Modélisation mathématique et analyse numérique

Tunc Geveci

On the application of mixed finite element methods to the wave equations

Modélisation mathématique et analyse numérique, tome 22, n° 2 (1988), p. 243-250

<http://www.numdam.org/item?id=M2AN_1988__22_2_243_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 22, n° 2, 1988, p. 243 à 250)

ON THE APPLICATION OF MIXED FINITE ELEMENT METHODS TO THE WAVE EQUATIONS (*)

by Tunc GEVECI [1]

Abstract — *The convergence of certain semidiscrete approximation schemes based on the « velocity-stress » formulation of the wave equation and spaces such as those introduced by Raviart and Thomas is discussed The discussion also applies to similar schemes for the equations of elasticity*

Resumé. — *La convergence de certains schémas d'approximation semi-discrète basés sur la formulation « vitesse-contrainte » de l'équation d'onde et d'espace tel que ceux introduits par Raviart et Thomas est discuté La discussion s'applique également pour les schémas similaires aux équations d'élasticité*

1. THE « VELOCITY-STRESS » FORMULATION OF THE WAVE EQUATION AND A SEMIDISCRETE VERSION

Let us consider the following initial-boundary value problem for the wave equation :

$$\begin{aligned} &D_t^2 u(t,x) - \Delta u(t,x) = f(t,x), \quad t>0, \quad x \in \Omega \subset \mathbb{R}^2, \\ &u(t,x) = 0, \quad t>0, \quad x \in \Gamma, \\ &u(0,x) = u_0(x), \quad D_t u(0,x) = v_0(x), \quad x \in \Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain with boundary Γ, and f, u_0, v_0 are given functions. Introducing the « stress » $\sigma = \nabla u$, (1.1) may be reformulated as

$$\begin{aligned} &D_t^2 u(t,x) - \operatorname{div} \sigma(t,x) = f(t,x), \quad t>0, \quad x \in \Omega, \\ &\sigma(t,x) = \nabla u(t,x), \quad t>0, \quad x \in \Omega, \\ &u(t,x) = 0, \quad t>0, \quad x \in \Gamma, \\ &u(0,x) = u_0(x), \quad D_t u(0,x) = v_0(x), \quad x \in \Omega. \end{aligned} \tag{1.2}$$

(*) Received in October 1986.

[1] Department of Mathematical Sciences, San Diego State University San Diego, California 92182

We use the notation of Johnson and Thomée [12] :

$$V = L_2(\Omega)\,, \quad H = \{\chi \in L_2(\Omega)^2 : \operatorname{div} \chi \in L_2(\Omega)\}\,.$$

Using Green's formula

$$\int_\Omega u \operatorname{div} \chi \, dx = \int_\Gamma u\chi \,.\, n \, ds - \int_\Omega \nabla u \,.\, \chi \, dx$$

where n is the unit exterior normal to Γ, a Galerkin version of (1.2) is to seek $u(t) \in V$, $\sigma(t) \in H$, $t > 0$, satisfying

$$\begin{aligned} &(D_t^2 u(t), w) - (\operatorname{div} \sigma(t), w) = (f(t), w)\,, \quad w \in V\,, \\ &(\sigma(t)\,\chi) + (u(t), \operatorname{div} \chi) = 0\,, \quad \chi \in H\,, \\ &u(0) = u_0\,, \quad D_t u(0) = v_0\,, \end{aligned} \tag{1.3}$$

where the parentheses denote the appropriate inner products (L_2-inner product in V, $L_2(\Omega)^2$-inner product in H). If $V_h \subset V$ and $H_h \subset H$ are finite dimensional subspaces, such as the spaces introduced by Raviart and Thomas [13], and by Brezzi, Douglas, Jr. and Marini [6], a semidiscrete version of (1.3) seeks $u_h(t) \in V_h$, $\sigma_h(t) \in H_h$, $t > 0$, satisfying

$$\begin{aligned} &(D_t^2 u_h(t), w_h) - (\operatorname{div} \sigma_h(t), w_h) = (f(t), w_h)\,, \quad w_h \in V_h\,, \\ &(\sigma_h(t), \chi_h) + (u_h(t), \operatorname{div} \chi_h) = 0\,, \quad \chi_h \in H_h\,, \\ &u_h(0) = u_{0,h}\,, \quad D_t u_h(0) = v_{0,h}\,, \end{aligned} \tag{1.3$_h$}$$

where $u_{0,h}$, $v_{0,h} \in V_h$ are approximations to u_0 and v_0, respectively.

Johnson and Thomée [12] have discussed the parabolic counterpart of (1.3_h). The analysis of convergence of (1.3_h) can be carried out along similar lines, parallel to Baker and Bramble [4], for example. (1.3_h) is treated essentially as a non-conforming « displacement » model for the wave equation (1.1). The purpose of this note is to discuss the convergence of the « velocity-stress » models based on pairs of spaces (V_h, H_h) such as those in [6], [12], [13]. Thus, defining $v = D_t u$, $v_h = D_t u_h$, (1.3) and (1.3_h) are transformed, respectively, to

$$\begin{aligned} &(D_t v(t), w) - (\operatorname{div} \sigma(t), w) = (f(t), w)\,, \quad w \in V\,, \\ &(D_t \sigma(t), \chi) + (v(t), \operatorname{div} \chi) = 0\,, \quad \chi \in H\,, \\ &v(0) = v_0\,, \quad \sigma(0) = \sigma_0 = \nabla u_0\,, \end{aligned} \tag{1.4}$$

where $v(t) \in V$, $\sigma(t) \in H$, $t \geqslant 0$, and

$$(D_t v_h(t), w_h) - (\operatorname{div} \sigma_h(t), w_h) = (f(t), w_h), w_h \in V_h\,,$$

$$(1.4_h) \qquad (D_t \sigma_h(t), \chi_h) + (v_h(t), \operatorname{div} \chi_h) = 0 , \quad \chi_h \in H_h ,$$
$$v_h(0) = v_{0,h} , \quad \sigma_h(0) = \sigma_{0,h} ,$$

where $v_h(t) \in V_h$, $\sigma_h(t) \in H_h$, $t \geqslant 0$.

We now list the basic features of the space V_h, H_h which lead to a straightforward analysis of the convergence of v_h to v and σ_h to σ :

(H.1) There exists a linear operator $\Pi_h : H \to H_h$ such that

$$(1.5) \qquad (\operatorname{div} \Pi_h \chi, w_h) = (\operatorname{div} \chi, w_h) \quad \forall w_h \in V_h , \quad \chi \in H ,$$

$$(1.6) \qquad \| \Pi_h \chi - \chi \| \leqslant C h^s \| \chi \|_s \quad \text{for} \quad 1 \leqslant s \leqslant r , \quad r \geqslant 2$$

($\| \ \|$ is the $L_2(\Omega)^2$-norm, and $\| . \|_s$ is the $H^s(\Omega)^2$-norm).

(H.2) There exists a linear operator $P_h : V \to V_h$ such that

$$(1.7) \qquad (P_h v, \operatorname{div} \chi_h) = (v, \operatorname{div} \chi_h) \quad \forall \chi_h \in H_h , \quad v \in V ,$$

$$(1.8) \qquad \| P_h v - v \| \leqslant C h^s \| v \|_s , \quad 1 \leqslant s \leqslant r , \quad r \geqslant 2$$

($\| \ \|$ is the $L_2(\Omega)$-norm, and $\| \ \|_s$ is the $H^s(\Omega)$-norm, and, as usual, C denotes a generic constant which depends only on the data and on the particular discretization scheme).

If Ω is a polygonal domain and V_h, H_h are the Raviart-Thomas spaces [12], [13], or if these spaces are the pairs introduced in the paper by Brezzi, Douglas, Jr., and Marini [6], div $\chi_h \in V_h$, and P_h can be taken to be the L_2-projection. For an example of a pair (V_h, H_h) satisfying the above hypotheses (with $r = 2$), where P_h is not the L_2-projection, we refer the reader to the paper by Johnson and Thomée [12]. We would also like to point out that (H.1) and (H.2) are valid for the mixed method that has been introduced by Arnold, Douglas, Jr., and Gupta [3] to approximate solution of plane elasticity problems. Our analysis is readily adapted to the corresponding (genuine) velocity-stress formulation of the time-dependent problem.

We can now state and prove our convergence result :

THEOREM : *If u is the solution of* (1.1), $v = D_t u$, $\sigma = \nabla u$, *and if the pair* $\{v_h, \sigma_h\}$ *is the solution of* (1.4_h), *under the hypotheses* (H.1) *and* (H.2) *we have, for* $1 \leqslant s \leqslant r$, $r \geqslant 2$,

$$(1.9) \quad \| v_h(t) - v(t) \| + \| \sigma_h(t) - \sigma(t) \| \leqslant C (\| v_0 - v_{0,h} \| + \| \sigma_0 - \sigma_{0,h} \|) + \\ + C h^s \left(\| v_0 \|_s + \| \sigma_0 \|_s + \int_0^t (\| D_\tau v(\tau) \|_s + \| D_\tau \sigma(\tau) \|_s) \, d\tau \right) .$$

Proof: Let us denote by X the space $V \times H$, the elements of which will be designated as $\xi = \{v, \sigma\}$ or $\zeta = \{w, \chi\}$ and set

$$((\xi, \zeta)) = (v, w) + (\sigma, \chi),$$
$$|||\xi||| = \sqrt{((\xi, \xi))}.$$

Let $X_h = V_h \times H_h$ be equipped with $((.\,,.))$ and the induced norm $|||\cdot|||$. Elements of X_h will be designated as $\xi_h = \{v_h, \sigma_h\}$ or $\zeta_h = \{w_h, \chi_h\}$.

We define the bilinear form $a(.\,,.)$ on X by

$$a(\xi, \zeta) = -(\operatorname{div} \sigma, w) + (v, \operatorname{div} \chi) \tag{1.10}$$

for $\xi = \{v, \sigma\}$, $\zeta = \{w, \chi\}$.

We can now express (1.4) as

$$((D_t\, \xi(t), \zeta)) + a(\xi(t), \zeta) = (f(t), w), \quad \zeta \in X \tag{1.11}$$
$$(\xi(t) = \{v(t), \sigma(t)\}, \zeta = \{w, \chi\}),$$

and we can express (1.4_h) as

$$((D_t\, \xi_h(t), \zeta_h)) + a(\xi_h(t), \zeta_h) = (f(t), w_h), \quad \zeta_h \in X_h \tag{1.11$_h$}$$
$$(\xi_h(t) = \{v_h(t), \sigma_h(t)\}, \zeta_h = \{w_h, \chi_h\}).$$

Let us define $\underline{P}_h\, \xi = \{P_h\, v, \Pi_h\, \sigma\}$ for $\xi = \{v, \sigma\} \in X$, and observe that

$$a(\underline{P}_h\, \xi, \zeta_h) = a(\xi, \zeta_h), \quad \zeta_h \in X_h \tag{1.12}$$

by (H.1) ((1.5)) and (H.2) ((1.7)).

Therefore we obtain from (1.11)

$$\begin{aligned} ((D_t\, \underline{P}_h\, \xi(t), \zeta_h)) &+ a(\underline{P}_h\, \xi(t), \zeta_h) \\ &= (f(t), w_h) + ((\underline{P}_h\, D_t\, \xi(t) - D_t\, \xi(t), \zeta_h)), \quad \zeta_h \in X_h. \end{aligned} \tag{1.13}$$

Setting $\varepsilon_h(t) = \underline{P}_h\, \xi(t) - \xi_h(t)$, (1.11_h) and (1.13) yield

$$\begin{aligned} ((D_t\, \varepsilon_h(t), \zeta_h)) &+ a(\varepsilon_h(t), \zeta_h) = \\ &= ((\underline{P}_h\, D_t\, \xi(t) - D_t\, \xi(t), \zeta_h)), \quad \zeta_h \in X_h. \end{aligned} \tag{1.14}$$

Let us define $\Lambda_h : X_h \to X_h$ by

$$((\Lambda_h\, \xi_h, \zeta_h)) = a(\xi_h, \zeta_h), \quad \xi_h, \zeta_h \in X_h. \tag{1.15}$$

Since

$$a(\xi_h, \zeta_h) = -a(\zeta_h, \xi_h), \quad \xi_h, \zeta_h \in X_h, \tag{1.16}$$

as is readily seen (*cf.* (1.10)), Λ_h is shew-adjoint,

$$((\Lambda_h \xi_h, \zeta_h)) = -((\xi_h, \Lambda_h \zeta_h)), \quad \xi_h, \zeta_h \in X_h, \tag{1.17}$$

and $-\Lambda_h$ generates the unitary group $e^{-t\Lambda_h}$. In particular,

$$|||e^{-t\Lambda_h}\xi_h(0)||| = |||\xi_h(0)|||, \quad t \in \mathbb{R}. \tag{1.18}$$

Let us denote by $\underline{P}_h^0 : X \to X_h$ the projection with respect $((.\,,.))$.
We can now express (1.14) as

$$D_t \varepsilon_h(t) + \Lambda_h \varepsilon_h(t) = \underline{P}_h^0(\underline{P}_h D_t \xi(t) - D_t \xi(t)) \tag{1.19}$$

so that

$$\varepsilon_h(t) = e^{-t\Lambda_h}\varepsilon_h(0) + \int_0^t e^{-(t-\tau)\Lambda_h}\underline{P}_h^0(\underline{P}_h D_\tau \xi(\tau) - D_\tau \xi(\tau))\, d\tau. \tag{1.20}$$

(1.18) and (1.20) yield the estimate

$$|||\varepsilon_h(t)||| \leqslant |||\varepsilon_h(0)||| + \int_0^t |||\underline{P}_h D_\tau \xi(\tau) - D_\tau \xi(\tau)|||\, d\tau \tag{1.21}$$

($\underline{P}_h^0$ is the $((.\,,.))$-projection).
(1.21) is readily translated to

$$\begin{aligned}
&\|P_h v(t) - v_h(t)\| + \|\Pi_h \sigma(t) - \sigma_h(t)\| \\
&\qquad \leqslant C(\|P_h v_0 - v_{0,h}\| + \|\Pi_h \sigma_0 - \sigma_{0,h}\| \\
&\qquad + \int_0^t (\|P_h D_\tau v(\tau) - D_\tau v(\tau)\| + \|\Pi_h D_\tau \sigma(\tau) - D_\tau \sigma(\tau)\|)\, d\tau) \\
&\qquad \leqslant C(\|v_0 - v_{0,h}\| + \|\sigma_0 - \sigma_{0,h}\| + \|P_h v_0 - v_0\| + \|\Pi_h \sigma_0 - \sigma_0\| \\
&\qquad + \int_0^t (\|P_h D_\tau v(\tau) - D_\tau v(\tau)\| + \|\Pi_h D_\tau \sigma(\tau) - D_\tau \sigma(\tau)\|)\, d\tau),
\end{aligned}$$

and this, together with (1.6) and (1.8), yields

$$\begin{aligned}
&\|P_h v(t) - v_h(t)\| + \|\Pi_h \sigma(t) - \sigma_h(t)\| \\
&\qquad \leqslant C(\|v_0 - v_{0,h}\| + \|\sigma_0 - \sigma_{0,h}\| + h^s(\|v_0\|_s + \|\sigma_0\|_s)) \\
&\qquad + Ch^s \int_0^t (\|D_\tau v(\tau)\|_s + \|D_\tau \sigma(\tau)\|_s)\, d\tau.
\end{aligned} \tag{1.22}$$

Since

$$\begin{aligned}\|v(t)-v_h(t)\| &+ \|\sigma(t)-\sigma_h(t)\| \\ &\leq \|v(t)-P_h v(t)\| + \|P_h v(t)-v_h(t)\| \\ &\quad + \|\sigma(t)-\Pi_h \sigma(t)\| + \|\Pi_h \sigma(t)-\sigma_h(t)\| \\ &\leq Ch^s(\|v(t)\|_s + \|\sigma(t)\|_s) + \|P_h v(t)-v_h(t)\| + \|\Pi_h \sigma(t)-\sigma_h(t)\|,\end{aligned}$$

by (1.6) and (1.8), and

$$\|v(t)\|_s \leq \|v_0\|_s + \int_0^t \|D_\tau v(\tau)\|_s \, d\tau ,$$

$$\|\sigma(t)\|_s \leq \|\sigma_0\|_s + \int_0^t \|D_\tau \sigma(\tau)\|_s \, d\tau ,$$

(1.22) leads to (1.9), the assertion of the theorem.

2. SOME OBSERVATIONS IN REGARD TO THE TIME-DIFFERENCING OF THE SEMIDISCRETE MODEL

(1.4_h) leads to a system of ordinary differential equations in the form

(2.1)
$$\begin{aligned} M_0 D_t W - D\Sigma &= F , \\ M_1 D_t \Sigma + D^T W &= 0 , \end{aligned}$$

where W corresponds to v_h, Σ corresponds to σ_h, M_0, M_1 are symmetric, positive-definite matrices, and D^T denotes the transpose of D. The application of implicit Euler time-differencing

(2.2)
$$\begin{aligned} M_0 \frac{W^{n+1} - W^n}{k} - D\Sigma^{n+1} &= F^{n+1} , \\ M_1 \frac{\Sigma^{n+1} - \Sigma^n}{k} + D^T W^{n+1} &= 0 , \end{aligned}$$

(k denotes the time step), necessitates the solution of

(2.3)
$$\begin{aligned} M_0 W^{n+1} - kD\Sigma^{n+1} &= kF^{n+1} + M_0 W^n , \\ M_1 \Sigma^{n+1} + kD^T W^{n+1} &= M_1 \Sigma^n . \end{aligned}$$

M_0 is in block-diagonal form if V_h consists of functions with no continuity requirement across inter-element boundaries, as is the case in [6], [12], [13], and the elimination of W^{n+1} in (2.3) is efficiently implementable. This leads to a system in the form

(2.4)
$$(M_1 + k^2 D^T M_0^{-1} D)\, \Sigma^{n+1} = G ,$$

where M_1 is symmetric, positive definite and $D^T M_0^{-1} D$ is symmetric, positive-semidefinite, for the determination of Σ^{n+1}.

On the other hand, (1.3_h) leads to

$$\begin{aligned} M_0 D_t^2 U - D\Sigma &= F , \\ M_1 \Sigma + D^T U &= 0 , \end{aligned} \tag{2.5}$$

where U corresponds to u_h. (2.5) can be expressed as

$$M_0 D_t^2 U + D M_1^{-1} D^T U = F , \tag{2.6}$$

where M_0, $D M_1^{-1} D^T$ are symmetric, positive-definite [12]. If (2.6) is expressed as a system in $\{U, W\}$,

$$\begin{aligned} D_t U - W &= 0 \\ M_0 D_t W + D M_1^{-1} D^T U &= F , \end{aligned} \tag{2.7}$$

and implicit Euler time-differencing is applied to (2.7),

$$\begin{aligned} \frac{U^{n+1} - U^n}{k} - W^{n+1} &= 0 \\ M_0 \frac{W^{n+1} - W^n}{k} + D M_1^{-1} D^T U^{n+1} &= F^{n+1} , \end{aligned} \tag{2.8}$$

elimination of W^{n+1} leads to a system in the form

$$(M_0 + k^2 D M_1^{-1} D^T)\, U^{n+1} = \tilde{G} . \tag{2.9}$$

The matrix in (2.9) is symmetric, positive-definite, so that (2.9) is solvable. But M_1 is not block-diagonal, unlike M_0, so that deriving the reduced system (2.9), which includes inverting M_1, is more expensive than forming the reduced system (2.4). The time-independent counterpart of (2.5),

$$\begin{aligned} - D\Sigma &= F , \\ M_1 \Sigma + D^T U &= 0 , \end{aligned} \tag{2.10}$$

led Arnold and Brezzi [2] to relax the requirement that $\operatorname{div} \sigma_h \in L_2(\Omega)$ in order to have a block-diagonal matrix instead of M_1 and be able to eliminate Σ efficiently. This approach has to introduce a multiplier corresponding to the relaxation of the requirement $\operatorname{div} \sigma_h \in L_2(\Omega)$.

The above considerations suggest that the « velocity-stress » formulation (1.4_h) may be preferable to (1.3_h) if the approximation of the « stress » σ is of primary concern.

The application of diagonally implicit Runge-Kutta methods (see, for example, Crouzeix [8], Crouzeix and Raviart [9], Alexander [1], Burrage

[7], Dougalis and Serbin [10]) to (2.1) leads to systems similar to (2.4) so that our discussion is relevant to higher-order time differencing as well. We will not prove error estimates for such full-descrete approximation schemes based on (1.4_h). Such estimates should be obtainable by employing techniques that have been utilized in [5] or [11], for example.

REFERENCES

[1] R. ALEXANDER, *Diagonally implicity Runge-Kutta methods for stiff O.D.E.'s,* SIAM J. Numer. Anal. 14 (1977), 1006-1021.

[2] D. N. ARNOLD and F. BREZZI, *Mixed and nonconforming finite element methods : Implementation, postprocessing and error estimates,* R.A.I.R.O. Math. Model. and Num. Anal. (M^2AN) 1 (1985), 7-32.

[3] D. N. ARNOLD, J. DOUGLAS, Jr., and C. P. GUPTA, *A family of higher order mixed finite element methods for plane elasticity,* Numer. Math. 45 (1984), 1-22.

[4] G. A. BAKER and J. H. BRAMBLE, *Semidiscrete and single step fully discrete approximations for second order hyperbolic equations,* R.A.I.R.O. Anal. Num. 13 (1979), 75-100.

[5] P. BRENNER, M. CROUZEIX and V. THOMÉE, *Single step methods for inhomogeneous linear differential equations in Banach spaces,* R.A.I.R.O. Anal. Num. 16 (1982), 5-26.

[6] F. BREZZI, J. DOUGLAS, Jr. and L. D. MARINI, *Two families of mixed finite elements for second order elliptic problems,* Numer. Math. 47 (1985), 217-235.

[7] K. BURRAGE, *Efficiently implementable algebraically stable Runge-Kutta methods,* SIAM J. Numer. Anal. 19 (1982), 245-258.

[8] M. CROUZEIX, *Sur l'approximation des équations différentielles opérationnelles linéaires par des méthodes de Runge-Kutta,* thèse, Paris (1975).

[9] M. CROUZEIX and P.-A. RAVIART, *Approximation des problèmes d'évolution,* preprint, Université de Rennes (1980).

[10] V. DOUGALIS and S. M. SERBIN, *On some unconditionally stable, higher order methods for numerical solution of the structural dynamics equations,* Int. J. Num. Meth. Eng. 18 (1982), 1613-1621.

[11] E. GEKELER, *Discretization Methods for Stable Initial Value Problems,* Springer Lecture Notes in Mathematics 1044 (1984), Springer-Verlag, Berlin, Heidelberg, New York.

[12] C. JOHNSON and V. THOMÉE, *Error estimates for some mixed finite element methods for parabolic type problems,* R.A.I.R.O. Anal. Num. 15 (1981), 41-78.

[13] P.-A. RAVIART and J. M. THOMAS, *A mixed finite element method for 2nd order problems, in Mathematical Aspects of the Finite Element Method,* Springer Lecture Notes in Mathematics 606 (1977), Springer-Verlag, Berlin-Heidelberg-New York.