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ON THE APPLICATION OF MIXED FINITE ELEMENT METHODS TO THE WAVE EQUATIONS (*)

by Tunc Geveci (1)

Abstract — The convergence of certain semidiscrete approximation schemes based on the « velocity-stress » formulation of the wave equation and spaces such as those introduced by Raviart and Thomas is discussed. The discussion also applies to similar schemes for the equations of elasticity.

Resumé. — La convergence de certains schémas d'approximation semi-discrète basés sur la formulation « vitesse-contrainte » de l'équation d'onde et d'espace tel que ceux introduits par Raviart et Thomas est discuté. La discussion s'applique également pour les schémas similaires aux équations d'élasticité.

1. THE « VELOCITY-STRESS » FORMULATION OF THE WAVE EQUATION AND A SEMIDISCRETE VERSION

Let us consider the following initial-boundary value problem for the wave equation:

\[ D_t^2 u(t, x) - \Delta u(t, x) = f(t, x), \quad t > 0, \quad x \in \Omega \subset \mathbb{R}^2, \]

\[ u(t, x) = 0, \quad t > 0, \quad x \in \Gamma, \]

\[ u(0, x) = u_0(x), \quad D_t u(0, x) = v_0(x), \quad x \in \Omega, \]

where \( \Omega \) is a bounded domain with boundary \( \Gamma \), and \( f, u_0, v_0 \) are given functions. Introducing the « stress » \( \sigma = \nabla u \), (1.1) may be reformulated as

\[ D_t^2 u(t, x) - \text{div} \sigma(t, x) = f(t, x), \quad t > 0, \quad x \in \Omega, \]

\[ \sigma(t, x) = \nabla u(t, x), \quad t > 0, \quad x \in \Omega, \]

\[ u(t, x) = 0, \quad t > 0, \quad x \in \Gamma, \]

\[ u(0, x) = u_0(x), \quad D_t u(0, x) = v_0(x), \quad x \in \Omega. \]

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We use the notation of Johnson and Thomée [12]:

\[ V = L^2(\Omega), \quad H = \{ \chi \in L^2(\Omega)^2 : \text{div} \, \chi \in L^2(\Omega) \}. \]

Using Green's formula

\[
\int_{\Omega} u \, \text{div} \, \chi \, dx = \int_{\Gamma} u \chi \cdot n \, ds - \int_{\Omega} \nabla u \cdot \chi \, dx
\]

where \( n \) is the unit exterior normal to \( \Gamma \), a Galerkin version of (1.2) is to seek \( u(t) \in V, \sigma(t) \in H, \, t > 0, \) satisfying

\[
(D_t^2 u(t), w) - (\text{div} \, \sigma(t), w) = (f(t), w), \quad w \in V, \quad (1.3)
\]

\[
(\sigma(t) \chi) + (u(t), \text{div} \, \chi) = 0, \quad \chi \in H,
\]

\[
u(0) = u_0, \quad \sigma(0) = 0,
\]

where the parentheses denote the appropriate inner products (\( L^2 \)-inner product in \( V \), \( L^2(\Omega)^2 \)-inner product in \( H \)). If \( V_h \subset V \) and \( H_h \subset H \) are finite dimensional subspaces, such as the spaces introduced by Raviart and Thomas [13], and by Brezzi, Douglas, Jr. and Marini [6], a semidiscrete version of (1.3) seeks \( u_h(t) \in V_h, \sigma_h(t) \in H_h, \, t > 0, \) satisfying

\[
(D_t^2 u_h(t), w_h) - (\text{div} \, \sigma_h(t), w_h) = (f(t), w_h), \quad w_h \in V_h, \quad (1.3_h)
\]

\[
(\sigma_h(t), \chi_h) + (u_h(t), \text{div} \, \chi_h) = 0, \quad \chi_h \in H_h,
\]

\[
u_h(0) = u_{0,h}, \quad \sigma_h(0) = 0h,
\]

where \( u_{0,h}, v_{0,h} \in V_h \) are approximations to \( u_0 \) and \( v_0 \), respectively.

Johnson and Thomée [12] have discussed the parabolic counterpart of (1.3). The analysis of convergence of (1.3) can be carried out along similar lines, parallel to Baker and Bramble [4], for example. (1.3) is treated essentially as a non-conforming « displacement » model for the wave equation (1.1). The purpose of this note is to discuss the convergence of the « velocity-stress » models based on pairs of spaces \( (V_h, H_h) \) such as those in [6], [12], [13]. Thus, defining \( v = D_t u, \sigma_h = D_t u_h, \) (1.3) and (1.3) are transformed, respectively, to

\[
(D_t v(t), w) - (\text{div} \, \sigma(t), w) = (f(t), w), \quad w \in V, \quad (1.4)
\]

\[
(D_t \sigma(t), \chi) + (v(t), \text{div} \, \chi) = 0, \quad \chi \in H,
\]

\[
v(0) = v_0, \quad \sigma(0) = \sigma = \text{div} \, u_0,
\]

where \( v(t) \in V, \sigma(t) \in H, \, t \geq 0, \) and

\[
(D_t v_h(t), w_h) - (\text{div} \, \sigma_h(t), w_h) = (f(t), w_h), \quad w_h \in V_h,
\]

\[
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\[(1.4_h) \quad (D_h \sigma_h(t), \chi_h) + (v_h(t), \text{div} \chi_h) = 0, \quad \chi_h \in H_h, \quad v_h(0) = v_{0,h}, \quad \sigma_h(0) = \sigma_{0,h},\]

where \(v_h(t) \in V_h, \sigma_h(t) \in H_h, t \geq 0.\)

We now list the basic features of the space \(V_h, H_h\) which lead to a straightforward analysis of the convergence of \(v_h\) to \(v\) and \(\sigma_h\) to \(\sigma:\)

(H.1) There exists a linear operator \(\Pi_h : H \rightarrow H_h\) such that

\[(1.5) \quad (\text{div} \Pi_h \chi, w_h) = (\text{div} \chi, w_h), \quad \forall w_h \in V_h, \quad \chi \in H,\]

\[(1.6) \quad \|\Pi_h \chi - \chi\| \leq Ch^s \|\chi\|_s \quad \text{for} \quad 1 \leq s \leq r, \quad r \geq 2\]

(\(\| \|\) is the \(L_2(\Omega)^2\)-norm, and \(\| . \|_s\) is the \(H^s(\Omega)^2\)-norm).

(H.2) There exists a linear operator \(P_h : V \rightarrow V_h\) such that

\[(1.7) \quad (P_h v, \text{div} \chi_h) = (v, \text{div} \chi_h), \quad \forall \chi_h \in H_h, \quad v \in V,\]

\[(1.8) \quad \|P_h v - v\| \leq Ch^s \|v\|_s, \quad 1 \leq s \leq r, \quad r \geq 2\]

(\(\| \|\) is the \(L_2(\Omega)\)-norm, and \(\| . \|_s\) is the \(H^s(\Omega)\)-norm, and, as usual, \(C\) denotes a generic constant which depends only on the data and on the particular discretization scheme).

If \(\Omega\) is a polygonal domain and \(V_h, H_h\) are the Raviart-Thomas spaces \([12], [13]\), or if these spaces are the pairs introduced in the paper by Brezzi, Douglas, Jr., and Marini \([6]\), \(\text{div} \chi_h \in V_h\), and \(P_h\) can be taken to be the \(L_2^*\)-projection. For an example of a pair \((V_h, H_h)\) satisfying the above hypotheses (with \(r = 2\)), where \(P_h\) is not the \(L_2^*\)-projection, we refer the reader to the paper by Johnson and Thomée \([12]\). We would also like to point out that (H.1) and (H.2) are valid for the mixed method that has been introduced by Arnold, Douglas, Jr., and Gupta \([3]\) to approximate solution of plane elasticity problems. Our analysis is readily adapted to the corresponding (genuine) velocity-stress formulation of the time-dependent problem.

We can now state and prove our convergence result:

**Theorem:** If \(u\) is the solution of \((1.1)\), \(v = D_t u, \sigma = \nabla u\), and if the pair \(\{v_h, \sigma_h\}\) is the solution of \((1.4_h)\), under the hypotheses (H.1) and (H.2) we have, for \(1 \leq s \leq r, \quad r \geq 2,\)

\[(1.9) \quad \|v_h(t) - v(t)\| + \|\sigma_h(t) - \sigma(t)\| \leq C (\|v_0 - v_{0,h}\| + \|\sigma_0 - \sigma_{0,h}\|) + \]

\[+ Ch^s \left( \|v_0\|_s + \|\sigma_0\|_s + \int_0^t \|D_t v(\tau)\|_s + \|D_t \sigma(\tau)\|_s \) d\tau \right) .\]
Proof : Let us denote by $X$ the space $V \times H$, the elements of which will be designated as $\xi = \{v, \sigma\}$ or $\zeta = \{w, \chi\}$ and set
\[
((\xi, \zeta)) = (v, w) + (\sigma, \chi), \quad \|\xi\| = \sqrt{((\xi, \xi))}.
\]

Let $X_h = V_h \times H_h$ be equipped with $((., .))$ and the induced norm $\|\cdot\|$. Elements of $X_h$ will be designated as $\xi_h = \{v_h, \sigma_h\}$ or $\zeta_h = \{w_h, \chi_h\}$.

We define the bilinear form $a(. , .)$ on $X$ by
\[
a(\xi, \zeta) = -(\text{div } \sigma, w) + (v, \text{div } \chi)
\]
for $\xi = \{v, \sigma\}$, $\zeta = \{w, \chi\}$.

We can now express (1.4) as
\[
((D, \xi(t), \zeta)) + a(\xi(t), \zeta) = (f(t), w), \quad \zeta \in X
\]
\[
(\xi(t) = \{v(t), \sigma(t)\}, \zeta = \{w, \chi\})
\]
and we can express (1.4$_h$) as
\[
((D, \xi_h(t), \zeta_h)) + a(\xi_h(t), \zeta_h) = (f(t), w_h), \quad \zeta_h \in X_h
\]
\[
(\xi_h(t) = \{v_h(t), \sigma_h(t)\}, \zeta_h = \{w_h, \chi_h\})
\]

Let us define $P_h \xi = \{P_h v, \Pi_h \sigma\}$ for $\xi = \{v, \sigma\} \in X$, and observe that
\[
a(P_h \xi, \zeta_h) = a(\xi, \zeta_h), \quad \zeta_h \in X_h
\]
by (H.1) ((1.5)) and (H.2) ((1.7)).

Therefore we obtain from (1.11)
\[
((D, P_h \xi(t), \zeta_h)) + a(P_h \xi(t), \zeta_h) = (f(t), w_h) + ((P_h D, \xi(t) - D, \xi_h(t), \zeta_h)), \quad \zeta_h \in X_h.
\]
Setting $\varepsilon_h(t) = P_h \xi(t) - \xi_h(t)$, (1.11$_h$) and (1.13) yield
\[
((D, \varepsilon_h(t), \zeta_h)) + a(\varepsilon_h(t), \zeta_h) = (f(t), w_h) + ((P_h D, \xi(t) - D, \xi_h(t), \zeta_h)), \quad \zeta_h \in X_h.
\]

Let us define $\Lambda_h : X_h \to X_h$ by
\[
((\Lambda_h \xi_h, \zeta_h)) = a(\xi_h, \zeta_h), \quad \zeta_h, \xi_h \in X_h.
\]
Since
\[
a(\xi_h, \zeta_h) = -a(\zeta_h, \xi_h), \quad \xi_h, \zeta_h \in X_h
\]
as is readily seen (cf. (1.10)), $\Lambda_h$ is shew-adjoint,

$$(1.17) \quad ((\Lambda_h \xi_h, \zeta_h)) = - ((\xi_h, \Lambda_h \zeta_h)), \quad \xi_h, \zeta_h \in X_h,$$

and $-\Lambda_h$ generates the unitary group $e^{-t\Lambda_h}$. In particular,

$$\| e^{-t\Lambda_h} \xi_h(0) \| = \| \xi_h(0) \|, \quad t \in \mathbb{R}. \quad (1.18)$$

Let us denote by $P_h^0: X \to X_h$ the projection with respect ((., .)).

We can now express (1.14) as

$$D_t \varepsilon_h(t) + \Lambda_h \varepsilon_h(t) = P_h^0 (P_h D_t \xi(t) - D_t \xi(t)) \quad (1.19)$$

so that

$$\varepsilon_h(t) = e^{-t\Lambda_h} \varepsilon_h(0) + \int_0^t e^{-(t-\tau)\Lambda_h} P_h^0 (P_h D_\tau \xi(\tau) - D_\tau \xi(\tau)) \, d\tau. \quad (1.20)$$

(1.18) and (1.20) yield the estimate

$$\| \varepsilon_h(t) \| \leq \| \varepsilon_h(0) \| + \int_0^t \| P_h D_\tau \xi(\tau) - D_\tau \xi(\tau) \| \, d\tau. \quad (1.21)$$

($P_h^0$ is the ((., .))-projection).

(1.21) is readily translated to

$$\| P_h v(t) - v_h(t) \| + \| \Pi_h \sigma(t) - \sigma_h(t) \|$$

$$\leq C \left( \| P_h v_0 - v_{0,h} \| + \| \Pi_h \sigma_0 - \sigma_{0,h} \| \right.$$

$$+ \int_0^t \left( \| P_h D_\tau v(\tau) - D_\tau v(\tau) \| + \| \Pi_h D_\tau \sigma(\tau) - D_\tau \sigma(\tau) \| \right) \, d\tau)$$

$$\leq C \left( \| v_0 - v_{0,h} \| + \| \sigma_0 - \sigma_{0,h} \| + \| P_h v_0 - v_0 \| + \| \Pi_h \sigma_0 - \sigma_0 \| \right.$$\n
$$+ \int_0^t \left( \| P_h D_\tau v(\tau) - D_\tau v(\tau) \| + \| \Pi_h D_\tau \sigma(\tau) - D_\tau \sigma(\tau) \| \right) \, d\tau),$$

and this, together with (1.6) and (1.8), yields

$$\| P_h v(t) - v_h(t) \| + \| \Pi_h \sigma(t) - \sigma_h(t) \|$$

$$\leq C \left( \| v_0 - v_{0,h} \| + \| \sigma_0 - \sigma_{0,h} \| + h^s (\| v_0 \|_s + \| \sigma_0 \|_s) \right)$$\n
$$+ C h^s \int_0^t \left( \| D_\tau v(\tau) \|_s + \| D_\tau \sigma(\tau) \|_s \right) \, d\tau. \quad (1.22)$$
Since
\[ \| v(t) - v_h(t) \| + \| \sigma(t) - \sigma_h(t) \| \leq \| v(t) - P_h v(t) \| + \| P_h v(t) - v_h(t) \| \\
+ \| \sigma(t) - \Pi_h \sigma(t) \| + \| \Pi_h \sigma(t) - \sigma_h(t) \| \\
\leq C h^t (\| v(t) \|_s + \| \sigma(t) \|_s ) + \| P_h v(t) - v_h(t) \| + \| \Pi_h \sigma(t) - \sigma_h(t) \|, \]

by (1.6) and (1.8), and
\[ \| v(t) \|_s \leq \| v_0 \|_s + \int_0^t \| D_v \sigma(\tau) \|_s d \tau, \]
\[ \| \sigma(t) \|_s \leq \| \sigma_0 \|_s + \int_0^t \| D_\sigma \sigma(\tau) \|_s d \tau, \]

(1.22) leads to (1.9), the assertion of the theorem.

2. SOME OBSERVATIONS IN REGARD TO THE TIME-DIFFERENCING OF THE SEMIDISCRETE MODEL

(1.4\_h) leads to a system of ordinary differential equations in the form

\[
\begin{align*}
M_0 D_t W - D \Sigma &= F, \\
M_1 D_t \Sigma + D^T W &= 0,
\end{align*}
\]

(2.1)

where $W$ corresponds to $v_h$, $\Sigma$ corresponds to $\sigma_h$, $M_0$, $M_1$ are symmetric, positive-definite matrices, and $D^T$ denotes the transpose of $D$. The application of implicit Euler time-differencing

\[
\begin{align*}
M_0 \frac{W^{n+1} - W^n}{k} - D \Sigma^{n+1} &= F^{n+1}, \\
M_1 \frac{\Sigma^{n+1} - \Sigma^n}{k} + D^T W^{n+1} &= 0,
\end{align*}
\]

(2.2)

($k$ denotes the time step), necessitates the solution of

\[
\begin{align*}
M_0 W^{n+1} - kD \Sigma^{n+1} &= kF^{n+1} + M_0 W^n, \\
M_1 \Sigma^{n+1} + kD^T W^{n+1} &= M_1 \Sigma^n.
\end{align*}
\]

(2.3)

$M_0$ is in block-diagonal form if $V_h$ consists of functions with no continuity requirement across inter-element boundaries, as is the case in [6], [12], [13], and the elimination of $W^{n+1}$ in (2.3) is efficiently implementable. This leads to a system in the form

\[
\begin{align*}
(M_1 + k^2 D^T M_0^{-1} D) \Sigma^{n+1} &= G,
\end{align*}
\]

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where $M_1$ is symmetric, positive definite and $D^T M_0^{-1} D$ is symmetric, positive-semidefinite, for the determination of $\Sigma^{n+1}$.

On the other hand, (1.3h) leads to

\begin{align}
M_0 D_t^2 U - D \Sigma &= F, \\
M_1 \Sigma + D^T U &= 0,
\end{align}

where $U$ corresponds to $u_h$. (2.5) can be expressed as

\begin{align}
M_0 D_t^2 U + D M_1^{-1} D^T U &= F, \\
M_0 D_t U - W &= 0, \\
M_0 D_t W + D M_1^{-1} D^T U &= F,
\end{align}

and implicit Euler time-differencing is applied to (2.7),

\begin{align}
\frac{U^{n+1} - U^n}{k} - W^{n+1} &= 0, \\
M_0 \frac{W^{n+1} - W^n}{k} + D M_1^{-1} D^T U^{n+1} &= F^{n+1},
\end{align}

elimination of $W^{n+1}$ leads to a system in the form

\begin{align}
(M_0 + k^2 D M_1^{-1} D^T) U^{n+1} &= \tilde{G}.
\end{align}

The matrix in (2.9) is symmetric, positive-definite, so that (2.9) is solvable. But $M_1$ is not block-diagonal, unlike $M_0$, so that deriving the reduced system (2.9), which includes inverting $M_1$, is more expensive than forming the reduced system (2.4). The time-independent counterpart of (2.5),

\begin{align}
- D \Sigma &= F, \\
M_1 \Sigma + D^T U &= 0,
\end{align}

led Arnold and Brezzi [2] to relax the requirement that $\text{div} \sigma_h \in L_2(\Omega)$ in order to have a block-diagonal matrix instead of $M_1$ and be able to eliminate $\Sigma$ efficiently. This approach has to introduce a multiplier corresponding to the relaxation of the requirement $\text{div} \sigma_h \in L_2(\Omega)$.

The above considerations suggest that the «velocity-stress» formulation (1.4h) may be preferable to (1.3h) if the approximation of the «stress» $\sigma$ is of primary concern.

The application of diagonally implicit Runge-Kutta methods (see, for example, Crouzeix [8], Crouzeix and Raviart [9], Alexander [1], Burrage vol. 22, n° 2, 1988
[7], Dougalis and Serbin [10]) to (2.1) leads to systems similar to (2.4) so that our discussion is relevant to higher-order time differencing as well. We will not prove error estimates for such full-discrete approximation schemes based on \( (1.4_h) \). Such estimates should be obtainable by employing techniques that have been utilized in [5] or [11], for example.

REFERENCES