G. Lube

Uniform in $\varepsilon$ discretization error estimates for convection dominated convection-diffusion problems


<http://www.numdam.org/item?id=M2AN_1988__22_3_477_0>
UNIFORM IN $\varepsilon$ DISCRETIZATION ERROR ESTIMATES FOR CONVECTION DOMINATED CONVECTION-DIFFUSION PROBLEMS (*)

by G. LUBE (1)

Communiqué par Ph CIARLET

Abstract. — Asymptotically fitted variants of the standard Galerkin finite element method and of the streamlined diffusion method of Hughes and Brooks for solving a mixed boundary value problem for convection dominated convection-diffusion flow problems are considered.

We discuss the existence of global and local $L^2$-discretization error estimates (and sometimes in energy norm) which are uniformly valid with respect to the diffusion parameter $\varepsilon$.

1. INTRODUCTION

In this paper, we consider convection-reaction-diffusion problems of the form

$$
\begin{align*}
(\mathcal{L}_\varepsilon) & \\
- \varepsilon \Delta u_\varepsilon + b \cdot \nabla u_\varepsilon + cu_\varepsilon &= f \quad \text{in} \quad \Omega \subset \mathbb{R}^N \\
u_\varepsilon &= g_1 \quad \text{on} \quad \Gamma_1, \\
\varepsilon \frac{\partial u_\varepsilon}{\partial \nu} + \sigma (u_\varepsilon - g_2) &= 0 \quad \text{on} \quad \Gamma_2
\end{align*}
$$

where $\Omega$ is a bounded domain with boundary $\partial \Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$. $\varepsilon$ is the diffusion parameter with $0 \leq \varepsilon \leq 1$. In the singularly perturbed case $0 < \varepsilon \ll 1$, the solution of $(\mathcal{L}_\varepsilon)$ is characterized by sharp boundary layers.

(*) Received in October 1986
(1) Technische Hochschule Magdeburg, Sektion Mathematik und Physik, P S F 124, 3040 Magdeburg, DDR
In general, the numerical solution of standard finite element methods (f.e.m.) applied to \( (\mathcal{L}_e) \) has undesired oscillations unless the mesh is very fine. In the case of the Galerkin f.e.m. (with piecewise polynomials of degree \( k \)), and for essential boundary conditions on the downstream layer part of \( \partial \Omega \), we have the \( L_2 \)-discretization error estimate
\[
\| u_\varepsilon - u_h \|_{0,2} \leq C \left( \frac{h}{\varepsilon} \right)^{k+1}.
\]

Well-known modifications of standard f.e.m. are mesh-refinements, up-wind-schemes and exponentially fitted schemes. In this paper, we derive for adapted finite element schemes global and local discretization error estimates in the \( L_2 \)-norm and sometimes in the energy norm which are uniformly valid with respect to the full range of \( \varepsilon \). More precisely, a family of approximations, \( u_{h, \varepsilon} \), \( 0 < h \leq h_\varepsilon \) converges uniformly (with respect to \( 0 \leq \varepsilon \leq \varepsilon_0 \)) to \( u_\varepsilon \) in the norm \( \| . \| \) of order \( p \) if with a constant \( C \) independent of \( \varepsilon \) and \( h \)
\[
\sup_{0 \leq \varepsilon \leq \varepsilon_0} \| u_\varepsilon - u_h \| \leq C h^p.
\]

An important consequence of such an uniform in \( \varepsilon \) discretization error estimate is that the given f.e.m. well approximates the solution \( u_0 \) of the limit problem
\[
(\mathcal{L}_0) \quad \begin{cases}
 b \cdot \nabla u_0 + cu_0 = f & \text{in } \bar{\Omega}\setminus\Gamma_-
 
 u_0 = g_1 & \text{on } \Gamma_-
\end{cases}
\]
if \( \varepsilon, h \to 0 \) (independent on the relation between \( \varepsilon \) and \( h \)). (Compare the discussion concerning finite difference schemes in Giles/Rose [6].) \( \Gamma_- = \{ x \in \partial \Omega | \exists v, b \cdot v < 0 \} \) denotes the inflow part of \( \partial \Omega \). For simplicity we assume \( \Gamma_- \subseteq \Gamma_1 \). In case of \( b \equiv 0 \), uniformly in \( \varepsilon \) convergent finite difference schemes are considered by several authors (Miller [12], Nijijima [15], Shishkin-Titov [21]). Jemeljanov [9], Kellogg [11] and Shishkin [20] considered special one- and two-dimensional cases with \( |b| > 0 \). Concerning uniformly in \( \varepsilon \) convergent finite element schemes in case of \( b \equiv 0 \), we refer to Schatz-Wahlbin [18]. In case of \( |b| > 0 \) Nävert [14] derived uniformly in \( \varepsilon \) valid local energy and \( L_2 \)-norm estimates for the streamlined diffusion f.e.m. The question of local \( L_\infty \)-estimates for this situation is studied in Risch [17] for a hybrid f.e.m. preserving the maximum principle. The problem of uniformly in \( \varepsilon \) convergent (with respect to global norms) f.e.m. in case of \( b \neq 0 \) seems to be open. In the present paper, we propose "switching" algorithms of the following kind : in case of \( \varepsilon \equiv h^\kappa \) (with some \( \kappa > 0 \)), the f.e.m. is exactly the standard Galerkin f.e.m. (SG-FEM).
first variant in the case $0 \leq \varepsilon < h^s$ (modified Galerkin f.e.m.-MG-FEM) consists on replacing the given boundary conditions on $\partial \Omega \setminus \Gamma_-$ by homogeneous conditions on Neumann type (cf. Axelsson [1] with no rigorous asymptotic analysis). A second variant is to approximate the limit problem $(\mathcal{L}_0)$ instead of $(\mathcal{L}_\varepsilon)$ by the streamlined diffusion f.e.m. of Hughes/Brooks [8] (scheme MSD.1) or to solve $(\mathcal{L}_\varepsilon)$ with homogeneous conditions of Neumann type on $\partial \Omega \setminus \Gamma_-$ approximately by the streamlined diffusion method (scheme MSD.2).

This paper is organized as follows: Section 2 is concerned with notations and with estimates for problem $(\mathcal{L}_\varepsilon)$. In Section 3 we consider the Galerkin f.e.m. and the modified Galerkin scheme MG-FEM. Section 4 contains error estimates for the streamlined diffusion f.e.m. and its modifications MSD.1 and MSD.2. In Section 5 sufficient conditions on the limit problem $(\mathcal{L}_0)$ which guarantee uniform in $\varepsilon$ error estimates are given. Concluding remarks are contained in Section 6.

2. ESTIMATES FOR THE CONTINUOUS PROBLEM

We use the conventional Sobolev spaces $W^k_p(G)$ on $G \subseteq \Omega$ and their norms (or seminorms) $\| \cdot \|_{k,p,G}$ (or $| \cdot |_{k,p,G}$). $\tilde{W}^k_p(G)$ denotes the closure of $C_0^\infty(G)$ in the norm of $W^k_p$. Further let $| \cdot |_{k,p,G'}$ be the norm on $W^k_p(G')$, $G' \subseteq \partial \Omega$. $(\cdot, \cdot)_G$ denotes the inner product in $L_2(G) = W_0^1(G)$. If there is no doubt we omit the index $\Omega$.

We shall consider the convection-reaction-diffusion problem

$$(\mathcal{L}_\varepsilon) \quad \begin{cases} L_\varepsilon u_\varepsilon = -\varepsilon \Delta u_\varepsilon + b \cdot \nabla u_\varepsilon + cu_\varepsilon = f & \text{in } \Omega \subset \mathbb{R}^N \\ u = g_1 & \text{on } \Gamma_1, \quad \varepsilon \frac{\partial u_\varepsilon}{\partial \nu} + \sigma (u_\varepsilon - g_2) = 0 & \text{on } \Gamma_2 \end{cases}$$

where some scalar field $u_\varepsilon$ (concentration of heat or a chemical) is driven by a velocity field $b$. $\varepsilon$ is the diffusion parameter with $0 \leq \varepsilon \leq 1$, $\nu$ is the outward pointing normal vector on $\partial \Omega$ and

$$\Gamma_{(\varepsilon)} = \left\{ x \in \partial \Omega \mid \exists \nu(x), (b \cdot \nu)(x) < 0 \right\},$$

$$\Gamma_0 = \left\{ x \in \partial \Omega \mid \exists \nu(x), (b \cdot \nu)(x) = 0 \right\}$$

are the inflow (outflow) and characteristic parts, respectively, of $\partial \Omega$.

For the given data we assume:

vol. 22, n° 3, 1988
Ω ⊂ ℝ^N bounded domain with the boundary ∂Ω ⊂ C^2 or a polyhedron
\[ \partial \Omega = \Gamma_1 \cup \Gamma_2, \quad \Gamma_1 \cap \Gamma_2 = \emptyset. \]

(H.2) \( b \in [C^1(\bar{\Omega})]^N, \quad c \in L_\infty(\Omega), \quad f \in L_2(\Omega); \quad g_i \in L_\infty(\Gamma_i), \quad i = 1, 2, \quad \sigma \in L_\infty(\Gamma_2) \) with \( 0 \leq \sigma(x, \varepsilon) \leq C \varepsilon^\tau, \tau \geq 0. \)

(H.3) \( \exists \bar{g} \in W^2_2(\Omega) \) with \( \bar{g} = g_1 \) on \( \Gamma_1, \quad \varepsilon \frac{\partial \bar{g}}{\partial \nu} + \sigma(\bar{g} - g_2) = 0 \) on \( \Gamma_2. \)

(H.4) \( \inf_{\Omega} \left\{ c(x) - \frac{1}{2} (\nabla \cdot b)(x) \right\} \geq \alpha_0 > 0. \)

(H.5) \( \tilde{\Gamma}_- \subseteq \Gamma_1. \)

Let \( W^\varepsilon = W^2_2(\Omega) \) if \( \varepsilon > 0, \) \( W^0 = \{ w \in L_2(\Omega) \mid L_0 w \in L_2(\Omega) \}, \Sigma_\varepsilon = \Gamma_1 \) if \( \varepsilon > 0 \) and \( \Sigma_0 = \Gamma_- \). Further let \( V^\varepsilon = \{ w \in W^\varepsilon \mid w = g_1 \) on \( \Sigma_\varepsilon \} \) and \( \dot{V}^\varepsilon = \{ w \in W^\varepsilon \mid w = 0 \) on \( \Sigma_\varepsilon \}. \)

The variational formulation of \((\mathcal{L}_\varepsilon)\) is

\[ (2.1) \quad u_\varepsilon \in V^\varepsilon: \quad B_{\varepsilon, \sigma}(u_\varepsilon, v) = l_\sigma(v) \quad \forall v \in \dot{V}^\varepsilon \]

where

\[ (2.2) \quad B_{\varepsilon, \sigma}(u, v) = \varepsilon (\nabla u, \nabla v) + (b \cdot \nabla u + cu, v) + \int_{\Gamma_2} \sigma uv \ ds \]

\[ (2.3) \quad l_\sigma(v) = (f, v) + \int_{\Gamma_2} \sigma g_2 v \ ds. \]

With respect to the weighted norm \( \| \cdot \|_{\varepsilon, 0} \) defined by

\[ (2.4) \quad \| u \|_{\varepsilon, 0} = (\varepsilon \| \nabla u \|_{0,2}^2 + \alpha_0 \| u \|_{0,2}^2 \]

\[ + \int_{\Gamma_-} \sigma u^2 \ ds + \frac{1}{2} \int_{\Gamma_+} (b \cdot v) u^2 \ ds)^{1/2} \]

we obtain using Green's formula and standard inequalities.

**Lemma 2.1:** Under the hypotheses (H.1)-(H.5) it holds for any \( u, v \in W^2_2(\Omega) \) and \( 0 \leq \varepsilon \leq 1. \)

(i) \( B_{\varepsilon, \sigma}(u, u) \geq \| u \|_{\varepsilon, 0}^2 - \frac{1}{2} \int_{\Gamma_-} \frac{b \cdot v}{\| u \|_{0,2}} u^2 \ ds. \)

(ii) \( | B_{\varepsilon, \sigma}(u, v) | \leq C ( \| u \|_{\varepsilon, 0} \| v \|_{\varepsilon, 0} + \| u \|_{0,2} \| b \cdot \nabla v \|_{0,2} + \]

\[ + | \int_{\Gamma_-} \frac{b \cdot v}{\| u \|_{0,2}} uv \ ds | ). \]
As a consequence of Lemma 2.1 (i) there exists an unique solution \( u_\varepsilon \) of (2.1), \( 0 \leq \varepsilon \leq 1 \).

Let us now assume a sharper hypothesis as (H.3)\(^+\).

(H.3) The solution of (2.1) belongs to \( W^2_2(\Omega) \) and satisfies
\[
\varepsilon | u_\varepsilon |_{2,2} \leq C \left( \| L_\varepsilon u_\varepsilon \|_{0,2} + \| u_\varepsilon \|_{1,2} \right).
\]

Remark 2.1: Consider the following imbedded form of (2.1) with \( 0 \leq \delta \leq 1 \)

\[ (2.1)_\delta \quad u \in V^\varepsilon : \quad B_{\varepsilon, \delta}(u, v) = l_\varepsilon^\delta(v) \quad \forall v \in \hat{V}^\varepsilon \]
with
\[
(2.2)_\delta \quad B_{\varepsilon, \delta}(u, v) = \varepsilon (\nabla u, \nabla v) - \varepsilon \delta \sum_{i=1}^l (\Delta u, b \cdot \nabla v)_{\Omega_i} + \\
\quad + (L_0 u, v + \delta b \cdot \nabla v) + \int_{\Gamma_2} \sigma u v \, ds.
\]

(2.3)_\delta \quad l_\varepsilon^\delta(v) = (f, v + \delta b \cdot \nabla v) + \int_{\Gamma_2} \sigma g_2 v \, ds
\]

where \( \tilde{\Omega} = \bigcup_{i=1}^l \tilde{\Omega}_i \), \( \Omega_i \cap \Omega_j = \emptyset \) if \( i \neq j \). Under our hypotheses the solutions of (2.1) and (2.1)_\delta are identical. ■

Further we have the following a priori estimates.

**Lemma 2.2:** Under the hypotheses (H.1)-(H.5) it holds for the solution \( u_\varepsilon \) of (2.1) with a constant \( K(f, \tilde{g}) \) independent on \( \varepsilon \).

(i) \[ \varepsilon^{3/2} | u_\varepsilon |_{2,2} + \| u_\varepsilon \|_{\varepsilon, 0} \leq K(f, \tilde{g}) . \]

In case of \( u_\varepsilon \in W^l_{2} + 1(\Omega) \), it is

(ii) \[ \varepsilon^{l+1/2} | u_\varepsilon |_{l+1,2} \leq K(f, \tilde{g}) , \quad l \geq 1 . \]

**Proof:** The estimate \( \| u_\varepsilon \|_{\varepsilon, 0} \leq K(f, \tilde{g}) \) follows from Lemma 2.1 with foregoing homogenization of \( (\mathcal{L}_\varepsilon) \) by the aid of (H.3)\(^+\). Hence the assertion (i) follows by (H.3).

The second assertion follows by induction. ■

Remark 2.2: The estimates of Lemma 2.2 are sharp in the case \( \Gamma_1 \cap \Gamma_1 \neq \emptyset \). ■
In the singularly perturbed case $0 \leq \varepsilon \leq \varepsilon_0 \ll 1$, the properties of the solution $u_\varepsilon$ of $(\mathcal{L}_\varepsilon)$ are depending on the properties of the solution $u_0$ of the limit problem $(\mathcal{L}_0)$. It holds
\[
\lim_{\varepsilon \to 0} \|u_\varepsilon - u_0\|_{0,2} = 0
\]
if $\partial \Omega = \Gamma_1$ (Bardos-Rauch [3], Th. 1). Note that we cannot ensure in general that $u_0 \in W^1_2(\Omega)$. In regard of uniform in $\varepsilon$ discretization error estimates we need the following sharpened regularity and convergence hypotheses on the solutions of $(\mathcal{L}_\varepsilon)$ and $(\mathcal{L}_0)$, respectively.

(H.6) \[ \exists r \in N_0: \ u_0 \in W^{r+1}_p(\Omega), \ 2 \leq p \leq + \infty. \]

(H.7) \[ \exists \beta > 0: \ \|u_\varepsilon - u_0\|_{0,2} \leq K(f, \tilde{g}) \varepsilon^\beta \]

with a constant $K(f, \tilde{g})$ independent on $\varepsilon$.

Sometimes it holds even

(H.8) \[ \exists \beta > 0: \ \|u_\varepsilon - u_0\|_{\varepsilon,0} \leq K(f, \tilde{g}) \varepsilon^\beta \]

with a constant $K(f, \tilde{g})$ independent on $\varepsilon$.

The question of sufficient conditions for (H.6)-(H.8) will be discussed in Section 5.

For subdomains $G \subseteq \Omega$ we denote by $(\partial G)_-$, $(\partial G)_+$ and $(\partial G)_0$, respectively, the inflow, outflow and characteristic parts of $\partial G$, respectively. In preparation of interior discretization error estimates, we give the following result for subdomains of special type. First of all, we need some notations. Let $\xi_x(\tau)$ be the solution of
\[ \dot{\xi}(\tau) = b(\xi(\tau)), \ \xi(0) = x \in \bar{\Omega} \]
(the "streamline" passing through $x \in \bar{\Omega}$). For any point $x \in \Omega \cup \bar{\Gamma}_-$
\[ \tau_+(x) = \inf \{ \tau > 0 | \xi_x(\tau) \notin \Omega \} \]
denotes the first exit time of $\xi_x(\tau)$ from $\Omega$. Let for $\Gamma' \subseteq \bar{\Gamma}_-$ be
\[ E(\Gamma') = \{ \xi_x(\tau) | x \in \Gamma', \ 0 \leq \tau \leq \tau_+(x) \} \].

DEFINITION: A domain $\Omega$ is of "channel type" if $\bar{\Omega} = E(\bar{\Gamma}_-)$. In particular, this property implies that all streamlines $\xi_x(\tau)$, $x \in \bar{\Omega}$ leave $\Omega$ in finite time.

The result announced above is:

M² AN Modélisation mathématique et Analyse numérique
Mathematical Modelling and Numerical Analysis
LEMMA 2.3: Let $\Omega_i$, $i = 1, 2$ be simply connected domains of "channel type" with $\Omega_1 \subseteq \Omega_2 \subseteq \Omega$. Further let

$$\text{dist } ((\partial\Omega_1)_+, (\partial\Omega_2)_+) \geq C_1(s) \ln q,$$

$$\text{dist } ((\partial\Omega_1)_0, (\partial\Omega_2)_0) \geq C_2(s) \sqrt{q} \ln \sqrt{q}$$

and $b \cdot v \big|_{(\partial\Omega_1)_+} \equiv b_0 > 0$. Then it holds with $q = \varepsilon$ and constants $K(f, \bar{g})$ independent on $\varepsilon$

(i) $\|u_\varepsilon\|_{l+1,2,\Omega_1} \leq K(f, \bar{g})$ if $u_\varepsilon \in W_2^{l+1}(\Omega_2)$.

(ii) $\|u_\varepsilon\|_{0,2,\Omega_1} \leq K(f, \bar{g})\varepsilon$ if $L u_\varepsilon = 0$ in $\Omega_2$.

(iii) $\|u_\varepsilon - u_0\|_{0,p,\Omega_1} \leq K(f, \bar{g})\varepsilon$ with $2 \leq p \leq +\infty$

if $u_\varepsilon, u_0 \in W_2^0(\Omega_2)$. ■

Proof: Assertion (i) holds in case of $\partial\Omega = \Gamma_1$ (cf. Nävert [14], Th. 2.3). We can generalize the proof given there to our case since another structure of the boundary values on $\partial\Omega \setminus \Gamma_-$ is not essential.

(ii) We use a modification of the proof of Lemma 2.2 in Schatz-Wahlbin [18] in the case $b \equiv 0$. The proof is somewhat technical and therefore omitted.

(iii) We can generalize the proof given in Goering et al. [7], Theorem 4.3 using a generalized maximum principle (cf. Lemma 5.3). ■

3. AN ASYMPTOTICALLY FITTED GALERKIN FINITE ELEMENT METHOD

Let $\mathcal{C}_h$ be a quasi-uniform triangulation of $\bar{\Omega}$ with (for simplicity)

$\bar{\Omega} = \bigcup_{i=1}^{l(h)} \bar{\tau}_i$. Let $S_h \subset W_\infty^1(\Omega)$ be a finite element space of piecewise polynomials of degree $k$ satisfying

(A.1) (inverse property)

$$\forall v \in S_h, \forall \tau_i \in \mathcal{C}_h : \|v\|_{m,2,\tau_i} \leq Ch^{-1}\|v\|_{m-1,2,\tau_i}, \quad m \geq 1.$$

(A.2) (approximation property)

$$\forall v \in W_2^{l+1}(\Omega), \ 1 \leq l \leq k \text{ with } v = g \text{ on } \Gamma' \subseteq \partial\Omega$$

$$\exists \Pi_h v \in S_h, \quad \Pi_h v = g_h \text{ on } \Gamma' \text{ with } |g_h|_{0,\infty,\Gamma'} \leq |g|_{0,\infty,\Gamma'}$$

vol. 22, n° 3, 1988
such that
\[ h^2 \| v - \Pi_h v \|_{2,2,h} + h \| v - \Pi_h v \|_{1,2} + \| v - \Pi_h v \|_{0,2} + \sqrt{h} \| v - \Pi_h v \|_{0,2,\Gamma} \leq C h^{l+1} \| v \|_{l+1,2} \]
where
\[ \| \cdot \|_{2,2,h} = \left( \sum_{i=1}^{I(h)} \| \cdot \|_{2,2,\tau_i}^2 \right)^{1/2}. \]

Let \( V^\varepsilon_h = \{ w \in S_h | w = g_1, h \text{ on } \Sigma^\varepsilon \} \) with \( g_1, h = \Pi_h g_1 \) and \( \hat{V}^\varepsilon_h = S_h \cap \hat{V}^\varepsilon. \)
The standard Galerkin finite element method (SG-FEM) is
\[(3.1) \quad u_{\varepsilon,h} \in V^\varepsilon_h : B_{\varepsilon,\sigma}(u_e, u_{\varepsilon,h}, v) = 0 \quad \forall \varepsilon \in \hat{V}^\varepsilon_h \]
and it holds

**Lemma 3.1:** Suppose that the hypotheses (H.1)-(H.5) and (A.1), (A.2) are valid. Then we have the bounds for the scheme SG-FEM

(i) \[ \| u_{\varepsilon,0} - u_{\varepsilon,h} \|_{\varepsilon,0} \leq C \inf_{w \in V^\varepsilon_h} \left\{ \| u_{\varepsilon,0} - w \|_{\varepsilon,0} + \frac{1}{\varepsilon} \| u_{\varepsilon,0} - w \|_{0,2} \right\}. \]

(ii) \[ \| u_{\varepsilon} - u_{\varepsilon,h} \|_{0,2} \leq C \frac{h}{\varepsilon} \left( 1 + \frac{\varepsilon}{h} \right) \| u_{\varepsilon} - u_{\varepsilon,h} \|_{\varepsilon,0}. \]

Moreover if \( u_{\varepsilon} \in W^{l+1}_2(\Omega), 1 \leq l \leq k \) and \( \varepsilon \leq h \), we find
\[ \| u_{\varepsilon} - u_{\varepsilon,h} \|_{0,2} + \frac{h}{\varepsilon} \| u_{\varepsilon} - u_{\varepsilon,h} \|_{\varepsilon,0} \leq C \frac{h^{l+1}}{\varepsilon} \| u_{\varepsilon} \|_{l+1,2}. \]

**Proof:** It is a straightforward generalization of the result given in Nävert [14] for the special case \( \partial \Omega = \Gamma_1 \) using Lemma 2.1 and Aubin's duality trick. ■

**Remark 3.1:** Lemma 2.2 implies that the estimates in Lemma 3.1 are uniformly valid on \( h^\kappa \leq \varepsilon \leq 1, \ 0 \leq \kappa < 1 \)
\[ \sup_{h^\kappa \leq \varepsilon \leq 1} \left( \| u_{\varepsilon,0} - u_{\varepsilon} \|_{0,2,0} + \frac{h}{\varepsilon} \| u_{\varepsilon,0} - u_{\varepsilon,h} \|_{\varepsilon,0} \right) \leq K(f, \tilde{g}) h^{(l+1)(1-\kappa)}. \]

**Remark 3.2:** For the scheme SG-FEM we obtain with respect to the adapted norm \( \| \cdot \| \) (cf. Schieweck [19]) defined by
\[ \| u \| = \| u \|_{\varepsilon,0} + \sup_{0 \neq w \in V^\varepsilon_h} \frac{B_{\varepsilon,\sigma}(u, w)}{\| w \|_{\varepsilon,0}} \]

\[ M^2 \text{ AN Modélisation mathématique et Analyse numérique} \]

Mathematical Modelling and Numerical Analysis
the following error estimate

\[
\| u_{\epsilon} - u_{\epsilon,h} \|_{\tau,0} \leq C \inf_{w \in V_h} \| u_{\epsilon} - w \|.
\]

Exponentially fitted schemes with adding boundary layer-like functions to the trial space are one way to obtain useful error estimates in case of \(0 \leq \epsilon \ll 1\) (cf. Schieweck [19] with \(N \equiv 2\)). Because of the difficulties arising for such schemes in case of a more complex geometry, we propose another modified Galerkin scheme. Let \(u_{\epsilon}\) be the solution of \((\mathcal{L}_\epsilon)\) in the special case \(\Gamma_- = \Gamma_1, \Gamma_2 = \Gamma_+ \cup \Gamma_0, \sigma = g_2 = 0\):

\[
(3.3) \quad u_{\epsilon} \in W_2^1(\Omega) \cap V^0: \quad B_{\epsilon,0}(u_{\epsilon}, v) = (f, v) \quad \forall v \in W_2^1(\Omega) \cap \tilde{V}^0
\]

and let \(u_{\epsilon,h}\) be the projection corresponding to SG-FEM

\[
(3.4) \quad u_{\epsilon,h} \in V_h^0: \quad B_{\epsilon,0}(u_{\epsilon} - u_{\epsilon,h}, v) = 0 \quad \forall v \in \tilde{V}_h^0.
\]

Then we define the modified Galerkin scheme MG-FEM

\[
(MG-FEM) \quad u_{h}^{(M)} = \begin{cases} 
\text{defined by (3.1) if } h^\kappa \leq \epsilon \leq 1 \\
\text{defined by (3.4) if } 0 \leq \epsilon \leq h^\kappa.
\end{cases}
\]

We obtain the following result.

**Lemma 3.2:** Under the hypotheses (H.1)-(H.5) and (A.1), (A.2) it holds for the scheme MG-FEM with \(0 \leq \epsilon < h^\kappa, \kappa > 0\)

\[
(3.5) \quad \begin{cases} 
\| u_{\epsilon} - u_{h}^{(M)} \|_{0,2} \leq \| u_{\epsilon} - \tilde{u}_{\epsilon} \|_{0,2} + \\
+ \| u_{\epsilon} - u_{h}^{(M)} \|_{\epsilon,0} \leq \| u_{\epsilon} - \tilde{u}_{\epsilon} \|_{\epsilon,0} + \\
+ C \left( \| \tilde{u}_{\epsilon} - u_0 \| + \inf_{w \in V_h^0} \| u_0 - w \| \right).
\end{cases}
\]

Moreover, if (H.6), (H.7) with \(r \equiv 1\) and (H.6), (H.8) with \(r \equiv 1\), respectively, are satisfied, we have the bounds

(i) \(\| u_{\epsilon} - u_{h}^{(M)} \|_{0,2} \leq K(f, \tilde{\mathcal{G}}) \left( h^r + \epsilon^\min \{1/2, \tilde{\beta} \} \right)\)

and

(ii) \(\| u_{\epsilon} - u_{h}^{(M)} \|_{\epsilon,0} \leq K(f, \tilde{\mathcal{G}}) \left( h^r + \epsilon^\min \{1/2, \tilde{\beta} \} \right),\)

respectively.
Proof: (3.5) follows from triangle inequality and (3.2). By Lemma 5.2 we have

\[ \| u_\epsilon - u_0 \| \leq C \sqrt{\epsilon} |u_0|_{1,2}. \]

Further it is

\[ \| u_0 - w \| \leq \| u_0 - w_{\epsilon,0} + C \| u_0 - w \|_{1,2} \]

and with the approximation property (A.2)

\[ \inf_{w \in V_h^0} \| u_0 - w \| \leq Ch^r |u_0|_{r+1,2}. \]

The assertions (i), (ii) are obtained combining (3.5)-(3.7) and (H.6), (H.7) or (H.6), (H.8).

Regarding uniform in $\epsilon$ discretization error estimates we obtain now from Remark 3.1 and Lemma 3.2.

**Theorem 3.1**: Let the hypotheses of Lemma 3.2 be satisfied and let $u_\epsilon \in W_2^{l+1}(\Omega)$ and $u_0 \in W_2^r(\Omega)$, $1 \leq l \leq k$, $1 \leq r \leq k$ be the solutions of $(\mathcal{L}_\epsilon)$ and $(\mathcal{L}_0)$, respectively. Then it holds for the scheme MG-FEM

(i) $\| u_\epsilon - u_h^{(M)} \|_{0,2} \leq K(f, \bar{g}) \min \left\{ \left( \frac{h}{\epsilon} \right)^{l+1}; h^r + \epsilon \min \{1/2; \beta\} \right\}$

with $\kappa = \kappa_1 = \frac{l + 1}{l + 1 + \min \{1/2; \beta\}}$ if (H.6), (H.7) are valid.

(ii) $\| u_\epsilon - u_h^{(M)} \|_{\epsilon,0} \leq K(f, \bar{g}) \min \left\{ \left( \frac{h}{\epsilon} \right)^{l}; h^r + \epsilon \min \{1/2; \beta\} \right\}$

with $\kappa = \kappa_2 = \frac{l}{l + \min \{1/2; \beta\}}$ if (H.6), (H.8) are valid.

**Conclusion 3.1**: Theorem 3.1 implies the uniform in $\epsilon$ estimates with the corresponding assumptions of Theorem 3.1 (i), (ii)

(i) $\sup_{0 \leq \epsilon \leq 1} \| u_\epsilon - u_h^{(M)} \|_{0,2} \leq K(f, \bar{g}) h^{\kappa_1 \min \{1/2; \beta\}}$

(ii) $\sup_{0 \leq \epsilon \leq 1} \| u_\epsilon - u_h^{(M)} \|_{\epsilon,0} \leq K(f, \bar{g}) h^{\kappa_2 \min \{1/2; \beta\}}$. ⊠
4. ASYMPTOTICALLY FITTED STREAMLINED DIFFUSION FINITE ELEMENT SCHEMES

We consider now the streamlined diffusion scheme (SD-FEM) of Hughes-Brooks [8] starting from the imbedded form (2.1) with $0 \leq \varepsilon, \delta \leq 1$.

(4.1) $u^\delta_{\varepsilon, h} \in V_h^\delta : \quad B^\delta_{\varepsilon, \sigma}(u^\varepsilon - u^\delta_{\varepsilon, h}, v) = 0 \quad \forall v \in \tilde{V}_h^\varepsilon$.

Additionally we assume

(H.4) $\forall \delta \in [0 ; 1] : \quad \varepsilon \delta \leq C h^2, \quad \inf_{\Omega} \{1 + \delta c(x)\} \equiv c_\delta \equiv 0$

$$\inf_{\Omega} \left\{ c(x) - \frac{1}{2} (\nabla \cdot b)(x) - \frac{\delta}{2} (\nabla \cdot bc)(x) \right\} \equiv \alpha_\delta > 0.$$

Further let $\| \cdot \|_{\varepsilon, \delta}$ be the norm defined by

(4.2) $\| u \|_{\varepsilon, \delta} = (\varepsilon \| \nabla u \|_{0, 2}^2 + \delta \| b \cdot \nabla u \|_{0, 2}^2 + \alpha_\delta \| u \|_{0, 2}^2 +$

$$+ \int_{\Gamma_2} \sigma u^2 \, ds + \int_{\Gamma_+} (b \cdot v)(1 + \delta c) u^2 \, ds)^{1/2}.$$

**Remark 4.1:** The term $\sqrt{\delta} \| b \cdot \nabla u \|_{0, 2}$ in (4.2) represents an essential effect of the scheme SD-FEM with $\delta > 0$. There is additionally control on the derivative $b \cdot \nabla u$. ■

First of all, we prove $S_h$-ellipticity and continuity of the bilinearform $B^\delta_{\varepsilon, \sigma}(\cdot, \cdot)$.

**Lemma 4.1:** Under the assumptions (H.1)-(H.5) and (A.1), (A.2) it holds form the scheme SD-FEM with $0 \leq \delta \leq 1$

(i) $\forall u_h \in S_h : \quad B^\delta_{\varepsilon, \sigma}(u_h, u_h) \geq \frac{1}{2} \| u_h \|_{\varepsilon, \delta}^2 + \frac{c_\delta}{2} \int_{\Gamma_2} (b \cdot v) u_h^2 \, ds$

(ii) $\forall u, v \in W^2_0(\Omega)$:

$$| B^\delta_{\varepsilon, \sigma}(u, v) | \leq C_1 \| u \|_{\varepsilon, \delta} \| v \|_{\varepsilon, \delta} + C_2 (\varepsilon \delta \| u \|_{2, 2, h} + \| u \|_{0, 2}) \times$$

$$\times \| b \cdot \nabla v \|_{0, 2} + \left( \int_{\Gamma_2} | b \cdot v | u^2 \, ds \right)^{1/2} \left( \int_{\Gamma_+} | b \cdot v | v^2 \, ds \right)^{1/2}. ■$$

**Proof:** It is a straightforward generalization of the result in Johnson et al. [10] for the special case $\partial \Omega = \Gamma_1$. ■

A first consequence is the stability result.
**Lemma 4.2**: Under the hypotheses of Lemma 4.1 there is a unique solution \( u_{e,h}^\delta \in S_h \) of the scheme SD-FEM, \( 0 \leq \delta \leq 1 \) satisfying

\[
\| u_{e,h}^\delta \|_{\epsilon,\delta} \leq C \left( \| f \|_{0,2} + \| \Pi_h \tilde{g} \|_{1,2} + \right.
\]
\[
+ \left| \sqrt{\alpha} g_2 \right|_{0,2,\tau_2} + \left| \sqrt{b \cdot v} \right|_{0,2,\Gamma^-} \right) .
\]

The following error estimates generalize the results given in Johnson et al. [10] for \( \partial \Omega = \Gamma_1 \) and Axelsson [2] for \( \Gamma_1 = \Gamma_- \), \( \Gamma_2 = \Gamma_0 \cup \Gamma_+ \), \( \sigma = 0 \).

**Lemma 4.3**: Under the assumptions of Lemma 4.1 the solutions \( u_{e,h}^\delta \) of SD-FEM with \( \delta > 0 \) and \( u_e \) of \( (\mathcal{L}_e) \), respectively, satisfy

\[
\| u_{e,h}^\delta - u_{e,h}^\delta \|_{\epsilon,\delta} \leq C \inf_{w \in \mathcal{V}_h^\delta} \left\{ \| u_e - w \|_{\epsilon,\delta} + \epsilon \sqrt{\delta} \| u_e - w \|_{2,2,h} + \right.
\]
\[
\left. + \frac{1}{\sqrt{\delta}} \| u_{e,h}^\delta - w \|_{0,2} \right\} .
\]

Together with approximation property

\[
\| u - \Pi_h u \|_{\epsilon,\delta} \leq C h^l (\sqrt{\epsilon} + \sqrt{\delta} + \sqrt{h}) |u|_{l+1,2}
\]

Lemma 4.3 implies:

**Conclusion 4.1**: Let the assumptions of Lemma 4.3 be valid. Moreover, if \( u_e \in W^{l+1}_2(\Omega) \), \( 1 \leq l \leq k \) holds with \( \delta = C h \equiv \epsilon \equiv 0 \), we have the bound

\[
\| u_{e,h}^\delta - u_{e,h}^\delta \|_{\epsilon,\delta} \leq C h^{l+1/2} |u_e|_{l+1,2} .
\]

**Remark 4.2**: According to Lemma 2.2 the estimates of Conclusion 4.1 are not uniformly valid on \( 0 \leq \epsilon \leq 1 \) in the general case setting \( \delta = 0 \) if \( \epsilon \equiv C h \).

Regarding discretization error estimates, uniformly on \( 0 \leq \epsilon \leq 1 \), we propose asymptotically fitted schemes of the following kind. For \( \epsilon \equiv h^k \) we solve \( (\mathcal{L}_e) \) by the scheme SG-FEM. In case of \( 0 \leq \epsilon < h^k \), we solve in a first variant the limit problem \( (\mathcal{L}_0) \) instead of \( (\mathcal{L}_e) \) by SD-FEM

\[
(4.5) \quad u_{e,0}^\delta \in V_h^0 : \quad B_{0,0}^\delta (u - u_{0,h}^\delta, v) = 0 \quad \forall v \in \hat{V}_h^0 .
\]

A second variant is analogously to scheme MG-FEM. We solve problem \( (3.3) \) instead of \( (\mathcal{L}_e) \) by SD-FEM

\[
(4.6) \quad \bar{u}_{e}^\delta \in V_h^0 : \quad B_{e,0}^\delta (\bar{u}_e - \bar{u}_{e,h}^\delta, v) = 0 \quad \forall v \in \hat{V}_h^0 .
\]
Thus, we define the modified streamlined diffusion schemes

\[(\text{MSD.1}) \quad u_h^{(S1)} = \begin{cases} u_{\varepsilon, h} & \text{defined by (3.1) if } h^\kappa \equiv \varepsilon \equiv 1 \\ u_{0, h}^\delta & \text{defined by (4.5) if } 0 \equiv \varepsilon < h^\kappa \end{cases}\]

and

\[(\text{MSD.2}) \quad u_h^{(S2)} = \begin{cases} u_{\varepsilon, h} & \text{defined by (3.1) if } h^\kappa \equiv \varepsilon \equiv 1 \\ u_{0, h}^\delta & \text{defined by (4.6) if } 0 \equiv \varepsilon < h^\kappa \end{cases}\]

Because of definition of the adapted schemes, we have to perform the analysis only in the case $0 \equiv \varepsilon < h^\kappa$. First of all consider scheme (MSD.1).

**Lemma 4.4:** Under the hypotheses of Lemma 4.1 it holds for the scheme MSD.1 with $0 \equiv \varepsilon < h^\kappa$, $0 < \kappa < 1$, $\delta = Ch$

\[(4.7) \quad \begin{align*} \| u_{\varepsilon} - u_h^{(S1)} \|_{0,2} & \equiv \| u_{\varepsilon} - u_0 \|_{0,2} + \\ & + C \inf_{w \in V_h^\delta} \left\{ \| u_0 - w \|_{0,h} + \frac{1}{\sqrt{h}} \| u_0 - w \|_{0,2} \right\} . \end{align*}\]

Moreover, if (H.6), (H.7) and (H.6), (H.8) respectively, are valid, it holds

(i) \[\| u_{\varepsilon} - u_h^{(S1)} \|_{0,2} \equiv K(f, \bar{g})(h^{r + 1/2} + \varepsilon^\beta)\]

and

(ii) \[\| u_{\varepsilon} - u_h^{(S1)} \|_{0,1/\kappa} \equiv K(f, \bar{g})(h^{r + 1/2} + \varepsilon^\beta) ,\]

respectively.

**Proof:** (4.7) follows from triangle inequality and Lemma 4.3 with $\varepsilon = 0$. Assertion (i) is now a consequence of (4.7), (4.3) with $\varepsilon = 0$ and (H.6), (H.7). Further it holds for $0 \equiv \varepsilon < h^\kappa$ with $0 < \kappa < 1$

\[(4.8) \quad \| u_{\varepsilon} - u_0 \|_{0,1/\kappa} \equiv \| u_{\varepsilon} - u_0 \|_{0,\varepsilon} \equiv C \| u_{\varepsilon} - u_0 \|_{\varepsilon,0} \equiv K(f, \bar{g}) \varepsilon^\beta \]

and

\[(4.9) \quad \| u_0 - u_{0,h}^\delta \|_{0,1/\kappa} \equiv \| u_0 - u_{0,h}^\delta \|_{0,1,2} \equiv C h^{r + 1/2} \| u_0 \|_{r + 1/2} .\]

Thus we obtain (ii) combining (4.7)-(4.9).

Summarizing Lemma 4.4 and Lemma 3.1, we obtain:
THEOREM 4.1: Let the hypotheses of Lemma 3.2 be satisfied and let $u_\varepsilon \in W^{l+1}_2(\Omega)$ and $u_0 \in W^{r+1}_2(\Omega)$, $1 \leq l, r \leq k$ be the solutions of $(\mathcal{L}_\varepsilon)$ and $(\mathcal{L}_0)$, respectively. Then it holds for the streamlined diffusion scheme MSD.1

(i) $\| u_\varepsilon - u_h^{(S1)} \|_{0, r} \leq K(f, \bar{g}) \min \left\{ \left( \frac{h}{\varepsilon} \right)^{l+1}; h^{r+1/2} + \varepsilon^\beta \right\}$

with $\kappa = \kappa_3 = \frac{l + 1}{l + \beta + 1}$ if (H.6), (H.7) are valid and

(ii) $\| u_\varepsilon - u_h^{(S1)} \|_{0, \varepsilon^{l/\kappa}} \leq K(f, \bar{g}) \min \left\{ \left( \frac{h}{\varepsilon} \right)^{l}; h^{r+1/2} + \varepsilon^\beta \right\}$

with $\kappa = \kappa_4 = \frac{l}{l + \beta}$ if (H.6), (H.8) are valid.

Conclusion 4.2: Theorem 4.1 implies the uniform in $\varepsilon$ estimates (with the corresponding assumptions of Theorem 4.1 (i) (ii))

(i) $\sup_{0 \leq \varepsilon \leq 1} \| u_\varepsilon - u_h^{(S1)} \|_{0, r} \leq K(f, \bar{g}) h^{\kappa_3 \beta}$

(ii) $\sup_{0 \leq \varepsilon \leq 1} \| u_\varepsilon - u_h^{(S1)} \|_{0, \varepsilon^{l/\kappa}} \leq K(f, \bar{g}) h^{\kappa_4 \beta}$.

Considering scheme MSD.2, we obtain the following global error estimate for sufficiently small $\varepsilon$.

THEOREM 4.2: Under the hypotheses of Lemma 4.1 it holds for the modified streamlined diffusion scheme MSD.2 with

$0 \leq \varepsilon \leq h^\kappa$, $\kappa > 2$ and $\delta = Ch$

(i) $\| u_\varepsilon - u_h^{(S2)} \|_{0, 2} \leq K(f, \bar{g}) \left( h^{r+1/2} + \varepsilon^\beta + \frac{\sqrt{\varepsilon}}{h} \right)$

if (H.6), (H.7) are valid and

(ii) $\| u_\varepsilon - u_h^{(S2)} \|_{0, \varepsilon^{l/\kappa}} \leq K(f, \bar{g}) \left( h^{r+1/2} + \varepsilon^\beta + \frac{\sqrt{\varepsilon}}{h} \right)$

if (H.6), (H.8) are valid.

Proof: We have to modify the proof of Lemma 4.4.

(4.10) $| u_\varepsilon - u_h^{(S2)} | \leq | u_\varepsilon - u_0 | + | u_0 - u_h^\delta | + | u_0^\delta - u_h^\delta |$. 

M² AN Modélisation mathématique et Analyse numérique
Mathematical Modelling and Numerical Analysis
The first right hand side term is bounded by (H.7) or (H.8). For the second term we obtain from Conclusion 4.1 with \( \varepsilon = 0 \)

\[
(4.11) \quad \left\| u_0 - u_{0,h}^\delta \right\|_{0,h} \leq C h^{r+1/2} |u_0|_{r+1,2}.
\]

For the third term we estimate with \( v = \tilde{u}_h^\delta - u_{0,h}^\delta \in \tilde{V}_h^0 \) and using Lemma 4.1 and inverse properties (A.1)

\[
\frac{1}{2} \left\| v \right\|_{0,8} \leq B_{0,0}^\delta(v, v) = B_{v,0}^\delta(\tilde{u}_h^\delta, v) - \varepsilon (\nabla \tilde{u}_h^\delta, \nabla v) + \\
+ \varepsilon \delta \sum_{i=1}^{l(h)} (\Delta \tilde{u}_h^\delta, b \cdot \nabla v)_{\xi_i} - B_{0,0}^\delta(u_{0,h}^\delta, v)
\]

\[
= -\varepsilon (\nabla \tilde{u}_h^\delta, v) + \varepsilon \delta \sum_{i=1}^{l(h)} (\Delta \tilde{u}_h^\delta, b \cdot \nabla v)_{\xi_i}
\]

\[
\leq \varepsilon \left\| \nabla \tilde{u}_h^\delta \right\|_{0,2} \cdot \| v \|_{0,2} + \varepsilon \delta \left\| \tilde{u}_h^\delta \right\|_{2,2,h} \| b \cdot \nabla v \|_{0,2}
\]

\[
\leq \frac{C \varepsilon}{h} \left\| \tilde{u}_h^\delta \right\|_{1,2} \| v \|_{0,2} + \frac{C \varepsilon \delta}{h} \| \tilde{u}_h^\delta \|_{1,2} \| b \cdot \nabla v \|_{0,2}
\]

\[
\leq \frac{\alpha_0}{4} \left\| v \right\|_{0,2}^2 + \frac{C \varepsilon^2}{h^2} \left\| \tilde{u}_h^\delta \right\|_{1,2}^2 + \frac{\delta}{4} \| b \cdot \nabla v \|_{0,2}^2
\]

\[
+ \frac{C \varepsilon^2 \delta}{h^2} \left\| \tilde{u}_h^\delta \right\|_{1,2}^2.
\]

Hence, we obtain together with Lemma 4.2

\[
\frac{1}{4} \left\| v \right\|_{0,8}^2 \leq C \frac{\varepsilon^2}{h^2} (1 + \delta) \left\| \tilde{u}_h^\delta \right\|_{1,2}^2 \leq K(f, \tilde{g}) \frac{\varepsilon}{h^2} (1 + \delta)
\]

and thus

\[
(4.12) \quad \left\| \tilde{u}_h^\delta - u_{0,h}^\delta \right\|_{0,8} \leq K(f, \tilde{g}) \frac{\sqrt{\varepsilon}}{h}.
\]

Summarizing (4.10)-(4.12), we have with (H.6), (H.7) or (H.6), (H.8)

\[
\left\| u_{\varepsilon} - \tilde{u}_h^\delta \right\|_{0,2} \leq K(f, \tilde{g}) \left( \varepsilon^\beta + h^{r+1/2} + \frac{\sqrt{\varepsilon}}{h} \right)
\]

if (H.6), (H.7) are valid and

\[
\left\| u_{\varepsilon} - \tilde{u}_h^\delta \right\|_{0,\varepsilon} \leq C \left\| u_{\varepsilon} - u_0 \right\|_{\varepsilon,0} + \left\| u_0 - \tilde{u}_h^\delta \right\|_{0,h}
\]

\[
\leq K(f, \tilde{g}) \left( \varepsilon^\beta + h^{r+1/2} + \frac{\sqrt{\varepsilon}}{h} \right)
\]

if (H.6), (H.8) are valid. ■

vol. 22, n° 3, 1988
Conclusion 4.3: Theorem 4.2 implies in case of $0 \leq \varepsilon < h^\star$, $\kappa > 2$ the uniform in $\varepsilon$ estimates

$$(i) \quad \sup_{0 \leq \varepsilon \leq h^\star} \| u_\varepsilon - u_h^{(S2)} \|_{0,2,\Omega_1} \leq K(f, \bar{g}) h^{\min \{r+1/2, \kappa/2-1, \beta \kappa \}}$$

in case of (H.6), (H.7)

$$\quad \sup_{0 \leq \varepsilon \leq h^\star} \| u_\varepsilon - u_h^{(S2)} \|_{0,\varepsilon,\Omega_1} \leq K(f, \bar{g}) h^{\min \{r+1/2, \kappa/2-1, \beta \kappa \}}$$

in case of (H.6), (H.8).

Remark 4.3: The question of global error estimates, uniformly on $0 \leq \varepsilon \leq 1$, for the modified streamlined diffusion scheme MSD.2 is open.

Lastly, we consider local discretization error estimates in case of sufficiently small $\varepsilon$, say $0 \leq \varepsilon < h$. Let additionally be the local interpolation property (A.3) of Nävert [14] be satisfied which is valid in case of Lagrangian triangular elements or tensor products of one-dimensional Lagrangian elements. Nävert [14] proved in case of $\partial \Omega = \Gamma_1$ for subdomains of "channel type":

**Lemma 4.5**: Let $\partial \Omega = \Gamma_1$ and let the hypotheses of Lemma 4.1 and (A.3) be valid. Let $\Omega_i$, $i = 1, 2$ be subdomains of $\Omega$ satisfying the hypotheses of Lemma 2.3 with $q = h$. Then it holds for the streamlined diffusion scheme SD-FEM, $0 \leq \varepsilon < h$, $\delta = Ch$ with $u_\varepsilon \in W_2^{l+1}(\Omega_2)$, $1 \leq l \leq k$

$$\| u_\varepsilon - u_\varepsilon^{(S1)} \|_{\varepsilon, h, \Omega_1} \leq K(f, \bar{g}) h^{\min \{s, l+1/2 \}}.$$

By the aid of Lemma 4.5 which is valid also for the problem $(\mathcal{L}_\varepsilon)$, we obtain for the modified streamlined diffusion schemes MSD.1 and MSD.2.

**Theorem 4.3**: Under the assumption of Lemma 4.5 (without the restriction $\partial \Omega = \Gamma_1$) it holds for the solutions $u_h^{(S1)}$ and $u_h^{(S2)}$, respectively, of scheme MSD.1 and MSD.2, respectively, and with $s \geq l + 1/2$.

$$(i) \quad \| u_\varepsilon - u_h^{(S1)} \|_{0,2,\Omega_1} \leq K(f, \bar{g}) (h^{l+1/2} + \varepsilon)$$

$$(ii) \quad \| u_\varepsilon - u_h^{(S2)} \|_{0,2,\Omega_1} \leq K(f, \bar{g}) (h^{l+1/2}).$$

**Proof**: Assertion (i) follows from Lemma 2.3 (iii), Lemma 4.5 with $\varepsilon = 0$ and triangle inequality. Assertion (ii) is a consequence of the estimate
where the first right hand side term is bounded by Lemma 2.3 (ii) (because of \( L^e(u_e - \bar{u}_e) = 0 \)) and the second term by Lemma 4.5 with \( s \equiv l + 1/2 \).

**Remark 4.4**: Assume that the inverse property

\[
\| v_h \|_{0, \infty, \tau_i} \leq C h^{-N/2} \| v_h \|_{0, 2, \tau_i}, \quad \forall v_h \in S_h, \quad \forall \tau_i \in \mathcal{G}_h
\]

holds. Then we obtain the (non-optimal) \( L^\infty \)-estimate

\[
\| u_e - u_h^{(S2)} \|_{0, \infty, \Omega_1} \leq C \left\{ \inf_{w \in V_h^t} \| u_e - w \|_{0, \infty, \Omega_1} + h^{-N/2} \| u_e - u_h^{(S2)} \|_{0, 2, \Omega_1} \right\}
\]

\[
\leq K(f, \bar{g}) h^{l + 1/2 - N/2}
\]

if \( u_e \in C(\bar{\Omega}_1) \cap W^{l+1}_\infty(\Omega_1) \). The boundedness of \( |u_e|_{l+1, \infty, \Omega_1} \) follows from Lemma 2.3 (i) and Sobolev's imbedding theorem.

5. SOME ASYMPTOTIC ERROR ESTIMATES

It remains to give conditions which are sufficient for the hypotheses (H.6)-(H.8). Clearly, such conditions depend essentially on the asymptotic behaviour of the solution \( u_e \) of \((\mathcal{L}_e)\) and on the limit problem \((\mathcal{L}_0)\).

We restrict the considerations to domains of "channel type" or to situations where \( \Gamma^- \) is closed in \( \partial \Omega \). More precisely, let one of the following conditions be satisfied:

(C.1) \( \Gamma^- \) closed in \( \partial \Omega \); \( c(x) \equiv c_0(r) \) with sufficiently large \( c_0 \) (estimates of \( c_0 \) are given in [7], Th. 3.1).

(C.2) \( \Gamma^- \) closed in \( \partial \Omega \); \( \exists \eta(x) : b \cdot \nabla \eta \equiv b_0 > 0 \) in \( \bar{\Omega} \), \( c(x) \equiv c_0 \) with \( c_0 \) arbitrary.

(C.3) \( \Gamma^- \) simply connected; \( \bar{\Omega} = E(\bar{\Gamma}_-), b \cdot \nu \neq 0 \) on \( \bar{\Gamma}_- \cup \bar{\Gamma}_+ \), \( c(x) \equiv c_0 \) with \( c_0 \) arbitrary.

Lemma 5.1 yields sufficient conditions for the regularity assumption (H.6) (cf. [7], Sect. 3).

**Lemma 5.1**: Let the data of \((\mathcal{L}_0)\) and \( \Gamma_- , \Gamma_+ , \Gamma_0 \) be sufficiently smooth. Then any of the conditions (C.1)-(C.3) is sufficient for (H.6).
Remark 5.1: Condition on \( r \) in (C.2) is equivalent to the fact that all streamlines \( \xi(\tau), \ x \in \Omega \) leave \( \bar{\Omega} \) in finite time (Devinatz-Ellis-Friedman [4]).

Remark 5.2: We find local regularity statements of type (H.6) applying Lemma 5.1 on subdomains \( G \subseteq \Omega \), especially in case of \( G \subseteq E(\bar{\Gamma}_-) \).

Remark 5.3: Sometimes we need instead of (H.6) the weaker hypothesis. (H.6)\( ^* \) \( u_0 \) is semiregular in \( \Omega \) (there exists a constant \( M > 0 \) such that \( \Delta u_0 \leq u \) in the sense of \( W_2^{-1}(\Omega) \)).

Felgenhauer ([5], Th. 5) gives sufficient conditions for (H.6)\( ^* \) in case of (C.3) and piecewise smooth \( \bar{\Gamma}_- \).

Consider now the assumptions (H.7) and (H.8) of asymptotic convergence. In case of the special mixed boundary value problem

\[
\begin{align*}
\begin{cases}
L_{\epsilon} u_{\epsilon} = f & \text{in} \quad \Omega \subset \mathbb{R}^N \\
u_{\epsilon} = g & \text{on} \quad \Gamma_1 = \Gamma_- , \quad \epsilon \frac{\partial u_{\epsilon}}{\partial v} + \sigma u = 0 & \text{on} \quad \Gamma_2 = \Gamma_+ \cup \Gamma_0 \\
\text{with} \quad 0 \leq \sigma(x, \epsilon) \leq C \epsilon^\tau , \quad \tau > 0
\end{cases}
\end{align*}
\]  

(5.1)

it holds

**Lemma 5.2:** Under the hypotheses (H.1)-(H.5) and (H.6) with \( r = 0, \ p = 2 \) the solution of (5.1) satisfies the hypotheses (H.7), (H.8) with \( \beta = \beta = 1/2 \min \{1 \ ; \tau \} \).

Verification of (H.7), (H.8) is more complicated if essential boundary conditions are given on \( \Gamma' \subseteq \Gamma_0 \cup \Gamma_+ \). Let us consider for simplicity Dirichlet's problem

\[
\begin{align*}
\begin{cases}
L_{\epsilon} u_{\epsilon} = f & \text{in} \quad \Omega \subset \mathbb{R}^N \\
u_{\epsilon} = g & \text{on} \quad \partial \Omega = \Gamma_1 .
\end{cases}
\end{align*}
\]  

(5.2)

As a rule, we need some more information about the asymptotic behaviour of \( u_{\epsilon} \) for \( 0 < \epsilon \leq \epsilon_0 \ll 1 \). We discuss two variants.

In case of sufficient smooth datas of \( (\mathcal{L}_{\epsilon}) \) and of (H.6) with sufficiently large \( r \), there exists an asymptotic expansion

\[
u_{\epsilon}(x, \epsilon) = u_0(x) + \sum_{j=1}^{j} v_j(x, \epsilon)
\]
where $u_0$ denotes the solution of $(\mathcal{L}_0)$ and the $v_j$ are boundary layer terms. If asymptotic error estimates and estimates of the $v_j$ in integrals norms are available we find

$$\|u_\varepsilon - u_0\|_{0,2} \lesssim \|u_\varepsilon - u_{as}\|_{0,2} + \sum_{j=1}^{J} \|v_j\|_{0,2}$$

or

$$\|u_\varepsilon - u_0\|_{\varepsilon,0} \lesssim \|u_\varepsilon - u_{as}\|_{\varepsilon,0} + \sum_{j=1}^{J} \|v_j\|_{\varepsilon,0} .$$

**Remark 5.4**: Let $\Gamma_1 \cap \Gamma_+ \neq \emptyset$. Then there exists an index $j$ such that $\|v_j\|_{\varepsilon,0} \neq 0$ if $\varepsilon \to 0$. Hence (H.8) cannot be valid.

The disadvantages of the first method are the smoothness assumptions and the explicit construction of boundary layer terms. Another way is to use a generalized maximum principle and boundary layer-like barrier functions.

**Lemma 5.3** (cf. Felgenhauer [5], Th. 2): Under the hypotheses (H.1)-(H.5) let be $v \leq s$ on $\partial G$. Further let $L_{\varepsilon} v \leq L_{\varepsilon} s$ in the sense of $W_2^{-1}(G)$, $G \subseteq \Omega$. Then it holds $v \equiv s$ a.e. in $G$.

Hence we prove (H.7) by the aid of a pointwise valid estimate

$$\left| (u_\varepsilon - u_0)(x) \right| \lesssim s(x, \varepsilon) \text{ a.e. in } \Omega$$

via $\|u_\varepsilon - u_0\|_{0,2} \lesssim \|s\|_{0,2}$. Pointwise estimates of type (5.3) are given in [7], Section 4.1 by the aid of classical maximum principle (cf. Protter-Weinberger [16]). Using Lemma 5.3 we can weaken the smoothness assumptions. We omit the somewhat technical details referring to [7] and give only results.

**Lemma 5.4**: Let $\Gamma_-$ be closed in $\partial \Omega$ (cf. (C.1), (C.2)) and let $\overline{\Gamma}_-$, $\overline{\Gamma}_+$, $\Gamma_0$ be sufficiently smooth. Let $u_0 \in W_\infty^0(\Omega)$ or let $u_0$ be semiregular in $\Omega$. Then one of the following conditions is sufficient for (H.7):

(i) $\partial \Omega = \Gamma_-$ (hence $\Gamma_+ = \Gamma_0 = \emptyset$) with $\beta = 1$.

(ii) $\partial \Omega = \Gamma_- \cup \Gamma_+$ (hence $\Gamma_0 = \emptyset$) with $\beta = 1/2$.

(iii) Let for $F(x) = - \text{dist}(x, \overline{\Gamma}_+ \cup \Gamma_0)$ in $U_\gamma(\overline{\Gamma}_+ \cup \Gamma_0) = \{x \in \Omega \mid F(x) \leq \gamma\}$ be $b \cdot \nabla F \geq 0$. Then it is $\beta = 1/4$.

**Lemma 5.5**: Let (C.3) be valid and let $\overline{\Gamma}_+$, $\Gamma_0$ be sufficiently smooth. Let $u_0 \in W_\infty^0(\Omega)$ or let $u_0$ be semiregular in $\Omega$. Then it holds (H.7) with $\beta = 1/2$ if $\Gamma_0 = \emptyset$ and $\beta = 1/4$ else.
Remark 5.5: In case of a convex polyhedron $\Omega$ (or convex $\Omega$ with piecewise smooth $\Gamma_+ \cup \Gamma_0$), the result of Lemma 5.5 remains valid.

Remark 5.6: In case of interior boundary layers ("shocks") we can prove estimates of type (H.7), (H.8) in subdomains away from the shock.

6. CONCLUDING REMARKS.

Under certain assumptions concerning the asymptotic behaviour of the solution $u_\varepsilon$ of $(\mathcal{L}_\varepsilon)$ and the limit solution $u_0$ of $(\mathcal{L}_0)$ we proved global $L_2$-error estimates which are uniformly valid on $0 \leq \varepsilon \leq \varepsilon_0$ for the modified Galerkin f.e.m. MG-FEM and for the modified streamlined diffusion f.e.m.'s MSD.1 and MSD.2. Sometimes error estimates in the weighted norm $\| \cdot \|_{\varepsilon,\delta}$ are possible. Obviously, the modified streamlined diffusion schemes MSD.1 and MSD.2 seem to be favourable because of the improved global $L_2$-error estimate $O(h^{r+1/2})$ and the high order and uniform in $\varepsilon$ local error estimates in case of small $\varepsilon$.

The question of quasioptimal interior $L_\infty$-estimates which are uniformly in $\varepsilon$ valid has been discussed in Schatz-Wahlbin [18] for $b \equiv 0$, $N \leq 2$. In case of $b \not\equiv 0$ this seems to be an open problem. One way to overcome this situation is perhaps given by the scheme proposed in Mizukami-Hughes [13], where a discrete maximum principle holds in the special case $N = 2$, $k = 1$.

REFERENCES


vol. 22, n° 3, 1988
