant Ch^2 |\log h|^2 (\|f\|_{L^\infty(\Omega)} + \|f\|_{W^{1,n}(\Omega)}) . \quad \square$$

The superconvergence estimate (5.24) leads to the following l^{∞}-superconvergence results.

COROLLARY 5.6 : *Let τ_h be a decomposition of Ω into n-rectangles T and let S_T be the set of Gauss-Legendre points of T. Then we have for $k \geq 0$*

$$(5.26) \quad \|u - u_h\|_{l^{\infty}(S_T)} \leq C h^{k+2} |\log h|^2 (\|f\|_{W^{k,\infty}(\Omega)} + \delta_{k0} \|f\|_{W^{1,n}(\Omega)}), \quad \forall T \in \tau_h .$$

Moreover, if τ_h is a decomposition of Ω into n-simplices T and x_T is the barycenter of $T \in \tau_h$, then for $k = 0$

$$(5.27) \quad |(u - u_h)(x_T)| \leq C h^2 |\log h|^2 (\|f\|_{L^{\infty}(\Omega)} + \|f\|_{W^{1,n}(\Omega)}), \quad \forall T \in \tau_h . \quad \square$$

Before concluding this section several comments are in order.

Remark 5.2 : Let us consider the lowest order case in estimates (5.11) and (5.18). Assuming that f is piecewise continuous with a modulus of continuity $w(t) \leq C |\log t|^{-1}$, the logarithmic factors can be removed.

Remark 5.3 : The key argument in proving l^{∞}-superconvergence is first to demonstrate the superconvergence of $P_h^k u - u_h$ and second to localize a set of points S_T in each finite element $T \in \tau_h$, so that $(P_h^k v - v)(x) = 0$ $\forall v \in \mathbf{P}^{j+1}(T)$, $x \in S_T$, $T \in \tau_h$. This invariance property is no longer true for $j > 0$ and the families defined in (5.1), (5.3), (5.5), (5.7) and (5.9) (see [12]).

Remark 5.4 : Variable degree mixed methods were introduced by Brezzi-Douglas-Marini as generalizations of the families discussed above [7]. Now the degree of interpolant polynomials is allowed to vary from one element to another by means of transition elements. Since the operators Π_h^k and P_h^k are local, our main results (4.3) and (4.4) still apply extending to L^{∞} the error analysis presented in [7].

Remark 5.5 : The abstract analysis applies also to the finite elements introduced by Brezzi-Douglas-Fortin-Marini [5], giving the same rates of convergence as RTN spaces.

6. L^{∞}-ERROR ANALYSIS OF THE HYBRID FORM

The linear algebraic system associated with (2.5) is generally indefinite ; so standard solvers may fail to work. However, it can be reduced to a positive definite system through the use of Lagrange multipliers to relax the continuity constraint on the normal component of the vector field. This computational trick can be further exploited to produce a better approximation to the scalar field u. Indeed, the multipliers approximate u along the interelement boundaries in a sense to be specified later on (see Remark 6.1). This information can be postprocessed to get a more accurate approximation $\tilde{u}_h$ to u, as first noted by Arnold-Brezzi [2].

Let $\mathbf{Y}_h^k$ be the space of all functions defined on the interelement boundaries which restrict to polynomials of degree k on each edge $e \not\subset \partial\Omega$ and vanish on $\partial\Omega$. Moreover let $\mathbf{W}_h^k$ indicate the discrete space for the vector unknown without continuity constraints ; hence $\mathbf{W}_h^k|_T = \mathbf{X}_h^k|_T$ for all $T \in \tau_h$. Then the extended problem reads as follows : *find* $(z_h, \sigma_h, \lambda_h) \in \mathbf{M}_h^k \times \mathbf{W}_h^k \times \mathbf{Y}_h^k$ *such that*

$$
(6.1) \qquad \begin{aligned}
\langle \sigma_h, q_h \rangle - \langle \operatorname{div} q_h, z_h \rangle_h + \langle\!\langle \lambda_h, q_h \cdot n_T \rangle\!\rangle_h &= 0, & \forall q_h &\in \mathbf{W}_h^k , \\
\langle \operatorname{div} \sigma_h, v_h \rangle_h &= \langle f, v_h \rangle , & \forall v_h &\in \mathbf{M}_h^k , \\
\langle\!\langle \mu_h, \sigma_h \cdot n_T \rangle\!\rangle_h &= 0, & \forall \mu_h &\in \mathbf{Y}_h^k ,
\end{aligned}
$$

where $\langle f, g \rangle_h = \sum_T \int_T fg$, $\langle\!\langle f, g \rangle\!\rangle_h = \sum_T \int_{\partial T} fg$ and n_T is the outward normal to ∂T. Note that $v \in \mathbf{W}_h^k$ belongs to $\mathbf{X}_h^k$ if and only if

$$
(6.2) \qquad \langle\!\langle \mu_h, v \cdot n_T \rangle\!\rangle_h = 0, \quad \forall \mu_h \in \mathbf{Y}_h^k .
$$

This implies that the function σ_h given by (6.1) coincides with p_h given by (2.5) and, consequently, $z_h = u_h$.

The Lagrange multipliers λ_h approximate u on the interelement boundaries in the sense that

$$
(6.3) \quad \|\lambda_h - Q_h^k u\|_{L^2(e)} \leq C\,(h^{1/2}\|p - p_h\|_{L^2(T)} + h^{-1/2}\|P_h^k u - u_h\|_{L^2(T)}) ,
$$

for all $T \in \tau_h$ and $e \in \partial T \backslash \partial\Omega_h$, where $Q_h^k u$ is the L^2-projection of $u|_e$ onto $\mathbf{Y}_h^k|_e$ [2 ; 4, 6, 7]. The corresponding L^∞-error estimate for Lagrange multipliers follows from simply applying inverse inequalities to (6.3) on each element.

LEMMA 6.1 : *There exists a positive constant $C > 0$ independent of u and h, such that for every $T \in \tau_h$ and every edge e of T not belonging to $\partial\Omega$ we have*

$$
(6.4) \quad \|\lambda_h - Q_h^k u\|_{L^\infty(e)} \leq C\,(h\|p - p_h\|_{L^\infty(T)} + \|P_h^k u - u_h\|_{L^\infty(T)}) . \quad \square
$$

Remark 6.1 : The error estimate (6.4) combined with those in the previous section yields the *superconvergence rate* $\|\lambda_h - Q_h^k u\|_{L^\infty(e)} =$

$0(h^{k+2}|\log h|^2)$ and, therefore, the error bound on edges $\|u - \lambda_h\|_{L^\infty(e)} = 0(h^{k+1})$. Moreover, an l^∞-superconvergence result can be derived arguing as in Corollary 5.3. For $n = 2$ let S_e be the set of Gauss-Legendre points on e ; then

$$(6.5) \qquad \|u - \lambda_h\|_{l^\infty(S_e)} = 0(h^{k+2}|\log h|^2), \quad \forall e \subset \partial T \backslash \partial\Omega, \quad T \in \tau_h .$$

If $n = 3$ the estimate (6.5) still holds for either $j = 0$ or Raviart-Thomas-Nedelec cubic elements.

Let us now conclude by describing how a modified scalar field $\tilde{u}_h$ can be constructed in some relevant examples. Take first $n = 2$. For the lowest order triangular elements of Raviart-Thomas-Nedelec recalled in (5.9) let $\tilde{u}_h|_T \in \mathbf{P}^1(T) =: \mathbf{V}$ be defined by $Q_h^0(\tilde{u}_h - \lambda_h) = 0$, [2]. For the lowest order triangular elements of Brezzi-Douglas-Marini reminded in (5.1), instead, let $\tilde{u}_h|_T \in \mathbf{P}^2(T) =: \mathbf{V}$ be defined by the conditions $Q_h^0(\tilde{u}_h - \lambda_h) = 0$ and $P_h^1(\tilde{u}_h - u_h) = 0$, [6]. Suppose now $n = 3$, $k = 1$ and that Ω is decomposed into tetrahedra. For the discrete spaces defined in (5.5) let $\tilde{u}_h$ satisfy $\tilde{u}_h| \in \mathbf{V} := \mathbf{P}^2(T) \oplus \text{span}\ \{z^2(x-y),\ x^2(y-z),\ y^2(x-z)\}$, $Q_h^1(\tilde{u}_h - \lambda_h) = 0$, and $P_h^0(\tilde{u}_h - u_h) = 0$. For the construction of such an approximation in the general case, we refer to [4].

Such approximations satisfy [2, 4, 6]

$$(6.6) \quad \|u - \tilde{u}_h\|_{L^2(T)} \leq C\,(\|u - \hat{u}\|_{L^2(T)} + \|P_h^k u - u_h\|_{L^2(T)} + h^{1/2}\|\lambda_h - Q_h^k u\|_{L^2(\partial T)})$$

where $\hat{u}|_T \in \mathbf{V}$ is an appropriate interpolant of u. Due to a simple application of inverse inequalities to (6.6), the following local L^∞-estimate holds

$$(6.7) \quad \|u - \tilde{u}_h\|_{L^\infty(T)} \leq C\,(\|u - \hat{u}\|_{L^\infty(T)} + \|P_h^k u - u_h\|_{L^\infty(T)} + \|\lambda_h - Q_h^k u\|_{L^\infty(\partial T)}) .$$

Finally, combining (6.4) with Corollaries 5.2 and 5.5 results in

COROLLARY 6.1. *Let* $k \geq 1$ *and* $\mathbf{M}_h^k$, $\mathbf{X}_h^k$ *be defined in either* (5.1), (5.3), (5.5) *or* (5.7). *Assume that* $f \in W^{k,\infty}$. *Then*

$$(6.8) \qquad \|u - \tilde{u}_h\|_{L^\infty(\Omega)} \leq C h^{k+2-\delta_{k1}}|\log h|^{2-\delta_{k1}}\|f\|_{W^{k,\infty}(\Omega)} .$$

Now let $k \geq 0$ *and* $\mathbf{M}_h^k$, $\mathbf{X}_h^k$ *be defined in either* (5.9) *or* (5.10) *and* $f \in W^{k,\infty}(\Omega) \cap W^{1,n}(\Omega)$. *Then*

$$(6.9) \qquad \|u - \tilde{u}_h\|_{L^\infty(\Omega)} \leq C h^{k+2}|\log h|^2(\|f\|_{W^{k,\infty}(\Omega)} + \delta_{k0}\|f\|_{W^{1,n}(\Omega)}) . \quad \square$$

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